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Munkres §16

Ex. 16.1 (Morten Poulsen). Let $(X, \mathcal{T})$ be a topological space, $(Y, \mathcal{T}_Y)$ be a subspace and let $A \subset Y$.

Let $\mathcal{T}_A^Y$ be the subspace topology on A as a subset of Y and let $\mathcal{T}_A^X$ be the subspace topology on A as a subset of X. Since

$$\begin{aligned} U \in \mathcal{T}_A^Y &\Leftrightarrow \exists U_Y \in \mathcal{T}_Y : U = A \cap U_Y \\ &\Leftrightarrow \exists U_X \in \mathcal{T} : U = A \cap (Y \cap U_X) \\ &\Leftrightarrow \exists U_X \in \mathcal{T} : U = A \cap U_X \\ &\Leftrightarrow U \in \mathcal{T}_A^X \end{aligned}$$

it follows that $\mathcal{T}_A^Y = \mathcal{T}_A^X$.

Ex. 16.3 (Morten Poulsen). Consider $Y = [-1, 1]$ as a subspace of $\mathbf{R}$ with the standard topology. By lemma 16.1 a basis for the subspace topology on Y is sets of the form:

$$Y \cap (a, b) = \begin{cases} (a, b), & a, b \in Y \\ [-1, b), & a \notin Y, b \in Y \\ (a, 1], & a \in Y, b \notin Y \\ Y, \emptyset, & a, b \notin Y. \end{cases}$$

Note that intervals of the form $[a, b)$ are not open in $\mathbf{R}$, since there are no basis element (c, d) such that $a \in (c, d) \subset [a, b)$. Similarly are intervals of the form $(a, b]$ and $[a, b]$ not open in $\mathbf{R}$.

$A = \{ x \mid \frac{1}{2} < |x| < 1 \}$: $A = (-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1)$, hence A open in $\mathbf{R}$. Since $A = Y \cap A$ it follows that A open in Y.

$B = \{ x \mid \frac{1}{2} < |x| \leq 1 \}$: Since $B = Y \cap (-2, -\frac{1}{2}) \cup (\frac{1}{2}, 2)$ it follows that B open in Y. Another argument is that $B = [-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1]$, i.e. an union of basis elements, hence open in Y. The set B is not open in $\mathbf{R}$, since if B is open in $\mathbf{R}$ then $B \cap (0, 2) = (\frac{1}{2}, 1]$ open in $\mathbf{R}$, contradicting that $(\frac{1}{2}, 1]$ not open in $\mathbf{R}$.

$C = \{ x \mid \frac{1}{2} \leq |x| < 1 \}$: Since there clearly is no basis element U for the subspace topology on Y such that $\frac{1}{2} \in U \subset C$, it follows that C is not open in Y. By an argument similar to the one for the set B, it follows that C not open in $\mathbf{R}$.

$D = \{ x \mid \frac{1}{2} \leq |x| \leq 1 \}$: By arguments similar to the ones above it is easily seen that D is not open in either Y or $\mathbf{R}$.

$E = \{ x \mid 0 < |x| < 1, \frac{1}{x} \notin \mathbf{Z}_+ \}$: Note that $E = (-1, 0) \cup ((0, 1) - K)$, where $K = \{ \frac{1}{n} \mid n \in \mathbf{Z}_+ \}$. Since $E = (-1, 0) \cup \bigcup_{n=1}^{\infty} (\frac{1}{n+1}, \frac{1}{n})$ it follows that E is open in $\mathbf{R}$ and Y.

Ex. 16.4. Let $\pi_k \colon \prod_{j \in J} X_j \to X_k$ be the projection map onto X_k. Observe that π_k maps any basis set, $\prod V_j$ where $V_j \subset X_j$ is open and $V_j = X_j$ for all but finitely many $j \in J$, to an open set in X_k, $\pi_k(\prod V_j) = V_k$. Since maps preserve unions of sets [Ex 2.2], it follows that π_k maps open sets in the product $\prod X_j$ to open sets in X_k.

Ex. 16.6 (Morten Poulsen). The set

$$\mathcal{B} = \{ (a, b) \times (c, d) \mid a < b, c < d \text{ and } a, b, c, d \in \mathbf{Q} \}$$

is a basis for $\mathbf{R}^2$: The set

$$\{ (r, s) \mid r < s \text{ and } r, s \in \mathbf{Q} \}$$

is a basis for $\mathbf{R}$, by Ex. 13.8(a). From Theorem 15.1 it follows that $\mathcal{B}$ is a basis for $\mathbf{R}^2$.

Ex. 16.7. $\mathbf{R}_+ \times \mathbf{R}$ is a convex subset of the linearly ordered set $\mathbf{R} \times \mathbf{R}$ that is not an interval nor a ray.

Ex. 16.9 (Morten Poulsen). Let $\mathbf{R}^2_{dict}$ be $\mathbf{R}^2$ with the dictionary order topology and let $\mathbf{R}_d \times \mathbf{R}$ be the product topology, where $\mathbf{R}_d$ is $\mathbf{R}$ with the discrete topology and $\mathbf{R}$ is $\mathbf{R}$ with the standard topology.

The set
$$\{\, (a \times b, c \times d) \mid a, b \in \mathbf{R} \text{ and } (a < c) \vee (a = c \wedge b < d) \,\}$$
is a basis for $\mathbf{R}^2_{dict}$.

The set
$$\{\, \{a\} \mid a \in \mathbf{R} \,\}$$
is a basis for $\mathbf{R}_d$, c.f. §13 Example 3.

The set
$$\{\, (a, b) \mid a, b \in \mathbf{R} \text{ and } a < b \,\}$$
is a basis for $\mathbf{R}$.

By Theorem 15.1 the set
$$\{\, \{a\} \times (b, c) \mid a, b, c \in \mathbf{R} \text{ and } b < c \,\}$$
is a basis for $\mathbf{R}_d \times \mathbf{R}$.

Claim 1. $\mathbf{R}^2_{dict} = \mathbf{R}_d \times \mathbf{R}$.

Proof. "$\subset$": Given basis element $(a \times b, c \times d) \in \mathbf{R}^2_{dict}$ and $x \times y \in (a \times b, c \times d)$ then $x \times y \in \{x\} \times I \subset (a \times b, c \times d)$, where I is an open interval in $\mathbf{R}$ containing y. By Lemma 13.3 it follows that $\mathbf{R}_d \times \mathbf{R}$ is finer than $\mathbf{R}^2_{dict}$.

"$\supset$": Clear, since every basis element $\{a\} \times (b, c) \in \mathbf{R}_d \times \mathbf{R}$ is a basis element in $\mathbf{R}^2_{dict}$. $\square$

References