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Munkres §25

Ex. 25.1. $\mathbf{R}_\ell$ is totally disconnected [Ex 23.7]; its components and path components [Thm 25.5] are points. The only continuous maps $f\colon \mathbf{R} \to \mathbf{R}_\ell$ are the constant maps as continuous maps on connected spaces have connected images.

Ex. 25.2.

$\mathbf{R}^\omega$ in product topology: Let X be $\mathbf{R}^\omega$ in the product topology. Then X is is path connected (any product of path connected spaces is path connected [Ex 24.8]) and hence also connected.

$\mathbf{R}^\omega$ in uniform topology: Let X be $\mathbf{R}^\omega$ in the uniform topology. Then X is not connected for $X = B \cup U$ where both B, the set of bounded sequences, and U, the complementary set of unbounded sequences, are open as any sequence within distance $\frac{1}{2}$ of a bounded (unbounded) sequence is bounded (unbounded).

We shall now determine the path components of X. Note first that for any sequence (y_n) we have

$$(0) \text{ and } (y_n) \text{ are in the same path component} \Leftrightarrow (y_n) \text{ is a bounded sequence}$$

$\Rightarrow$: Let $u\colon [0,1] \to X$ be a path from (0) to (y_n). Since $u(0) = (0)$ is bounded, also $u(1) = (y_n)$ is bounded for the connected set $u([0,1])$ can not intersect both subsets in a separation of X.

$\Leftarrow$: The formula $u(t) = (ty_n)$ is a path from (0) to (y_n). To see that u is continuous note that $d(u(t_1), u(t_0)) = \sup\{n \in \mathbf{Z}_+ \mid \min(|(t_1-t_0)y_n|, 1)\} = |t_1-t_0|M$ when $|t_1-t_0| < M^{-1}$ where $M = \sup\{|y_n| \mid n \in \mathbf{Z}_+\}$ and d is the uniform metric.

Next observe that $(y_n) \to (x_n) + (y_n)$ is an isometry or X to itself [Ex 20.7]. It follows that in fact

$$(x_n) \text{ and } (y_n) \text{ are in the same path component} \Leftrightarrow (y_n - x_n) \text{ is a bounded sequence}$$

for any two sequences $(x_n), (y_n) \in \mathbf{R}^\omega$.

This describes the path components of X. It also shows that balls of radius < 1 are path connected. Therefore X is locally path connected so that the path components are the components [Thm 25.5].

$\mathbf{R}^\omega$ in box topology: Let X be $\mathbf{R}^\omega$ in the box topology. Then X is not connected for the box topology is finer than the uniform topology [1, Thm 20.4, Ex 23.1]; in fact, $X = B \cup U$ where both B, the set of bounded sequences, and U, the complementary set of unbounded sequences, are open as they are open in the uniform topology or as any sequence in the neighborhood $\prod(x_n - 1, x_n + 1)$ is bounded (unbounded) if (x_n) is bounded (unbounded), see [1, Example 6, p 151].

The (path) components of X can be described as follows:

$$(x_n) \text{ and } (y_n) \text{ in the same (path) component} \Leftrightarrow x_n = y_n \text{ for all but finitely many } n$$

$\Rightarrow$: Suppose that x_n and y_n are different for infintely many $n \in \mathbf{Z}_+$. For each n, choose a homeomorphism $h_n\colon \mathbf{R} \to \mathbf{R}$ such that $h_n(x_n) = 0$ and $h_n(y_n) = n$ in case $x_n \neq y_n$. Then $h = \prod h_n\colon X \to X$ is a homeomorphism with $h(x_n) = (0)$ and $h(y_n) = n$ for infinitely many n. Since a homeomorphism takes (path) components to (path) components and $h(x_n) = (0) \in B$ and $h(y_n) \in U$ are not in the same (path) component, (x_n) and (y_n) are not in the same (path) component either.

$\Leftarrow$: The map $u(t) = ((1-t)x_n + ty_n)$, $t \in [0,1]$, is constant in all but finitely many coordinates. From this we see that $u\colon [0,1] \to X$ is a continuous path from (x_n) to (y_n). Therefore, (x_n) and (y_n) are in the same (path) component.

X is not locally connected since the components are not open [1, Thm 25.3]. The component of the constant sequence (0) is $\mathbf{R}^\infty$.

$\mathbf{R}^\omega$ in the box topology is an example of a space where the components and the path components are the same even though the space is not locally path connected, cf [1, Thm 25.5].

Ex. 25.3. A connected and not path connected space can not be locally path connected [Thm 25.5]. Any linear continuum is locally connected (the topology basis consists of intervals which are connected in a linear continuum [Thm 24.1]). The subsets $\{x\} \times [0,1] = [x \times 0, x \times 1]$, $x \in [0,1]$, are path connected for they are homeomorphic to $[0,1]$ in the usual order topology [Thm 16.4]. There is no continuous path starting in $[x \times 0, x \times 1]$ and and ending in $[y \times 0, y \times 1]$ when $x \neq y$ for the same reason as there is no path from 0×0 to 1×1 [Example 6, p 156]. Therefore these sets are the path components of I_o^2. Since the path components are not open we see once again that I_o^2 is not locally path connected [Thm 25.4]. (${I_o}^2$ is an example of a space with one component and uncountable many path components.)

Ex. 25.4. Any open subset of a locally path connected space is locally path connected. In a locally path connected space, the components and the path components are the same [Thm 25.5].

Ex. 25.8. Let $p\colon X \to Y$ be a quotient map where X is locally (path-)connected. The claim is that Y is locally (path-)connected.

Let U be an open subspace of Y and C a (path-)component of U. We must show that C is open in Y, ie that that $p^{-1}(C)$ is open in X. But $p^{-1}(C)$ is a union of (path-)components of the open set $p^{-1}(U)$ and in the locally (path-)connected space X open sets have open (path-)components.

References

[1] James R. Munkres, *Topology. Second edition*, Prentice-Hall Inc., Englewood Cliffs, N.J., 2000. MR 57 #4063