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Munkres §18

Ex. 18.1 (Morten Poulsen). Recall the ε-δ-definition of continuity: A function $f : \mathbf{R} \to \mathbf{R}$ is said to be continuous if

$$\forall a \in \mathbf{R} \, \forall \varepsilon \in \mathbf{R}_+ \, \exists \delta \in \mathbf{R}_+ \, \forall x \in \mathbf{R} : |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

Let $\mathcal{T}$ be the standard topology on $\mathbf{R}$ generated by the open intervals.

Theorem 1. *For functions $f : \mathbf{R} \to \mathbf{R}$ the following are equivalent:*

(i) $\forall a \in \mathbf{R} \, \forall \varepsilon \in \mathbf{R}_+ \, \exists \delta \in \mathbf{R}_+ \, \forall x \in \mathbf{R} : |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$.
(ii) $\forall a \in \mathbf{R} \, \forall \varepsilon \in \mathbf{R}_+ \, \exists \delta \in \mathbf{R}_+ : f((a - \delta, a + \delta)) \subset (f(a) - \varepsilon, f(a) + \varepsilon)$.
(iii) $\forall U \in \mathcal{T} : f^{-1}(U) \in \mathcal{T}$.

Proof. "$(i) \Leftrightarrow (ii)$": Clear.

"$(ii) \Rightarrow (iii)$": Let $U \in \mathcal{T}$. If $a \in f^{-1}(U)$ then $f(a) \in U$. Since U is open there exists $\varepsilon \in \mathbf{R}_+$ such that $(f(a) - \varepsilon, f(a) + \varepsilon) \subset U$. By assumption there exists $\delta_a \in \mathbf{R}_+$ such that $f((a - \delta_a, a + \delta_a)) \subset (f(a) - \varepsilon, f(a) + \varepsilon)$, hence $(a - \delta_a, a + \delta_a) \subset f^{-1}(U)$. It follows that $f^{-1}(U) = \bigcup_{a \in f^{-1}(U)} (a - \delta_a, a + \delta_a)$, i.e. open.

"$(iii) \Rightarrow (ii)$": Let $a \in \mathbf{R}$. Given $\varepsilon \in \mathbf{R}_+$ then $f^{-1}((f(a) - \varepsilon, f(a) + \varepsilon))$ is open and contains a. Hence there exists $\delta \in \mathbf{R}_+$ such that $(a - \delta, a + \delta) \subset f^{-1}((f(a) - \varepsilon, f(a) + \varepsilon))$. It follows that $f((a - \delta, a + \delta)) \subset (f(a) - \varepsilon, f(a) + \varepsilon)$. □

Ex. 18.2. Let $f : \mathbf{R} \to \{0\}$ be the constant map. Then 2004 is a limit point of $\mathbf{R}$ but $f(2004) = 0$ is not a limit point of $f(\mathbf{R}) = \{0\}$. (The question is if $f(A') \subset f(A)'$ in general. The answer is no: In the above example $\mathbf{R}' = \mathbf{R}$ so that $f(\mathbf{R}') = f(\mathbf{R}) = \{0\}$ but $f(\mathbf{R})' = \emptyset$.)

Ex. 18.6 (Morten Poulsen).

Claim 2. *The map $f : \mathbf{R} \to \mathbf{R}$, defined by*

$$f(x) = \begin{cases} x, & x \in \mathbf{R} - \mathbf{Q} \\ 0, & x \in \mathbf{Q}, \end{cases}$$

is continuous only at 0.

Proof. Since $|f(x)| \leq |x|$ for all x it follows that f is continuous at 0.

Let $x_0 \in \mathbf{R} - \{0\}$. Since $\lim_{x \to x_0, x \in \mathbf{Q}} f(x) = 0$ and $\lim_{x \to x_0, x \in \mathbf{R} - \mathbf{Q}} f(x) = x$ it follows that f is not continuous at x_0. □

Ex. 18.7 (Morten Poulsen).

(a). The following lemma describes the continuous maps $\mathbf{R}_\ell \to \mathbf{R}$.

Lemma 3. *Let $f : \mathbf{R} \to \mathbf{R}$. The following are equivalent:*

(i) *$f : \mathbf{R} \to \mathbf{R}$ is right continuous.*
(ii) $\forall x \in \mathbf{R} \, \forall \varepsilon \in \mathbf{R}_+ \, \exists \delta \in \mathbf{R}_+ : f([x, x + \delta)) \subset (f(x) - \varepsilon, f(x) + \varepsilon)$.
(iii) *For each $x \in \mathbf{R}$ and each interval (a, b) containing $f(x)$ there exists an interval $[c, d)$ containing x such that $f([c, d)) \subset (a, b)$.*
(iv) *$f : \mathbf{R}_\ell \to \mathbf{R}$ is continuous.*

Proof. "$(i) \Leftrightarrow (ii)$": By definition.

"$(ii) \Rightarrow (iii)$": Let $x \in \mathbf{R}$. Assume $f(x) \in (a, b)$. Set $\varepsilon = \min\{f(x) - a, b - f(x)\} \in \mathbf{R}_+$. Then $(f(x) - \varepsilon, f(x) + \varepsilon) \subset (a, b)$. By (ii) there exists $\delta \in \mathbf{R}_+$ such that $f([x, x+\delta)) \subset (f(x) - \varepsilon, f(x) + \varepsilon)$, hence $[c, d) = [x, x + \delta)$ does the trick.

"$(iii) \Rightarrow (ii)$": Let $x \in \mathbf{R}$ and $\varepsilon \in \mathbf{R}_+$. By (iii) there exists $[c, d)$ such that $x \in [c, d)$ and $f([c, d)) \subset (f(x) - \varepsilon, f(x) + \varepsilon)$. Set $\delta = d - x \in \mathbf{R}_+$. Then $[x, x + \delta) = [x, d) \subset [c, d)$, hence $f([x, x + \delta)) \subset (x - \varepsilon, x + \varepsilon)$.

"$(iii) \Leftrightarrow (iv)$": Clear. □

(b). The continuous maps from $\mathbf{R}$ to $\mathbf{R}_\ell$ are the constant maps, c.f. Ex. 25.1.

A map $f\colon \mathbf{R}_\ell \to \mathbf{R}_\ell$ is continuous if and only if for any x and $\varepsilon > 0$ there exists $\delta > 0$ such that $f([x, x+\delta)) \subset [f(x), f(x)+\varepsilon)$. Hence $f\colon \mathbf{R}_\ell \to \mathbf{R}_\ell$ is continuous if and only if $f\colon \mathbf{R} \to \mathbf{R}$ is right continuous and $\forall x \in \mathbf{R} \exists \delta > 0 \forall y \in [x, x+\delta)\colon f(x) \leq f(y)$. (Thanks to Prateek Karandikar for a correction and for this example of a continuous map $\mathbf{R}_\ell \to \mathbf{R}_\ell$.)

Ex. 18.8. Let Y be an ordered set. Give $Y \times Y$ the product topology. Consider the set

$$\Delta^- = \{y_1 \times y_2 \mid y_1 > y_2\}$$

of points below the diagonal. Let $(y_1, y_2) \in \Delta^-$ so that $y_1 > y_2$. If y_2 is the immediate predecessor of y_1 then

$$y_1 \times y_2 \in [y_1, \infty) \times (-\infty, y_2) = (y_2, \infty) \times (-\infty, y_1) \subset \Delta^-$$

and if $y_1 > y > y_2$ for some $y \in Y$ then

$$y_1 \times y_2 \in (y, \infty) \times (-\infty, y) \subset \Delta^-$$

This shows that Δ^- is open.

(a). Since the map $(f, g)\colon X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} Y \times Y$ is continuous, the preimage

$$(f, g)^{-1}(\Delta^-) = \{x \in X \mid f(x) > g(x)\}$$

is open and the complement $\{x \in X \mid f(x) \leq g(x)\}$ is closed.

(b). The map

$$\min\{f, g\}(x) = \begin{cases} f(x) & f(x) \leq g(x) \\ g(x) & f(x) \geq g(x) \end{cases}$$

is continuous according to [1, Thm 18.3].

Ex. 18.10. Let $(f_j\colon X_j \to Y_j)_{j \in J}$ be an indexed family of continuous maps. Define $\prod f_j : \prod X_j \to \prod Y_j$ to be the map that takes $(x_j) \in \prod X_j$ to $(f_j(x_j)) \in \prod Y_j$. The commutative diagram

$$\begin{tikzcd} \prod X_j \arrow[r, "\prod f_j"] \arrow[d, "\pi_k"'] & \prod Y_j \arrow[d, "\pi_k"] \\ X_k \arrow[r, "f_k"'] & Y_k \end{tikzcd}$$

shows that $\pi_k \circ \prod f_j = f_k \circ \pi_k$ is continuous for all $k \in J$. Thus $\prod f_j\colon \prod X_j \to \prod Y_j$ is continuous [1, Thm 18.4, Thm 19.6].

Ex. 18.13. Let $f, g\colon X \to Y$ be two continuous maps between topological spaces where the codomain, Y, is Hausdorff. The equalizer

$$\mathrm{Eq}(f, g) = \{x \in X \mid f(x) = g(x)\} = (f, g)^{-1}(\Delta)$$

is then a closed subset of X for it is the preimage under the continuous map $(f, g)\colon X \to Y \times Y$ of the diagonal $\Delta = \{(y, y) \in Y \times Y \mid y \in Y\}$ which is closed since Y is Hausdorff [1, Ex 17.13]. (This is [1, Ex 31.5].)

It follows that if f and g agree on the subset $A \subset X$ then they also agree on $\overline{A}$ for

$$A \subset \mathrm{Eq}(f, g) \Longrightarrow \overline{A} \subset \mathrm{Eq}(f, g)$$

In particular, if f and g agree on a dense subset of X, they are equal: Any continuous map into a Hausdorff space is determined by its values on a dense subset.

References

[1] James R. Munkres, *Topology. Second edition*, Prentice-Hall Inc., Englewood Cliffs, N.J., 2000. MR 57 #4063