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# Munkres §27

**Ex. 27.1 (Morten Poulsen).** Let $A \subset X$ be bounded from above by $b \in X$. For any $a \in A$ is $[a, b]$ compact.

The set $C = \overline{A} \cap [a, b]$ is closed in $[a, b]$, hence compact, c.f. theorem 26.2. The inclusion map $j : C \to X$ is continuous, c.f. theorem 18.2(b). By the extreme value theorem $C$ has a largest element $c \in C$. Clearly $c$ is an upper bound for $A$.

If $c \in A$ then clearly $c$ is the least upper bound. Suppose $c \notin A$. If $d < c$ then $(d, \infty)$ is an open set containing $c$, i.e. $A \cap (d, \infty) \neq \emptyset$, since $c$ is a limit point for $A$, since $c \in C \subset \overline{A}$. Thus $d$ is not an upper bound for $A$, hence $c$ is the least upper bound.

**Ex. 27.3.**

(a). $K$ is an infinite, discrete, closed subspace of $\mathbf{R}_K$, so $K$ can not be contained in any compact subspace of $\mathbf{R}_K$ [Thm 28.1].

(b). The subspaces $(-\infty, 0)$ and $(0, +\infty)$ inherit their standard topologies, so they are connected. Then also their closures, $(-\infty, 0]$ and $[0, +\infty)$ and their union, $\mathbf{R}_K$, are also connected [Thm 23.4, Thm 23.3].

(c). Since the topology $\mathbf{R}_K$ is finer than the standard topology [Lemma 13.4] on $\mathbf{R}$ we have

$$U \text{ is connected in } \mathbf{R}_K \overset{\text{Ex 23.1}}{\Rightarrow} U \text{ is connected in } \mathbf{R} \overset{\text{Thm 24.1}}{\Leftrightarrow} U \text{ is convex}$$

for any subspace $U$ of $\mathbf{R}_K$.

Let now $f \colon [0, 1] \to \mathbf{R}_K$ be a path from $f(0) = 0$ to $f(1) = 1$. The image $f([0, 1])$ is convex since it is connected as a subspace of $\mathbf{R}_K$ [Thm 23.5], and connected subspaces of $\mathbf{R}_K$ are convex as we just noted. Therefore the interval $[0, 1]$ and the its subset $K$ is contained in $f([0, 1])$. The image $f([0, 1])$ is also compact in the subspace topology from $\mathbf{R}_K$ [Thm 26.5]. Thus the image is a compact subspace of $\mathbf{R}_K$ containing $K$; this is a contradiction (see (a)). We conclude that there can not exist any path in $\mathbf{R}_K$ from 0 to 1.

**Ex. 27.5.** I first repeat Thm 27.7 in order to emphasize the similarity between the two statements.

**Theorem 1** (Thm 27.7)**.** *Let $X$ be a compact Hausdorff space with no isolated points. Then $X$ contains uncountably many points.*

*Proof.* Let $A = \{a_1, a_2, \dots\}$ be a countable subset of $X$. We must find a point in $X$ outside $A$.

We have $X \neq \{a_1\}$ for $\{a_1\}$ is not open. So the open set $X - \{a_1\}$ is nonempty. By regularity [Lemma 26.4, Lemma 31.1], we can find an open nonempty set $U_1$ such that

$$U_1 \subset \overline{U}_1 \subset X - \{a_1\} \subset X$$

We have $U_1 \neq \{a_2\}$ for $\{a_2\}$ is not open. So the open set $U_1 - \{a_2\}$ is nonempty. By regularity [Lemma 26.4, Lemma 31.1], we can find an open nonempty set $U_2$ such that

$$U_2 \subset \overline{U}_2 \subset U_1 - \{a_2\} \subset U_1$$

Continuing this way we find a descending sequence of nonempty open sets $U_n$ such that

$$U_n \subset \overline{U}_n \subset U_{n-1} - \{a_n\} \subset U_{n-1}$$

for all $n$.

Because $X$ is compact, the intersection $\bigcap U_n = \bigcap \overline{U}_n$ is nonempty [p. 170] and contained in $\bigcap (X - \{a_n\}) = X - \bigcup \{a_n\} = X - A$. $\square$

**Theorem 2** (Baire category theorem)**.** *Let $X$ be a compact Hausdorff space and $\{A_n\}$ a sequence of closed subspaces. If $\operatorname{Int} A_n = \emptyset$ for all $n$, then $\operatorname{Int} \bigcup A_n = \emptyset$.*

*Proof.* (See Thm 48.2.) Let $U_0$ be any nonempty subspace of $X$. We must find a point in $U_0$ outside $\bigcup A_n$.

We have $U_0 \not\subset A_1$ for $A_1$ has no interior. So the open set $U_0 - A_1$ is nonempty. By regularity [Lemma 26.4, Lemma 31.1], we can find a nonempty open set $U_1$ such that

$$U_1 \subset \overline{U}_1 \subset U_0 - A_1 \subset U_0$$

We have $U_1 \not\subset A_2$ for $A_2$ has no interior. So the open set $U_1 - A_2$ is nonempty. By regularity [Lemma 26.4, Lemma 31.1], we can find a nonempty open set $U_2$ such that

$$U_2 \subset \overline{U}_2 \subset U_1 - A_2 \subset U_1$$

Continuing this way, we find a descending sequence of nonempty open sets $U_n$ such that

$$U_n \subset \overline{U}_n \subset U_{n-1} - A_n \subset U_{n-1}$$

for all $n$.

Because $X$ is compact, the intersection $\bigcap U_n = \bigcap \overline{U}_n$ is nonempty [p. 170] and contained in $U_0 \cap \bigcap (X - A_n) = U_0 - \bigcup A_n$.

□

**Ex. 27.6 (The Cantor set).**

(a). The set $A_n$ is a union of $2^n$ disjoint closed intervals of length $1/3^n$. Let $p$ and $q$ be two points in $C$. Choose $n$ so that $|p-q| > 1/3^n$. Then there is point $r$ between them that is not in $A_n$, so not in $C$. As in [Example 4, p. 149], this shows that any subspace of $C$ containing $p$ and $q$ has a separation.

(b). $C$ is compact because [Thm 26.2] it is closed subspace of the compact space $[0,1]$.

(c). $C$ is constructed from any of the $A_n$ by removing interior points only. Thus the boundary of $A_n$ is contained in $C$ for all $n$. Any interval of length $> 1/3^{n+1}$ around any point of $A_n$ contains a boundary point of $A_{n+1}$, hence a point of $C$. Thus $C$ has no isolated points.

(d). $C$ is a nonempty compact Hausdorff space with no isolated points, so it contains uncountably many points [Thm 27.7].

## References