1st December 2004

Munkres §20

Ex. 20.5. Consider $\mathbf{R}^\omega$ with the uniform topology and let d be the uniform metric. Let $C \subset \mathbf{R}^\omega$ be the set of sequences that converge to 0. Then

$$\overline{\mathbf{R}^\infty} = C.$$

$\subset$: Since clearly $\mathbf{R}^\infty \subset C$ it is enough to show that C is closed. Let $(x_n) \in \mathbf{R}^\omega - C$ be a sequence that does not converge to 0. This means that there is some $1 > \varepsilon > 0$ such that $|x_n| > \varepsilon$ for infinitely many n. Then $B_d((x_n), \frac{1}{2}\varepsilon) \subset \mathbf{R}^\omega - C$.

$\supset$: Let $(x_n) \in C$. For any $1 > \varepsilon > 0$ we have $|x_n| < \varepsilon$ for all but finitely many n. Thus $B_d((x_n), 2\varepsilon) \cap \mathbf{R}^\infty \neq \emptyset$.

References