27th January 2005

Munkres §23

Ex. 23.1. Any separation $X = U \cup V$ of $(X, \mathcal{T})$ is also a separation of $(X, \mathcal{T}')$. This means that

$$(X, \mathcal{T}) \text{ is disconnected} \Rightarrow (X, \mathcal{T}') \text{ is disconnected}$$

or, equivalently,

$$(X, \mathcal{T}') \text{ is connected} \Rightarrow (X, \mathcal{T}) \text{ is disconnected}$$

when $\mathcal{T}' \supset \mathcal{T}$.

Ex. 23.2. Using induction and [1, Thm 23.3] we see that $A(n) = A_1 \cup \cdots \cup A_n$ is connected for all $n \geq 1$. Since the spaces $A(n)$ have a point in common, namely any point of A_1, their union $\bigcup A(n) = \bigcup A_n$ is connected by [1, Thm 23.3] again.

Ex. 23.3. Let $A \cup \bigcup A_\alpha = C \cup D$ be a separation. The connected space A is [Lemma 23.2] entirely contained in C or D, let's say that $A \subset C$. Similarly, for each α, the connected [1, Thm 23.3] space $A \cup A_\alpha$ is contained entirely in C or D. Sine it does have something in common with C, namely A, it is entirely contained in C. We conclude that $A \cup \bigcup A_\alpha = C$ and $D = \emptyset$, contradicting the assumption that $C \cup D$ is a separation

Ex. 23.4 (Morten Poulsen). Suppose $\emptyset \subsetneq A \subsetneq X$ is open and closed. Since A is open it follows that $X - A$ is finite. Since A is closed it follows that $X - A$ open, hence $X - (X - A) = A$ is finite. Now $X = A \cup (X - A)$ is finite, contradicting that X is infinite. Thus X and $\emptyset$ are the only subsets of X that are both open and closed, hence X is connected.

Ex. 23.5. $\mathbf{Q}$ is totally disconnected [1, Example 4, p. 149].
$\mathbf{R}_\ell$ is totally disconnected for $\mathbf{R}_\ell = (-\infty, b) \cup [b, +\infty)$ for any real number b.
Any well-ordered set X is totally disconnected in the order topology for

$$X = (-\infty, \alpha + 1) \cup (\alpha, +\infty) = (-\infty, \alpha] \cup [\alpha + 1, +\infty)$$

for any $\alpha \in X$ and if $A \subset X$ contains $\alpha < \beta$ then $\alpha \in (-\infty, \alpha + 1)$ and $\beta \in (\alpha, +\infty)$.

Ex. 23.6. $X = \mathrm{Int}(A) \cup \mathrm{Bd}(A) \cup \mathrm{Int}(X - A)$ is a partition of X for any subset $A \subset X$ [1, Ex 17.19]. If the subspace $C \subset X$ intersects both A and $X - A$ but not $\mathrm{Bd}(A)$, then C intersects $A - \mathrm{Bd}(A) = \mathrm{Int}(A)$ and $(X - A) - \mathrm{Bd}(X - A) = \mathrm{Int}(X - A)$ and

$$C = (C \cap \mathrm{Int}(A)) \cup (C \cap \mathrm{Int}(X - A))$$

is a separation of C.

Ex. 23.7. $\mathbf{R} = (-\infty, r) \cup [r, +\infty)$ is a separation of $\mathbf{R}_\ell$ for any real number r. It follows [1, Lemma 23.1] that any subspace of $\mathbf{R}_\ell$ containing more than one point is disconnected: $\mathbf{R}_\ell$ is totally disconnected.

Ex. 23.11. Let $X = C \cup D$ be a separation of X. Since fibres are connected, $p^{-1}(p(x)) \subset C$ for any $x \in C$ and $p^{-1}(p(x)) \subset D$ for any $x \in D$ [1, Lemma 23.2]. Thus C and D are saturated open disjoint subspaces of X and therefore $p(C)$ and $p(D)$ are open disjoint subspace of Y. In other words, $Y = p(C) \cup p(D)$ is a separation.

Ex. 23.12. Assume that the subspace Y is connected. Let $X - Y = A \cup B$ be a separation of $X - Y$ and $Y \cup A = C \cup D$ a separation of $Y \cup A$. Then [1, Lemma 23.1]

$$\overline{A} \subset X - B, \quad \overline{B} \subset X - A, \quad \overline{C} \subset X - D, \quad \overline{D} \subset X - C$$

and

$$Y \cup A \cup B = X = B \cup C \cup D$$

are partitions [1, §3] of X.

The connected subspace Y is entirely contained in either C or D [1, Lemma 23.2]; let's say that $Y \subset C$. Then $D = C \cup D - C \subset Y \cup A - Y \subset A$ and $\overline{D} \subset \overline{A} \subset X - B$. From

$$\overline{B \cup C} \stackrel{[Ex17.6.(b)]}{=} \overline{B} \cup \overline{C} \subset (X - A) \cup (X - D) = (B \cup Y) \cup (B \cup C) \subset B \cup C$$

$$\overline{D} \subset (X - B) \cap (X - C) = D$$

we conclude that $B \cup C = \overline{B \cup C}$ and $D = \overline{D}$ are closed subspaces. Thus $X = (B \cup C) \cup D$ is a separation of X.

References

[1] James R. Munkres, *Topology. Second edition*, Prentice-Hall Inc., Englewood Cliffs, N.J., 2000. MR 57 #4063