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Munkres §11

Ex. 11.8 (Morten Poulsen). First recall some definitions: Let V be a vector space over a field K. Let A be a (possibly empty) subset of V. The subspace spanned by A is denoted $\operatorname{span}_K A$ and is defined by

$$\operatorname{span}_K A = \{ k_1 a_1 + \cdots + k_n a_n \mid n \in \mathbf{Z}_+ \cup \{0\},\ k_1, \ldots, k_n \in K,\ a_1, \ldots, a_n \in A \}$$

i.e. the set of all finite linear combinations of elements from W. If $n = 0$ then the linear combination is defined to be the zero element in V. If A is the empty set then $\operatorname{span}_K A$ is the subspace $\{0\}$.

The subset A of V is said to be (linearly) independent if $n \in \mathbf{Z}_+ \cup \{0\}, k_1, \ldots, k_n \in K$ and $a_1, \ldots, a_n \in A$ satisfy

$$k_1 a_1 + \cdots + k_n a_n = 0$$

then

$$k_1 = 0, \ldots, k_n = 0.$$

The empty set is (by definition) independent.

The subset A is said to be a basis if A is independent and $\operatorname{span}_K A = V$. Note that if A is a basis for V then every element has an unique representation as a finite linear combination of elements of A.

If $V = \{0\}$ then the empty set is a basis, thus assume in the following that $V \neq \{0\}$.

(a). Assume $v \notin \operatorname{span}_K A$. Suppose

$$k_1 a_1 + \cdots + k_n a_n + k_{n+1} v = 0,$$

where $n \in \mathbf{Z}_+ \cup \{0\}$, $k_1, \ldots, k_{n+1} \in K$ and $a_1, \ldots, a_n \in A$.

If $k_{n+1} = 0$ then $k_1 = 0, \ldots, k_n = 0$, since A is independent. If $k_{n+1} \neq 0$ then $v = \frac{1}{k_{n+1}} (k_1 a_1 + \cdots + k_n a_n)$, contradicting $v \notin \operatorname{span}_K A$. It follows that $A \cup \{v\}$ is independent.

(b). Let W be the set of independent subsets of V. Define a relation $\prec$ on W by

$$\forall\, w_1, w_2 \in W : w_1 \prec w_2 \Leftrightarrow w_1 \subsetneq w_2.$$

The relation $\prec$ is clearly a strict partial order on W.

Let W_0 be a simply ordered subset of W then

$$U = \bigcup_{w_0 \in W_0} w_0$$

is an independent subset of V: Suppose

$$k_1 u_1 + \cdots + k_n u_n = 0,$$

where $n \in \mathbf{Z}_+ \cup \{0\}$, $k_1, \ldots, k_n \in K$ and $u_1, \ldots, u_n \in U$. Then there are $w_1, \ldots, w_n \in W_0$, such that $u_i \in w_i$ for $1 \leq i \leq n$. Since W_0 is simply ordered the subset $\{w_1, \ldots, w_n\} \subset W_0$ has a largest element w, hence $u_1, \ldots, u_n \in w$. Since w is independent it follows that $k_1 = 0, \ldots, k_n = 0$, hence U is independent, i.e. $U \in W$.

The set U is clearly an upper bound for W_0, hence every simply ordered subset of W has an upper bound in W. Zorn's Lemma gives that W has a maximal element B.

(c). The maximal element B in W is basis for V: Suppose $\operatorname{span}_K B$ is a proper subspace of V. Thus there is an element $b \in V - \operatorname{span}_K B$, hence, by (a), $B \cup \{b\} \in W$, contradicting the maximality of B. It follows that B is a basis for V.

Thus we have proved:

Theorem 1. *Every vector space has a basis.*

A few remarks: We know that if a vector space V over the field K has a basis with $n \in \mathbf{Z}_+$ elements then every basis for V has n elements and n is called the dimension of V over K, and is denoted $\dim_K V = n$. The notion of dimension extends to vector spaces with infinite bases. First one proves the following theorem.

Theorem 2. *If A and B are bases for V then A and B have the same cardinality.*

In view of the previous theorem we define the dimension of a vector space V to be the cardinality of some basis A for V, i.e. $\dim_K V = \operatorname{card} A$.

Hamel bases. Regard the real numbers as a vector space over the rationals. Then this vector space has a basis, any basis for this vector space is called a Hamel basis. Furthermore one shows that $\dim_{\mathbf{Q}} \mathbf{R} = \operatorname{card} \mathbf{R}$.

References