1st December 2004

Munkres §17

Ex. 17.3. $A \times B$ is closed because its complement

$$(X \times Y) - (A \times B) = (X - A) \times Y \cup X \times (Y - B)$$

is open in the product topology.

Ex. 17.6.

(a). If $A \subset B$, then all limit points of A are also limit points of B, so [Thm 17.6] $\overline{A} \subset \overline{B}$.

(b). Since $A \cup B \subset \overline{A} \cup \overline{B}$ and $\overline{A} \cup \overline{B}$ is closed [Thm 17.1], we have $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$ by (a). Conversely, since $A \subset A \cup B \subset \overline{A \cup B}$, we have $\overline{A} \subset \overline{A \cup B}$ by (a) again. Similarly, $\overline{B} \subset \overline{A \cup B}$. Therefore $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$. This shows that closure commutes with *finite* unions.

(c). Since $\bigcup A_\alpha \supset A_\alpha$ we have $\overline{\bigcup A_\alpha} \supset \overline{A_\alpha}$ by (a) for all α and therefore $\overline{\bigcup A_\alpha} \supset \bigcup \overline{A_\alpha}$. In general we do not have equality as the example $A_q = \{q\}$, $q \in \mathbf{Q}$, in $\mathbf{R}$ shows.

Ex. 17.8.

(a). By [Ex 17.6.(a)], $\overline{A \cap B} \subset \overline{A}$ and $\overline{A \cap B} \subset \overline{B}$, so $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$. It is *not* true in general that $\overline{A \cap B} = \overline{A} \cap \overline{B}$ as the example $A = [0, 1)$, $B = [1, 2]$ in $\mathbf{R}$ shows. (However, if A is open and D is dense then $\overline{A \cap D} = \overline{A}$).

(b). Since $\bigcap A_\alpha \subset A_\alpha$ we have $\overline{\bigcap A_\alpha} \subset \overline{A_\alpha}$ for all α and therefore $\overline{\bigcap A_\alpha} \subset \bigcap \overline{A_\alpha}$. (In fact, (a) is a special case of (b)).

(c). Let $x \in \overline{A} - \overline{B}$. For any neighborhood of x, $U - \overline{B}$ is also a neighborhood of x so

$$U \cap (A - B) = (U - B) \cap A \supset (U - \overline{B}) \cap A \neq \emptyset$$

since x is in the closure of A [Thm 17.5]. So $x \in \overline{A - B}$. This shows that $\overline{A} - \overline{B} \subset \overline{A - B}$. Equality does not hold in general as $\overline{\mathbf{R}} - \overline{\{0\}} = \mathbf{R} - \{0\} \subsetneqq \overline{\mathbf{R} - \{0\}} = \mathbf{R}$.

Just to recap we have

(1) $A \subset B \Rightarrow \overline{A} \subset \overline{B}$ $\quad (A \subset B,\ B \text{ closed} \Rightarrow \overline{A} \subset B)$
(2) $\overline{A \cup B} = \overline{A} \cup \overline{B}$
(3) $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$ $\quad (\overline{A \cap D} = \overline{A}$ if D is dense.)
(4) $\overline{\bigcup A_\alpha} \supset \bigcup \overline{A_\alpha}$
(5) $\overline{\bigcap A_\alpha} \subset \bigcap \overline{A_\alpha}$
(6) $\overline{A} - \overline{B} \subset \overline{A - B}$

Dually,

(1) $A \subset B \Rightarrow \text{Int}\, A \subset \text{Int}\, B$ $\quad (A \subset B,\ A \text{ open} \Rightarrow A \subset \text{Int}\, B)$
(2) $\text{Int}\,(A \cap B) = \text{Int}\, A \cap \text{Int}\, B$
(3) $\text{Int}\,(A \cup B) \supset \text{Int}\, A \cup \text{Int}\, B$

These formulas are really the same because

$$\overline{X - A} = X - \text{Int}\, A, \quad \text{Int}\,(X - A) = X - \overline{A}$$

Ex. 17.9. [Thm 19.5] Since $\overline{A} \times \overline{B}$ is closed [Ex 17.3] and contains $A \times B$, it also contains the closure of $A \times B$ [Ex 17.6.(a)], i.e. $\overline{A \times B} \subset \overline{A} \times \overline{B}$.

Conversely, let $(x, y) \in \overline{A} \times \overline{B}$. Any neighborhood of (x, y) contains a product neighborhood of the form $U \times V$ where $U \subset X$ is a neighborhood of x and $V \subset Y$ a neighborhood of y. The intersection of this product neighborhood with $A \times B$

$$(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap Y)$$

is nonempty because $U \cap A \neq \emptyset$ as $x \in \overline{A}$ and $V \cap B \neq \emptyset$ as $y \in \overline{B}$. Since thus any neighborhood of (x, y) intersect $A \times B$ nontrivially, the point (x, y) lies in the closure of $A \times B$ [Thm 17.5]. This shows that $\overline{A} \times \overline{B} \subset \overline{A \times B}$.

Ex. 17.10 (Morten Poulsen).

Theorem 1. *Every order topology is Hausdorff.*

Proof. Let $(X, \leq)$ be a simply ordered set. Let X be equipped with the order topology induced by the simple order. Furthermore let a and b be two distinct points in X, may assume that $a < b$. Let

$$A = \{ x \in X \mid a < x < b \},$$

i.e. the set of elements between a and b.

If A is empty then $a \in (-\infty, b)$, $b \in (a, \infty)$ and $(-\infty, b) \cap (a, \infty) = \emptyset$, hence X is Hausdorff.

If A is nonempty then $a \in (-\infty, x)$, $b \in (x, \infty)$ and $(-\infty, x) \cap (x, \infty) = \emptyset$ for any element x in A, hence X is Hausdorff. □

Ex. 17.11 (Morten Poulsen).

Theorem 2. *The product of two Hausdorff spaces is Hausdorff.*

Proof. Let X and Y be Hausdorff spaces and let $a_1 \times b_1$ and $a_2 \times b_2$ be two distinct points in $X \times Y$. Note that either $a_1 \neq a_2$ or $b_1 \neq b_2$.

If $a_1 \neq a_2$ then, since X is Hausdorff, there exists open sets U_1 and U_2 in X such that $a_1 \in U_1$, $a_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$. It follows that $U_1 \times Y$ and $U_2 \times Y$ are open in $X \times Y$. Furthermore $a_1 \times b_1 \in U_1 \times Y$, $a_2 \times b_2 \in U_2 \times Y$ and $(U_1 \times Y) \cap (U_2 \times Y) = (U_1 \cap U_2) \times Y = \emptyset \times Y = \emptyset$, hence $X \times Y$ is Hausdorff.

The case $b_1 \neq b_2$ is similar. □

Ex. 17.12 (Morten Poulsen).

Theorem 3. *Every subspace of a Hausdorff space is Hausdorff.*

Proof. Let A be a subspace of a Haussdorff space X and let a and b be two distinct points in A.

Since X is Hausdorff there exists two open sets U and V in X such that $a \in U$, $b \in V$ and $U \cap V = \emptyset$. Hence $a \in A \cap U$, $b \in A \cap V$ and $(A \cap U) \cap (A \cap V) = (U \cap V) \cap A = \emptyset \cap A = \emptyset$. Since $A \cap U$ and $A \cap V$ are open in A, it follows that A is Hausdorff. □

Ex. 17.13 (Morten Poulsen).

Theorem 4. *A topological space X is Hausdorff if only if the diagonal*

$$\Delta = \{ x \times x \in X \times X \mid x \in X \}$$

is closed in $X \times X$.

Proof. Suppose X is Hausdorff. The diagonal Δ is closed if and only if the complement $\Delta^c = X \times X - \Delta$ is open. Let $a \times b \in \Delta^c$, i.e. a and b are distinct points in X. Since X is Hausdorff there exists open sets U and V in X such that $a \in U$, $b \in V$ and $U \cap V = \emptyset$. Hence $a \times b \in U \times V$ and $U \times V$ open in $X \times X$. Furthermore $(U \times V) \cap \Delta = \emptyset$, since U and V are disjoint. So for every point $a \times b \in \Delta^c$ there exists an open set $U_{a \times b}$ such that $a \times b \in U_{a \times b} \subset \Delta^c$. By Ex. 13.1 it follows that Δ^c open, i.e. Δ closed.

Now suppose Δ is closed. If a and b are two distinct points in X then $a \times b \in \Delta^c$. Since Δ^c is open there exists a basis element $U \times V$, U and V open in X, for the product topology, such that $a \times b \in U \times V \subset \Delta^c$. Since $U \times V \subset \Delta^c$ it follows that $U \cap V = \emptyset$. Hence U and V are open sets such that $a \in U$, $b \in V$ and $U \cap V = \emptyset$, i.e. X is Hausdorff. □

Ex. 17.14 (Morten Poulsen). The sequence converges to every real number, by the following result.

Theorem 5. *Let X be a set equipped with the finite complement topology. If $(x_n)_{n \in \mathbf{Z}_+}$ is an infinite sequence of distinct points in X then (x_n) converges to every x in X.*

Proof. Let U be a neighborhood of $x \in X$, i.e $X - U$ is finite. It follows that $x_n \in U$, for all, but finitely many, $n \in \mathbf{Z}_+$, i.e. (x_n) converges to x. □

Ex. 17.21 (Morten Poulsen). Let X be a topological space. Consider the three operations on $\mathcal{P}(X)$, namely closure $A \mapsto \overline{A}$, complement $A \mapsto X - A$ and interior $A \mapsto A^\circ$. Write A^- instead of $\overline{A}$ and A^c instead of $X - A$, e.g. $X - \overline{X - A} = A^{c-c}$.

Lemma 6. *If $A \subset X$ then $A^\circ = A^{c-c}$.*

Proof. $A^\circ \supset A^{c-c}$: Since $A^c \subset A^{c-}$, $A^{c-c} \subset A$ and A^{c-c} is open.

$A^\circ \subset A^{c-c}$: Since $A^\circ \subset A$, $A^c \subset A^{\circ c}$ and $A^{\circ c}$ is closed, it follows that $A^{c-} \subset A^{\circ c}$, hence $A^\circ \subset A^{c-c}$. □

This lemma shows that the interior operation can be expressed in terms of the closure and complement operations.

(a). The following theorem, also known as Kuratowski's Closure-Complement Problem, was first proved by Kuratowski in 1922.

Theorem 7. *Let X be a topological space and $A \subset X$. Then at most 14 distinct sets can be derived from A by repeated application of closure and complementation.*

Proof. Let $A_1 = A$ and set $B_1 = A_1^c$. Define $A_{2n} = A_{2n-1}^-$ and $A_{2n+1} = A_{2n}^c$ for $n \in \mathbf{Z}_+$. Define $B_{2n} = B_{2n-1}^-$ and $B_{2n+1} = B_{2n}^c$ for $n \in \mathbf{Z}_+$.

Note that every set obtainable from A by repeatedly applying the closure and complement operations is clearly one of the sets A_n or B_n.

Now $A_7 = A_4^{c-c} = A_4^\circ = A_3^{-\circ}$. Since $A_3 = A_1^{-c}$ it follows that A_3 is open, hence $A_3 \subset A_7 \subset A_3^-$, so $A_7^- = A_3^-$, i.e. $A_8 = A_4$, hence $A_{n+4} = A_n$ for $n \geq 4$. Similarly $B_{n+4} = B_n$ for $n \geq 4$.

Thus every A_n or B_n is equal to one of the 14 sets $A_1, \dots, A_7, B_1, \dots, B_7$, this proves the result. □

(b). An example:

$$A = ((-\infty, -1) - \{-2\}) \cup ([-1, 1] \cap \mathbf{Q}) \cup \{2\}.$$

The 14 different sets:

$$\begin{aligned}
A_1 &= ((-\infty, -1) - \{-2\}) \cup ([-1, 1] \cap \mathbf{Q}) \cup \{2\} \\
A_2 &= (-\infty, 1] \cup \{2\} \\
A_3 &= (1, \infty) - \{2\} \\
A_4 &= [1, \infty) \\
A_5 &= (-\infty, 1) \\
A_6 &= (-\infty, 1] \\
A_7 &= (1, \infty) \\
B_1 &= \{-2\} \cup ([-1, 1] - \mathbf{Q}) \cup ((1, \infty) - \{2\}) \\
B_2 &= \{-2\} \cup [-1, \infty) \\
B_3 &= (-\infty, -1) - \{-2\} \\
B_4 &= (-\infty, -1] \\
B_5 &= (-1, \infty) \\
B_6 &= [-1, \infty) \\
B_7 &= (-\infty, -1).
\end{aligned}$$

Another example:

$$A = \{\, 1/n \mid n \in \mathbf{Z}_+ \,\} \cup (2, 3) \cup (3, 4) \cup \{9/2\} \cup [5, 6] \cup \{\, x \mid x \in \mathbf{Q}, 7 \leq x < 8 \,\}.$$

References