1st December 2004

Munkres §28

Ex. 28.1 (Morten Poulsen). Let d denote the uniform metric. Choose $c \in (0, 1]$. Let $A = \{0, c\}^\omega \subset [0, 1]^\omega$. Note that if a and b are distinct points in A then $d(a, b) = c$. For any $x \in X$ the open ball $B_d(x, c/3)$ has diameter less than or equal $2c/3$, hence $B_d(x, c/3)$ cannot contain more than one point of A. It follows that x is not a limit point of A.

Ex. 28.6 (Morten Poulsen).

Theorem 1. *Let (X, d) be a compact metric space. If $f : X \to X$ is an isometry then f is a homeomorphism.*

Proof. Clearly any isometry is continuous and injective. If f surjective then f^{-1} is also an isometry, hence it suffices to show that f is surjective.

Suppose $f(X) \subsetneq X$ and let $a \in X - f(X)$. Note that $f(X)$ is compact, since X compact, hence $f(X)$ closed, since X Hausdorff, i.e. $X - f(X)$ is open. Thus there exists $\varepsilon > 0$ such that $a \in B_d(a, \varepsilon) \subset X - f(X)$.

Define a sequence (x_n) by

$$x_n = \begin{cases} a, & n = 1 \\ f(x_n), & n > 1. \end{cases}$$

If $n \neq m$ then $d(x_n, x_m) \geq \varepsilon$: Induction on $n \geq 1$. If $n = 1$ then clearly $d(a, x_m) \geq \varepsilon$, since $x_m \in f(X)$. Suppose $d(x_n, x_m) \geq \varepsilon$ for all $m \neq n$. If $m = 1$ then $d(x_{n+1}, x_1) = d(f(x_n), a) \geq \varepsilon$. If $m > 1$ then $d(x_{n+1}, x_m) = d(f(x_n), f(x_{m-1})) = d(x_n, x_{m-1}) \geq \varepsilon$.

For any $x \in X$ the open ball $B_d(x, \varepsilon/3)$ has diameter less than or equal to $2\varepsilon/3$, hence $B_d(x, \varepsilon/3)$ cannot contain more than one point of A. It follows that x is not a limit point of A. $\square$