1st December 2004

Munkres §30

Ex. 30.3 (Morten Poulsen). Let X be second-countable and let A be an uncountable subset of X. Suppose only countably many points of A are limit points of A and let $A_0 \subset A$ be the countable set of limit points.

For each $x \in A - A_0$ there exists a basis element U_x such that $x \in U_x$ and $U_x \cap A = \{x\}$. Hence if a and b are distinct points of $A - A_0$ then $U_a \neq U_b$, since $U_a \cap A = \{a\} \neq \{b\} = U_b \cap A$. It follows that there uncountably many basis elements, contradicting that X is second-countable.

Note that it also follows that the set of points of A that are not limit points of A are countable.

Ex. 30.4 (Morten Poulsen).

Theorem 1. *Every compact metrizable space is second-countable.*

Proof. Let X be a compact metrizable space, and let d be a metric on X that induces the topology on X.

For each $n \in \mathbf{Z}_+$ let $\mathcal{A}^n$ be an open covering of X with $1/n$-balls. By compactness of X there exists a finite subcovering $\mathcal{A}_n$.

Now $\mathcal{B} = \bigcup_{n \in \mathbf{Z}_+} \mathcal{A}_n$ is countable, being a countable union of finite sets.

$\mathcal{B}$ is a basis: Let U be an open set in X and $x \in U$. By definition of the metric topology there exists $\varepsilon > 0$ such that $B_d(x, \varepsilon) \subset U$. Choose $N \in \mathbf{Z}_+$ such that $2/N < \varepsilon$. Since $\mathcal{A}_N$ covers X there exists $B_d(y, 1/N)$ containing x. If $z \in B_d(y, 1/N)$ then

$$d(x, z) \leq d(x, y) + d(y, z) \leq 1/N + 1/N = 2/N < \varepsilon,$$

i.e. $z \in B_d(x, \varepsilon)$, hence $B_d(y, 1/N) \subset B_d(x, \varepsilon) \subset U$. It follows that $\mathcal{B}$ is a basis. $\square$

Ex. 30.5. Let X be a metrizable topological space.

Suppose that X has a countable dense subset A. The collection $\{B(a, r) \mid a \in A, r \in \mathbf{Q}_+\}$ of balls centered at points in A and with a rational radius is a countable basis for the topology: It suffices to show that for any $y \in B(x, \varepsilon)$ there are $a \in A$ and $r \in \mathbf{Q}_+$ such that $y \in B(a, r) \subset B(x, \varepsilon)$. Let r be a positive rational number such that $2r < \varepsilon - d(x, y)$ and let $a \in A \cap B(y, r)$. Then $y \in B(a, r)$, of course, and $B(a, r) \subset B(x, \varepsilon)$ for if $d(a, z) < r$ then $d(x, z) \leq d(x, y) + d(y, z) \leq d(x, y) + d(y, a) + d(a, z) < d(x, y) + 2r < \varepsilon$.

Suppose that X is Lindelöf. For each positive rational number r, let A_r be a countable subset of X such that $X = \bigcup_{a \in A_r} B(a, r)$. Then $A = \bigcup_{r \in \mathbf{Q}_+} A_r$ is a dense countable subset: Any open ball $B(x, \varepsilon)$ contains a point of A_r when $0 < r < \varepsilon$, $r \in \mathbf{Q}$.

We now have an extended version of Thm 30.3:

Theorem 2. *Let X be a topological space. Then*

$$\begin{array}{ccccc} X \text{ has a countable dense subset} & \Longleftarrow & X \text{ is 2nd countable} & \Longrightarrow & X \text{ is Lindelöf} \\ & & \Downarrow & & \\ & & X \text{ is 1st countable} & & \end{array}$$

If X is metrizable, the three conditions of the top line equivalent.

Ex. 30.6. $\mathbf{R}_\ell$ has a countable dense subset and is not 2nd countable. According to [Ex 30.5] such a space is not metrizable.

The ordered square I_o^2 is compact and not second countable. Any basis for the topology has uncountably many members because there are uncountably many disjoint open sets $(x \times 0, x \times 1)$, $x \in I$, and each of them contains a basis open set. (Alternatively, note that I_o^2 contains the uncountable discrete subspace $\{x \times \frac{1}{2} \mid x \in I\}$ so it can not be second countable by [Example 2 p 190].) According to [Ex 30.4] or [30.5(b)] a compact space with no countable basis is not metrizable.

Ex. 30.7. (Open ordinal space and closed ordinal space) Sets of the form (α,β), $-\infty \le \alpha < \beta \le +\infty$, form bases for the topologies on the *open ordinal space* $S_\Omega = [0,\Omega)$ and the *closed ordinal space* $\overline{S}_\Omega = [0,\Omega]$ [§14, Thm 16.4]. The sets $(\alpha,\beta] = (\alpha,\beta+1) = [\alpha+1,\beta]$ are closed and open. Let n denote the nth immediate successor of the first element, 0.

$[0,\Omega)$ **is first countable:** $\{0\} = [0,1)$ is open so clearly $[0,\Omega)$ is first countable at the point 0. For any other element, $\alpha > 0$, we can use the collection of neighborhoods of the form $(\beta,\alpha]$ for $\beta < \alpha$.

$[0,\Omega)$ **does not have a countable dense subset:** The complement of any countable subset contains [Thm 10.3] an interval of the form (α,Ω) (which is nonempty, even uncountable [Lemma 10.2]).

$[0,\Omega)$ **is not second countable:** If it were, there would be a countable dense subset [Thm 30.3].

$[0,\Omega)$ **is not Lindelöf:** The open covering consisting of the sets $[0,\alpha)$, $\alpha < \Omega$, does not contain a countable subcovering.

$[0,\Omega]$ **is not first countable at Ω:** This is a consequence of [Lemma 21.2] in that Ω is a limit point of $[0,\Omega)$ but not the limit point of any sequence in $[0,\Omega)$ for all such sequences are bounded [Example 3, p. 181].

$[0,\Omega]$ **does not have a dense countable subset:** for the same reason as for $[0,\Omega)$.

$[0,\Omega]$ **is not second countable:** It is not even first countable.

$[0,\Omega]$ **is Lindelöf:** It is even compact [Thm 27.1].

$S_\Omega = [0,\Omega)$ is limit point compact but not compact [Example 2, p. 179] so it can not be metrizable [Thm 28.2]. S_Ω is first countable and limit point compact so it is also sequentially compact [Thm 28.2].

$\overline{S}_\Omega = [0,\Omega]$ is not metrizable since it is not first countable.

Ex. 30.9. A space X is Lindelöf if and only if any collection of closed subsets of X with empty intersection contains a countable subcollection with empty intersection. Since closed subsets of closed subsets are closed, it follows immediately that closed subspaces of Lindelöf spaces are Lindelöf.

The anti-diagonal $L \subset \mathbf{R}_\ell \times \mathbf{R}_\ell$ is a closed discrete uncountable subspace [Example 4 p 193]. Thus the closed subset L does not have a countable dense subset even though $\mathbf{R}_\ell \times \mathbf{R}_\ell$ has a countable dense subset.

Ex. 30.12. Let $f\colon X \to Y$ be an open continuous map.

Let $\mathcal{B}$ be a neighborhood basis at the point $x \in X$. Let $f(\mathcal{B})$ be the collection of images $f(B) \subset f(X)$ of members B of the collection $\mathcal{B}$. The sets in $f(\mathcal{B})$ are open in Y, and hence also in $f(X)$, since f is an open map. Let $f(x)$ be a point in $f(X)$. Any neighborhood of $f(x)$ has the form $V \cap f(X)$ for some neighborhood $V \subset Y$ of $f(x)$. Since $p^{-1}(V)$ is a neighborhood of x there is a set B in the collection $\mathcal{B}$ such that $x \in B \subset f^{-1}(V)$. Then $x \in f(B) \subset V \cap f(X)$. This shows that $f(\mathcal{B})$ is a neighborhood basis at $f(x) \in f(X)$.

Let $\mathcal{B}$ be a basis for the topology on X. Let $f(\mathcal{B})$ be the collection of images $f(B) \subset f(X)$ of members B of the collection $\mathcal{B}$. The sets in $f(\mathcal{B})$ are open in Y, and hence also in $f(X)$, since f is an open map. Since $\mathcal{B}$ ia a covering of X, $f(\mathcal{B})$ is a covering of $f(X)$. Suppose that $f(x) \in f(B_1) \cap f(B_2)$ where $x \in X$ and B_1, B_2 are basis sets. Choose a basis set B_3 such that $x \in B_3 \subset f^{-1}(f(B_1) \cap f(B_2))$. Then $f(x) \in f(B_3) \subset f(B_1) \cap f(B_2)$. This shows that $f(\mathcal{B})$ is a basis for a topology $\mathcal{T}_{f(\mathcal{B})}$ on $f(X)$. This topology is coarser than the topology on $f(X)$ since the basis elements are open in $f(X)$. Conversely, let $f(x) \in V \cap f(X)$ where V is open in Y. Choose a basis element B such that $x \in B \subset f^{-1}(V)$. Then $f(x) \in f(B) \subset V \cap f(X)$. This shows that all open subsets of $f(X)$ are in $\mathcal{T}_{f(\mathcal{B})}$. We conclude that $f(\mathcal{B})$ is a basis for the topology on $f(X)$.

We conclude that continuous open maps preserve 1st and 2nd countability.

Ex. 30.13. Let D be a countable dense subset and $\mathcal{U}$ a collection of open disjoint subsets. Pick a member of D inside each of the open open sets in $\mathcal{U}$. This gives an injective map $\mathcal{U} \to D$. Since D is countable also $\mathcal{U}$ is countable.

Ex. 30.16. For each natural number $k \in \mathbf{Z}_+$, let D_k be the set of all finite sequences

$$(I_1, \dots, I_k, x_1, \dots, x_k)$$

where $I_1, \dots, I_k \subset I$ are disjoint closed subintervals of I with rational endpoints and $x_1, \dots, x_k \in \mathbf{Q}$ are rational numbers. Since D_k is a subset of a countable set,

$$D_k \hookrightarrow \overbrace{(\mathbf{Q} \times \mathbf{Q}) \times \cdots \times (\mathbf{Q} \times \mathbf{Q})}^{k} \times \overbrace{\mathbf{Q} \times \cdots \times \mathbf{Q}}^{k} = \mathbf{Q}^{3k},$$

D_k itself is countable [Cor 7.3]. Put $D = \bigcup_{k \in \mathbf{Z}_+} D_k$. As a countable union of countable sets, D is countable [Thm 7.5].

For each element $(I_1, \dots, I_k, x_1, \dots, x_k) \in D_k$, let $x(I_1, \dots, I_k, x_1, \dots, x_k) \in \mathbf{R}^I$ be the element given by

$$\pi_t x(I_1, \dots, I_k, x_1, \dots, x_k) = \begin{cases} x_j & t \in I_j \text{ for some } j \in \{1, \dots, k\} \\ 0 & t \notin I_1 \cup \cdots \cup I_k \end{cases}$$

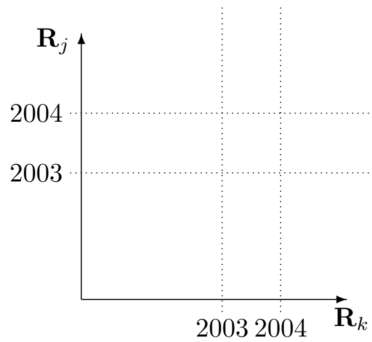
where $\pi_t \colon \mathbf{R}^I \to \mathbf{R}$, $t \in I$, is the projection map. This defines a map $x \colon D \to \mathbf{R}^I$.

(a). The basis open sets in $\mathbf{R}^I$ are finite intersections $\bigcap_{j=1}^k \pi_{i_j}^{-1}(U_{i_j})$ where $i_1, \dots, i_k$ are k distinct points in I and $U_{i_1}, \dots, U_{i_k}$ are k open subsets of $\mathbf{R}$. Choose disjoint closed subintervals I_j such that $i_j \in I_j$ and choose $x_j \in U_{i_j} \cap \mathbf{Q}$, $j = 1, \dots, k$. Then $x(I_1, \dots, I_k, x_1, \dots, x_k) \in \bigcap_{j=1}^k \pi_{i_j}^{-1}(U_{i_j})$ for $\pi_{i_j} x(I_1, \dots, I_k, x_1, \dots, x_k) = x_j \in U_{i_j}$ for all $j = 1, \dots, k$. This shows that any (basis) open set contains an element of $x(D)$, ie that the countable set $x(D)$ is dense in $\mathbf{R}^I$.

(b). Let D be a dense subset of $\mathbf{R}^J$ for some set J. Let $f \colon J \to \mathcal{P}(D)$ be the map from the index set J to the power set $\mathcal{P}(D)$ of D given by $f(j) = D \cap \pi_j^{-1}(2003, 2004)$. Let j and k be two distinct points of J. Then $f(j) \neq f(k)$ for

$$\begin{aligned} f(j) - f(k) &= \big(\pi_j^{-1}(2003, 2004) - \pi_k^{-1}(2003, 2004)\big) \cap D \\ &\supset \big(\pi_j^{-1}(2003, 2004) \cap \pi_k^{-1}(2002, 2003)\big) \cap D \neq \emptyset \end{aligned}$$

since D is dense. This shows that f is injective. Thus card $J \leq$ card $\mathcal{P}(D)$.



References