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Munkres §19

Ex. 19.7. Any nonempty basis open set in the product topology contains an element from $\mathbf{R}^\infty$, cf. Example 7p. 151. Therefore $\overline{\mathbf{R}^\infty} = \mathbf{R}^\omega$ in the product topology. ($\mathbf{R}^\infty$ is *dense* [Definition p. 191] in $\mathbf{R}^\omega$ with the product topology.)

Let (x_i) be any point in $\mathbf{R}^\omega - \mathbf{R}^\infty$. Put

$$U_i = \begin{cases} \mathbf{R} & \text{if } x_i = 0 \\ \mathbf{R} - \{0\} & \text{if } x_i \neq 0 \end{cases}$$

Then $\prod U_i$ is open in the box topology and $(x_i) \in \prod U_i \subset \mathbf{R}^\omega - \mathbf{R}^\infty$. This shows that $\mathbf{R}^\infty$ is closed so that $\overline{\mathbf{R}^\infty} = \mathbf{R}^\infty$ with the box topology on $\mathbf{R}^\omega$.

See [Ex 20.5] for the closure of $\mathbf{R}^\infty$ in $\mathbf{R}^\omega$ with the uniform topology.

Ex. 19.10.

(a). The topology $\mathcal{T}$ (the *initial topology* for the set maps $\{f_\alpha \mid \alpha \in J\}$) is the intersection [Ex 13.4] of all topologies on A for which all the maps f_α, $\alpha \in J$, are continuous.

(b). Since all the functions $f_\alpha \colon A \to X_\alpha$, $\alpha \in J$, are continuous, $\mathcal{S} = \bigcup \mathcal{S}_\alpha \subset \mathcal{T}$. The topology $\mathcal{T}_\mathcal{S}$ generated by $\mathcal{S}$, which is the coarsest topology containing $\mathcal{S}$ [Ex 13.5], is therefore also contained in $\mathcal{T}$. On the other hand, $\mathcal{T} \subset \mathcal{T}_\mathcal{S}$, for all the functions $f_\alpha \colon A \to X_\alpha$, $\alpha \in J$, are continuous in $\mathcal{T}_\mathcal{S}$ and $\mathcal{T}$ is the coarsest topology with this property. Thus $\mathcal{T} = \mathcal{T}_\mathcal{S}$.

(c). Let $g \colon Y \to A$ be any map. Then

$$\begin{aligned} g \colon Y \to A \text{ is continuous} &\Leftrightarrow \forall U \in \mathcal{S} \colon g^{-1}(U) \in \mathcal{T}_Y \\ &\Leftrightarrow \forall \alpha \in J\ \forall U_\alpha \in \mathcal{T}_\alpha \colon g^{-1}(f_\alpha^{-1} U_\alpha) \in \mathcal{T}_Y \\ &\Leftrightarrow \forall \alpha \in J\ \forall U_\alpha \in \mathcal{T}_\alpha \colon (f_\alpha \circ g)^{-1} U_\alpha \in \mathcal{T}_Y \\ &\Leftrightarrow \forall \alpha \in J \colon f_\alpha \circ g \colon Y \to X_\alpha \text{ is continuous} \\ &\Leftrightarrow f \circ g \colon Y \to \prod X_\alpha \text{ is continuous} \end{aligned}$$

where $\mathcal{T}_Y$ is the topology on Y and $\mathcal{T}_\alpha$ the topology on X_α.

(d). Consider first a single map $f \colon A \to X$, and give A the initial topology so that the open sets in A are the sets of the form $f^{-1}U$ for U open in X. Then $f \colon A \to f(A)$ is always continuous [Thm 18.2] and open because $f(A) \cap U = f(f^{-1}U)$ for all (open) subsets U of X.
Next, note that the initial topology for the set maps $\{f_\alpha \mid \alpha \in J\}$ is the initial topology for the single map $f = (f_\alpha) \colon A \to \prod X_\alpha$. As just observed, $f \colon A \to f(A)$ is continuous and open.

Example: The product topology on $\prod X_\alpha$ is the initial topology for the set of projections $\pi_\alpha \colon \prod X_\alpha \to X_\alpha$.

References