1st December 2004

Munkres §24

Ex. 24.2 (Morten Poulsen). Let $f : S^1 \to \mathbf{R}$ be a continuous map. Define $g : S^1 \to \mathbf{R}$ by $g(s) = f(s) - f(-s)$. Clearly g is continuous. Furthermore

$$g(s) = f(s) - f(-s) = -(f(-s) - f(s)) = -g(-s),$$

i.e. g is an odd map. By the Intermediate Value Theorem there exists $s_0 \in S^1$ such that $g(s_0) = 0$, i.e. $f(s_0) = f(-s_0)$.

This result is also known as the Borsuk-Ulam theorem in dimension one. Thus there are no injective continuous maps $S^1 \to \mathbf{R}$, hence S^1 is not homeomorphic to a subspace of $\mathbf{R}$, which is no surprise.

Ex. 24.4. [1, §17]. Suppose that X is a linearly ordered set that is not a linear continuum. Then there are nonempty, proper, clopen subsets of X:

- If $(x, y) = \emptyset$ for some points $x < y$ then $(-\infty, x] = (\infty, y)$ is clopen and $\neq \emptyset, X$.
- If $A \subset X$ is a nonempty subset bounded from above which has no least upper bound then the set of upper bounds $B = \bigcap_{a\in A}[a, \infty) = \bigcup_{b\in B}(b, \infty)$ is clopen and $\neq \emptyset, X$.

Therefore X is not connected [2, §23].

Ex. 24.8 (Morten Poulsen).

(a).

Theorem 1. *The product of an arbitrary collection of path connected spaces is path connected.*

Proof. Let $\{A_j\}_{j\in J}$ be a collection of path connected spaces. Let $x = (x_j)_{j\in J}$ and $y = (y_j)_{j\in J}$ be two points in $\prod_{j\in J} A_j$

For each $j \in J$ there exists a path $\gamma_j : [0, 1] \to A_j$ between x_j and y_j, since A_j is path connected for all j. Now the map $\gamma : [0, 1] \to \prod_{j\in J} A_j$ defined by $\gamma(t) = (\gamma_j(t))_{j\in J}$ is a path between x and y, hence the product is path connected. □

(b). This is not true in general: The set $S = \{x \times \sin(x^{-1} \mid 0 < x < \pi^{-1}\}$ is path connected, but $\overline{S} = S \cup (\{0\} \times [-1, 1])$ is not path connected, c.f. example 24.7.

(c).

Theorem 2. *If $f : X \to Y$ is a continuous map and X path connected then $f(X)$ is path connected.*

Proof. Clearly it suffices to consider the case where f is surjective. Let y_1 and y_2 be two points in Y. Then there exists x_1 and x_2 in X such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since X path connected there is a path $\gamma : [0, 1] \to X$ between x_1 and x_2. Now $\delta = f \circ g : [0, 1] \to Y$ is a path between $\delta(0) = f(\gamma(0)) = y_1$ and $\delta(1) = f(\gamma(1)) = y_2$, hence Y path connected. □

(d).

Theorem 3. *Let $\{A_j\}_{j\in J}$ be a collection of path connected spaces. If $\bigcap_{j\in J} A_j$ is nonempty then $\bigcup_{j\in J} A_j$ is path connected.*

Proof. Let a and b be two points in $\bigcup_{j\in J} A_j$ and let c be an element of $\bigcap_{j\in J} A_j$. Note that there exists s and t such that $a \in A_s$ and $b \in A_t$. Clearly $c \in A_s \cap A_t$. Since A_s and A_t are path connected there exists a path $f : [0, 1] \to A_s$ between a and c and a path $g : [0, 1] \to A_t$ between c and b.

Now $h : [0, 1] \to \bigcup_{j\in J} A_j$ defined by

$$h(t) = \begin{cases} f(2t), & 0 \leq t \leq 1/2 \\ g(2t - 1), & 1/2 \leq t \leq 1 \end{cases}$$

is a path, by the Pasting lemma, between a and b, hence $\bigcup_{j\in J} A_j$ is path connected.

□

Ex. 24.10 (Morten Poulsen). Let U be a nonempty, open and connected subspace of $\mathbf{R}^2$ and let $x_0 \in U$. Furthermore let A be the set of points in U that can be joined to x_0 by a path in U.

A open: Let $a_0 \in A \subset U$. Since U open there is an open rectangle $V(= (a,b) \times (c,d))$ such that $a_0 \in V \subset U$. Since V clearly is path connected it follows that $V \subset A$, hence A open.

A closed: Let $u_0 \in U - A$. If every open rectangle containing u_0 intersects A then clearly $u_0 \in A$, hence u_0 is not a limit point of A. Thus no point of $U - A$ is a limit point of A, hence A is closed.

Since U connected it follows that $A = U$, hence U path connected.

Ex. 24.11 (Morten Poulsen). Let A be a subspace of X.

A connected $\not\Rightarrow$ $\operatorname{Int} A$ and $\operatorname{Bd} A$ connected: If $A = [0,1]$ then $\operatorname{Bd} A = \{0,1\}$ is not connected. If $A = \overline{B(-1 \times 0, 1)} \cup \overline{B(1 \times 0, 1)} \subset \mathbf{R}^2$, then $\operatorname{Int} A = B(-1 \times 0, 1) \cup B(1 \times 0, 1)$ is not connected.

$\operatorname{Int} A$ connected $\not\Rightarrow$ A connected: If $A = (0,1) \cup \{2\}$ then $\operatorname{Int} A = (0,1)$ is connected, but A is not connected.

$\operatorname{Bd} A$ connected $\not\Rightarrow$ A connected: $A = \mathbf{Q}$ is not connected but $\operatorname{Bd} A = \mathbf{R}$ is connected.

$\operatorname{Int} A$ and $\operatorname{Bd} A$ connected $\not\Rightarrow$ A connected: One example is $A = \mathbf{Q}$. An example with nonempty interior is

$$A = ([0,1] \times [0,1]) \cup (\{0,1\} \times [1,2]) \cup (([0,1] \cap \mathbf{Q}) \times \{2\}) \subset \mathbf{R}^2$$

where

$$\operatorname{Bd} A = (\{0,1\} \times [0,2]) \cup (\{0,1,2\} \times [0,1])$$

and

$$\operatorname{Int} A = (0,1) \times (0,1).$$

both are connected but A is not connected.

Ex. 24.12 (The long line). The idea is that the two linear continua $S_\Omega \times [0,1)$ and $\mathbf{Z}_+ \times [0,1) = [1,\infty)$, or rather the long line $L = (S_\Omega \times [0,1)) - \{a_0 \times 0\}$ and the real line $\mathbf{R} = (\mathbf{Z}_+ \times [0,1)) - \{1 \times 0\}$, should have a great deal in common. L satisfies the conditions of a 1-dimensional manifold but 2nd countability. The long line is normal [Ex 32.8] but not metrizable [Ex 50.5].

(a) and (b). Easy.

(c). I do the hint first. Let $a > a_0$ be an element of S_Ω which has no immediate predecessor (there are uncountably many such elements [Ex 10.6]). The set of predecessors $S_a = \{b \in S_\Omega \mid b < a\} = \{b_1, b_2, \dots\}$ is countable and the sets $(b_n, a] = (b_n, a+1)$ is a neighborhood basis at a. Since b_1 is not an immediate predecessor, there is an element $a_1 \in (b_1, a]$. Since $\sup\{a_1, b_2\}$ is not an immediate predecessor, there is an element $a_2 \in (\sup\{a_1, b_2\}, a]$. Proceeding inductively, we find a sequence of elements $a_n < a$ such that $a_n > a_{n-1}$ and $a_n > b_n$ for all n. Then a_n is an increasing sequence and since $a_n > b_n, b_{n-1}, \dots, b_1$ for all n, the sequence a_n converges to a.

Let now J be the set of points a such that $[a_0 \times 0, a \times 1)$ has the order type of $[0,1)$. I claim that J is inductive. Suppose that $S_a \subset J$. If a has an immediate predecessor a_1 then $[a_0 \times 0, a \times 1) = [a_0 \times 0, (a-1) \times 1) \cup [a \times 0, a \times 1)$ has the order type of $[0,1)$ by (a). Otherwise, $[a_0 \times 0, a \times 1)$ has the order type of $[0,1)$ by the hint and (b) (let $x_0, x_1, x_2, \dots$ be the sequence $a_0 \times 0, a_1 \times 1, a_2 \times 1 \dots$).

(d,e). For every point $a \times t$ of $S_\Omega \times [0,1)$, the intervals $[a_0 \times 0, a \times 1)$ and $[a_0 \times 0, a \times t)$ have the order type of $[0,1)$ by (c) and (a). Then $(a_0 \times 0, a \times t)$ has the order type of $(0,1)$. Since intervals are convex, the subspace topology on $(a_0 \times 0, a \times t)$ is the order topology [Thm 16.4] so $(a_0 \times 0, a \times t)$ is homeomorphic to $(0,1)$. From this we see that any two points in L are contained in an interval homeomorphic to $(0,1)$ and therefore there is continuous path between them.

(f). Suppose that L is 2nd countable. Then also $S_\Omega - \{a_0\}$ is 2nd countable since this property is preserved under open continuous maps [Ex 16.4, Ex 30.12]. But $S_\Omega - \{a_0\}$ is not 2nd countable for it does not contain countable dense subsets. (Every countable subset of $S_\Omega - \{a_0\} \subset S_\Omega$ is bounded [Thm 10.3] so that the complement contains an interval of the form (α, Ω) which is non-empty, in fact, uncountable [Ex 10.6], cf. [Ex 30.7].)

References

[1] Jesper M. Møller, *General topology*, `http://www.math.ku.dk/ moller/e03/3gt/notes/gtnotes.dvi`.
[2] James R. Munkres, *Topology. Second edition*, Prentice-Hall Inc., Englewood Cliffs, N.J., 2000. MR 57 #4063