

## Munkres §13

**Ex. 13.1 (Morten Poulsen).** Let  $(X, \mathcal{T})$  be a topological space and  $A \subset X$ . The following are equivalent:

- (i)  $A \in \mathcal{T}$ .
- (ii)  $\forall x \in A \exists U_x \in \mathcal{T} : x \in U_x \subset A$ .

*Proof.* (i)  $\Rightarrow$  (ii): If  $x \in A$  then  $x \in A \subset A$  and  $A \in \mathcal{T}$ .

(ii)  $\Rightarrow$  (i):  $A = \bigcup_{x \in A} U_x$ , hence  $A \in \mathcal{T}$ . □

**Ex. 13.4 (Morten Poulsen).** Note that every collection of topologies on a set  $X$  is itself a set: A topologi is a subset of  $\mathcal{P}(X)$ , i.e. an element of  $\mathcal{P}(\mathcal{P}(X))$ , hence a collection of topologies is a subset of  $\mathcal{P}(\mathcal{P}(X))$ , i.e. a set.

Let  $\{\mathcal{T}_\alpha\}$  be a nonempty set of topologies on the set  $X$ .

(a). Since every  $\mathcal{T}_\alpha$  is a topology on  $X$  it is clear that the intersection  $\bigcap \mathcal{T}_\alpha$  is a topology on  $X$ .

The union  $\bigcup \mathcal{T}_\alpha$  is in general not a topology on  $X$ : Let  $X = \{a, b, c\}$ . It is straightforward to check that  $\mathcal{T}_1 = \{X, \emptyset, \{a\}, \{a, b\}\}$  and  $\mathcal{T}_2 = \{X, \emptyset, \{c\}, \{b, c\}\}$  are topologies on  $X$ . But  $\mathcal{T}_1 \cup \mathcal{T}_2$  is not a topology on  $X$ , since  $\{a, b\} \cap \{b, c\} = \{b\} \notin \mathcal{T}_1 \cup \mathcal{T}_2$ .

(b). The intersection of all topologies that are finer than all  $\mathcal{T}_\alpha$  is clearly the smallest topology containing all  $\mathcal{T}_\alpha$ .

The intersection of all  $\mathcal{T}_\alpha$  is clearly the largest topology that is contained in all  $\mathcal{T}_\alpha$ .

(c). The topology  $\mathcal{T}_3 = \mathcal{T}_1 \cap \mathcal{T}_2 = \{X, \emptyset, \{a\}\}$  is the largest topology on  $X$  contained in  $\mathcal{T}_1$  and  $\mathcal{T}_2$ .

The topology  $\mathcal{T}_4 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$  is the smallest topology that contains  $\mathcal{T}_1$  and  $\mathcal{T}_2$ .

**Ex. 13.5 (Morten Poulsen).** Let  $(X, \mathcal{T})$  be a topological space,  $\mathcal{A}$  basis for  $\mathcal{T}$  and let  $\{\mathcal{T}_\alpha\}$  be the set of topologies on  $X$  that contains  $\mathcal{A}$ .

**Claim 1.**  $\mathcal{T} = \bigcap \mathcal{T}_\alpha$ .

*Proof.* " $\subset$ ": Let  $U \in \mathcal{T}$ . By lemma 13.1,  $U$  is an union of elements of  $\mathcal{A}$ . Since  $\mathcal{T}_\alpha$  is a topology for all  $\alpha$ , it follows that  $U \in \mathcal{T}_\alpha$  for all  $\alpha$ , i.e.  $U \in \bigcap \mathcal{T}_\alpha$ .

" $\supset$ ": Clear since  $\mathcal{A} \subset \bigcap \mathcal{T}_\alpha \subset \mathcal{T}$ . □

Now assume  $\mathcal{A}$  is a subbasis.

**Claim 2.**  $\mathcal{T} = \bigcap \mathcal{T}_\alpha$ .

*Proof.* " $\subset$ ": Let  $U \in \mathcal{T}$ . By the definition of a subbasis and the remarks at the bottom on page 82,  $U$  is an union of finite intersections of elements of  $\mathcal{A}$ . Since  $\mathcal{T}_\alpha$  is a topology for all  $\alpha$ , it follows that  $U \in \mathcal{T}_\alpha$  for all  $\alpha$ , i.e.  $U \in \bigcap \mathcal{T}_\alpha$ .

" $\supset$ ": Clear since  $\mathcal{A} \subset \bigcap \mathcal{T}_\alpha \subset \mathcal{T}$ . □

**Ex. 13.6 (Morten Poulsen).** The topologies  $\mathbf{R}_l$  and  $\mathbf{R}_K$  on  $\mathbf{R}$  are not comparable:

$\mathbf{R}_l \not\subset \mathbf{R}_K$ : Consider  $[-1, 0) \in \mathbf{R}_l$ . Clearly no basis element  $B_K \in \mathbf{R}_K$  satisfy  $-1 \in B_K \subset [-1, 0)$ , hence  $\mathbf{R}_K$  is not finer than  $\mathbf{R}_l$ , by lemma 13.3.

$\mathbf{R}_K \not\subset \mathbf{R}_l$ : Consider  $(-1, 1) - K \in \mathbf{R}_K$ . Clearly no basis element  $B_l \in \mathbf{R}_l$  satisfy  $0 \in B_l \subset (-1, 1) - K$ , hence  $\mathbf{R}_l$  is not finer than  $\mathbf{R}_K$ , by lemma 13.3.

**Ex. 13.7 (Morten Poulsen).** We know that  $\mathcal{T}_1$  and  $\mathcal{T}_2$  are bases for topologies on  $\mathbf{R}$ . Furthermore  $\mathcal{T}_3$  is a topology on  $\mathbf{R}$ . It is straightforward to check that the last two sets are bases for topologies on  $\mathbf{R}$  as well.

The following table show the relationship between the given topologies on  $\mathbf{R}$ .

	$\mathcal{T}_1$	$\mathcal{T}_2$	$\mathcal{T}_3$	$\mathcal{T}_4$	$\mathcal{T}_5$
$\mathcal{T}_1$	=	$\subset$ (1)	$\not\subset$ (2)	$\subset$ (3)	$\not\subset$ (4)
$\mathcal{T}_2$	$\not\subset$ (5)	=	$\not\subset$ (6)	$\subset$ (7)	$\not\subset$ (8)
$\mathcal{T}_3$	$\subset$ (9)	$\subset$ (10)	=	$\subset$ (11)	$\not\subset$ (12)
$\mathcal{T}_4$	$\not\subset$ (13)	$\not\subset$ (14)	$\not\subset$ (15)	=	$\not\subset$ (16)
$\mathcal{T}_5$	$\subset$ (17)	$\subset$ (18)	$\not\subset$ (19)	$\subset$ (20)	=

- (1) Lemma 13.3.
- (2) Since  $\mathbf{R} - (0, 1)$  not finite.
- (3) Given basis element  $(a, b) \in \mathcal{T}_1$  and  $x \in (a, b)$  then the basis element  $(a, x] \in \mathcal{T}_4$  satisfy  $x \in (a, x] \subset (a, b)$ , hence  $\mathcal{T}_4$  is finer than  $\mathcal{T}_1$ , by lemma 13.3.
- (4) Given a basis element  $(a, b) \in \mathcal{T}_1$  and  $x \in (a, b)$  then there are clearly no basis element  $(-\infty, c) \in \mathcal{T}_5$  such that  $x \in (-\infty, c) \subset (a, b)$ , hence  $\mathcal{T}_5$  is not finer than  $\mathcal{T}_1$ , by lemma 13.3.
- (5) Lemma 13.3.
- (6) Since  $\mathbf{R} - (0, 1)$  not finite.
- (7) Given basis element  $(a, b) - K \in \mathcal{T}_2$  and  $x \in (a, b) - K$ . If  $x \in (0, 1)$  then there exists  $m \in \mathbf{Z}_+$  such that  $\frac{1}{m} < x < \frac{1}{m-1}$ , hence  $x \in (\frac{1}{m}, x] \subset (a, b) - K$ . If  $x \notin (0, 1)$  then  $x \in (a, x] \subset (a, b) - K$ . It follows from (4) and lemma 13.3 that  $\mathcal{T}_4$  is finer than  $\mathcal{T}_2$ .
- (8) Since  $\mathcal{T}_1 \not\subset \mathcal{T}_5$  and  $\mathcal{T}_1 \subset \mathcal{T}_2$ .
- (9) Let  $U \in \mathcal{T}_3$ ,  $U$  nonempty, i.e.  $\mathbf{R} - U = \{r_1, \dots, r_n\}$ ,  $r_1 < \dots < r_n$ . Since

$$U = \left( \bigcup_{i=1}^{\infty} (r_1 - i, r_1) \right) \cup \left( \bigcup_{j=1}^{n-1} (r_j, r_{j+1}) \right) \cup \left( \bigcup_{k=1}^{\infty} (r_n, r_n + k) \right)$$

it follows that  $U \in \mathcal{T}_1$ .

- (10) Since  $\mathcal{T}_3 \subset \mathcal{T}_1 \subset \mathcal{T}_2$ .
- (11) Let  $U \in \mathcal{T}_3$ ,  $U$  nonempty, i.e.  $\mathbf{R} - U = \{r_1, \dots, r_n\}$ ,  $r_1 < \dots < r_n$ , and let  $x \in U$ . If  $d = \min\{|x - r_i| \mid i \in \{1, \dots, n\}\} > 0$  then  $x \in (x - \frac{d}{2}, x + \frac{d}{2}) \subset U$ . It follows from lemma 13.3 that  $\mathcal{T}_4$  is finer than  $\mathcal{T}_3$ .
- (12) Consider  $U = \mathbf{R} - \{0\} \in \mathcal{T}_3$ . There are no basis element  $(-\infty, a) \in \mathcal{T}_5$  such that  $1 \in (-\infty, a) \subset U$ , hence  $\mathcal{T}_5$  is not finer than  $\mathcal{T}_4$ , by lemma 13.3.
- (13) Given basis element  $(c, x] \in \mathcal{T}_4$  there is clearly no basis element  $(a, b) \in \mathcal{T}_1$  such that  $x \in (a, b) \subset (c, x]$ , hence  $\mathcal{T}_1$  is not finer than  $\mathcal{T}_4$ , by lemma 13.3.
- (14) Given basis element  $(c, x] \in \mathcal{T}_4$  there is clearly no basis element  $B_K \in \mathcal{T}_2$  such that  $x \in B_K \subset (c, x]$ , hence  $\mathcal{T}_2$  is not finer than  $\mathcal{T}_4$ , by lemma 13.3.
- (15) Since  $\mathbf{R} - (0, 1]$  not finite.
- (16) Given basis element  $(c, x] \in \mathcal{T}_4$  there is clearly no basis element  $(-\infty, a) \in \mathcal{T}_5$  such that  $x \in (-\infty, a) \subset (c, x]$ , hence  $\mathcal{T}_5$  is not finer than  $\mathcal{T}_4$ , by lemma 13.3.
- (17) Since  $(-\infty, a) = \bigcup_{i=1}^{\infty} (a - i, a) \in \mathcal{T}_1$  for all  $a \in \mathbf{R}$ .
- (18) Since  $\mathcal{T}_5 \subset \mathcal{T}_1 \subset \mathcal{T}_2$ .
- (19) Since  $\mathbf{R} - (-\infty, 0)$  not finite.
- (20) Given basis element  $(-\infty, a) \in \mathcal{T}_5$  and  $x \in (-\infty, a)$  then clearly  $x \in (x - |x - a|, x + \frac{|x - a|}{2}) \subset (-\infty, a)$ , hence  $\mathcal{T}_4$  is finer than  $\mathcal{T}_5$ , by lemma 13.3.

**Ex. 13.8 (Morten Poulsen).**

(a). Let

$$\mathcal{B} = \{(a, b) \mid a, b \in \mathbf{Q}, a < b\}.$$

It is straightforward to check that  $\mathcal{B}$  is a basis. Let  $\mathcal{T}$  be the standard topology on  $\mathbf{R}$  generated by the basis:

$$\{(r, s) \mid r, s \in \mathbf{R}\}.$$

Let  $U \in \mathcal{T}$  and let  $x \in U$ . Then (by definition of an open set in a topology generated by a basis) there exists a basis element  $(r, s)$ ,  $r, s \in \mathbf{R}$ , such that  $x \in (r, s)$ . Furthermore there exists  $a, b \in \mathbf{Q}$  such that  $r \leq a < x < b \leq s$ , hence  $x \in (a, b) \subset (r, s)$ . It follows, by lemma 13.2, that  $\mathcal{B}$  is a basis for  $\mathcal{T}$ .

(b). Let

$$\mathcal{C} = \{[a, b) \mid a, b \in \mathbf{Q}, a < b\}.$$

It is straightforward to check that  $\mathcal{C}$  is a basis. Let  $\mathcal{T}_{\mathcal{C}}$  be the topology on  $\mathbf{R}$  generated by  $\mathcal{C}$ .

Consider  $[\sqrt{2}, 2) \in \mathbf{R}_l$ . There are clearly no basis element  $[a, b) \in \mathcal{C}$  such that  $\sqrt{2} \in [a, b) \subset [\sqrt{2}, 2)$ , hence  $\mathcal{T}_{\mathcal{C}}$  is not finer than  $\mathbf{R}_l$ , by lemma 13.3.

Since  $\mathbf{R}_l$  is clearly finer than  $\mathcal{T}_{\mathcal{C}}$ , it follows that  $\mathbf{R}_l$  is strictly finer than  $\mathcal{T}_{\mathcal{C}}$ .

#### REFERENCES