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Munkres §31

Ex. 31.1 (Morten Poulsen). Let a and b be distinct points of X. Note that X is Hausdorff, since X is regular. Thus there exists disjoint open sets A and B such that $a \in A$ and $b \in B$. By lemma 31.1(a) there exists open sets U and V such that

$$a \in U \subset \overline{U} \subset A \text{ and } b \in V \subset \overline{V} \subset B.$$

Clearly $\overline{U} \cap \overline{V} = \emptyset$.

Ex. 31.2 (Morten Poulsen). Let A and B be disjoint closed subsets of X. Since X normal there exists disjoint open sets U_0 and U_1 such that $A \subset U_0$ and $B \subset U_1$. By lemma 31.1(b) there exists open sets V_0 and V_1 such that

$$A \subset V_0 \subset \overline{V_0} \subset U_0 \text{ and } B \subset V_1 \subset \overline{V_1} \subset U_1$$

Clearly $\overline{U} \cap \overline{V} = \emptyset$.

Ex. 31.3 (Morten Poulsen).

Theorem 1. *Every order topology is regular.*

Proof. Let X be an ordered set. Let $x \in X$ and let U be a neighborhood of x, may assume $U = (a, b)$, $-\infty \leq a < b \leq \infty$. Set $A = (a, x)$ and $B = (x, b)$. Using the criterion for regularity in lemma 31.1(b) there are four cases:

(1) If $u \in A$ and $v \in B$ then $x \in (u, v) \subset \overline{(u, v)} \subset [u, v] \subset (a, b)$.
(2) If $A = B = \emptyset$ then $(a, b) = \{x\}$ is open and closed, since X Hausdorff, c.f. Ex. 17.10.
(3) If $A = \emptyset$ and $v \in B$ then $x \in (a, v) \subset= [x, v) \subset \overline{[x, v)} \subset [x, v] \subset (a, b)$.
(4) If $u \in A$ and $B = \emptyset$ then $x \in (u, b) \subset (u, x] \subset \overline{(u, x]} \subset [u, x] \subset (a, b)$.

Thus X is regular. □

Ex. 31.5. The diagonal $\Delta \subset Y \times Y$ is closed as Y is Hausdorff [Ex 17.13]. The map $(f, g) : X \to Y \times Y$ is continuous [Thm 18.4, Thm 19.6] so

$$\{x \in X \mid f(x) = g(x)\} = (f, g)^{-1}(\Delta)$$

is closed.

Ex. 31.6. Let $p\colon X \to Y$ be closed continuous surjective map. Then X normal $\Rightarrow Y$ normal.

For this exercise and the next we shall use the following lemma from [Ex 26.12].

Lemma 2. *Let $p\colon X \to Y$ be a closed map.*

(1) *If $p^{-1}(y) \subset U$ where U is an open subspace of X, then $p^{-1}(W) \subset U$ for some neighborhood $W \subset Y$ of y.*
(2) *If $p^{-1}(B) \subset U$ for some subspace B of Y and some open subspace U of X, then $p^{-1}(W) \subset U$ for some neighborhood $W \subset Y$ of B.*

Proof. Note that

$$p^{-1}(W) \subset U \Leftrightarrow \big[p(x) \in W \Rightarrow x \in U\big] \Leftrightarrow \big[x \notin U \Rightarrow p(x) \notin W\big] \Leftrightarrow p(X - U) \subset Y - W$$
$$\Leftrightarrow p(X - U) \cap W = \emptyset$$

(1) The point y does not belong to the closed set $p(X - U)$. Therefore a whole neighborhood $W \subset Y$ of y is disjoint from $p(X - U)$, i.e. $p^{-1}(W) \subset U$.
(2) Each point $y \in B$ has a neighborhood W_y such that $p^{-1}(W_y) \subset U$. The union $W = \bigcup W_y$ is then a neighborhood of B with $p^{-1}(W) \subset U$. □

Since points are closed in X and p is closed, all points in $p(X)$ are closed. All fibres $p^{-1}(y) \subset X$ are therefore also closed. Let y_1 and y_2 be two distinct points in Y. Since X is normal we can separate the disjoint closed sets $p^{-1}(y_1)$ and $p^{-1}(y_1)$ by disjoint neighborhoods U_1 and U_2. Using Lemma 2.(1), choose neighborhoods W_1 of y_1 and W_2 of y_2 such that $p^{-1}(W_1) \subset U_1$ and $p^{-1}(W_2) \subset U_2$. Then W_1 and W_2 are disjoint. Thus Y is Hausdorff.

Essentially the same argument, but now using Lemma 2.(2), shows that we can separate disjoint closed sets in Y by disjoint open sets. Thus Y is normal.

Alternatively, see [Lemma 73.3].

Example: If X is normal and $A \subset X$ is closed, then the quotient space X/A is normal.

Ex. 31.7. Let $p\colon X \to Y$ be closed continuous surjective map such that $p^{-1}(y)$ is compact for each $y \in Y$ (a perfect map).

(a). X Hausdorff $\Rightarrow$ Y Hausdorff.
Let y_1 and y_2 be two distinct points in Y. By an upgraded version [Ex 26.5] of [Lemma 26.4] we can separate the two disjoint compact subspaces $p^{-1}(y_1)$ and $p^{-1}(y_2)$ by disjoint open subspaces $U_1 \supset p^{-1}(y_1)$ and $U_2 \supset p^{-1}(y_2)$ of the Hausdorff space X. Choose (Lemma 2) open sets $W_1 \ni y_1$ and $W_2 \ni y_2$ such that $p^{-1}(W_1) \subset U_1$ and $p^{-1}(W_2) \subset U_2$. Then W_1 and W_2 are disjoint. This shows that Y is Hausdorff as well.

(b). X regular $\Rightarrow$ Y regular.
Y is Hausdorff by (a). Let $C \subset Y$ be a closed subspace and $y \in Y$ a point outside C. It is enough to separate the compact fibre $p^{-1}(y) \subset X$ and the closed set $p^{-1}(C) \subset X$ by disjoint open set. (Lemma 2 will provide open sets in Y separating y and C.) Each $x \in p^{-1}(y)$ can be separated by disjoint open sets from $p^{-1}(C)$ since X is regular. Using compactness of $p^{-1}(y)$ we obtain (as in the proof [Thm 26.3]) disjoint open sets $U \supset p^{-1}(y)$ and $V \supset p^{-1}(C)$ as required.

(c). X locally compact $\Rightarrow$ Y locally compact [1, 3.7.21].
Using compactness of $p^{-1}(y)$ and local compactness of X we construct an open subspace $U \subset X$ and a compact subspace $C \subset X$ such that $p^{-1}(y) \subset U \subset C$. In the process we need to know that a finite union of compact subspaces is compact [Ex 26.3]. By Lemma 2, there is an open set $W \ni y$ such that $p^{-1}(y) \subset p^{-1}(W) \subset U \subset C$. Then $y \in W \subset p(C)$ where $p(C)$ is compact [Thm 26.5]. Thus Y is locally compact.

(d). X 2nd countable $\Rightarrow$ Y 2nd countable.
Let $\{B_j\}_{j\in\mathbf{Z}_+}$ be countable basis for X. For each finite subset $J \subset \mathbf{Z}_+$, let $U_J \subset X$ be the union of all open sets of the form $p^{-1}(W)$ with open $W \subset Y$ and $p^{-1}(W) \subset \bigcup_{j\in J} B_j$. There are countably many open sets U_J. The image $p(U_J)$ is a union of open sets in Y, hence open. Let now $V \subset Y$ be any open subspace. The inverse image $p^{-1}(V) = \bigcup_{y\in V} p^{-1}(y)$ is a union of fibres. Since each fibre $p^{-1}(y)$ is compact, it can be covered by a finite union $\bigcup_{j\in J(y)} B_j$ of basis sets contained in $p^{-1}(V)$. By Lemma 2, there is an open set $W \subset Y$ such that $p^{-1}(y) \subset p^{-1}(W) \subset \bigcup_{j\in J(y)} B_j$. Taking the union of all these open sets W, we get $p^{-1}(y) \subset U_{J(y)} \subset \bigcup_{j\in J(y)} B_j \subset p^{-1}(V)$. We now have $p^{-1}(V) = \bigcup_{y\in V} U_{J(y)}$ so that $V = pp^{-1}(V) = \bigcup_{y\in V} p(U_{J(y)})$ is a union of sets from the countable collection $\{p(U_J)\}$ of open sets. Thus Y is 2nd countable.

Example: If Y is compact, then the projection map $\pi_2\colon X \times Y \to Y$ is perfect. (Show that π_2 is closed!)

Ex. 31.8. It is enough to show that $p\colon X \to G\backslash X$ is a perfect map [Ex 31.6, Ex 31.7]. We show that

(1) The saturation GA of any closed subspace $A \subset X$ is closed. (The map p is closed.)
(2) The orbit Gx of any point $x \in X$ is compact. (The fibres $p^{-1}(Gx) = Gx$ are compact.)

(1) Let $y \in X$ be any point outside $GA = \bigcup_{g\in G} gA$. For any $g \in G$, $g^{-1}y$ is outside the closed set $A \subset X$. By continuity of the action $G \times X \to X$,

$$U_g^{-1}V_g \subset X - A$$

for open sets $G \supset U_g \ni g$ and $X \subset V_g \ni y$. The compact space G can be covered by finitely many of the open sets U_g, say $G = U_1 \cup \cdots \cup U_n$. Let $V = V_1 \cap \cdots \cap V_n$ be the intersection of the corresponding neighborhoods of y. Then

$$G^{-1}V = \bigcup_i U_i^{-1}V \subset \bigcup_i U_i^{-1}V_i \subset X - A$$

so $y \in V \subset G(X - A) = X - GA$.
(2) The orbit Gx of a point $x \in X$ is compact because [Thm 26.5] it is the image of the compact space G under the continuous map $G \to X : g \to gx$.

References

[1] Ryszard Engelking, *General topology*, second ed., Sigma Series in Pure Mathematics, vol. 6, Heldermann Verlag, Berlin, 1989, Translated from the Polish by the author. MR **91c:**54001