Munkres §10

Ex. 10.1. If a subset of a well-ordered set has an upper bound, the smallest upper bound is a least upper bound (supremum) for the set. (This proof is a tautology!)

Ex. 10.2.

(a). The smallest successor x_+ of any element x is the immediate successor. (The iterated successors of x has the order type of a section of $\mathbf{Z}_+$.)

(b). $\mathbf{Z}$.

Ex. 10.4.

(a). Let A be a simply ordered set containing a subset with the order type of $\mathbf{Z}_-$. Then this subset does not have a smallest element so A is not well-ordered. Conversely, let A be simply ordered set containing a nonempty subset B with no smallest element. Let b_1 be any element of B. Since b_1 is not a smallest element of B there is some element b_2 of B such that $b_2 < b_1$. Continuing inductively we obtain an infinite descending chain $\cdots < b_{n+1} < b_n < \cdots < b_2 < b_1$ forming a subset of the same order type as $\mathbf{Z}_-$.

(b). A does not contain a subset with the order type of $\mathbf{Z}_-$.

Ex. 10.6.

(a). For any element α of S_Ω, the set $\{x \in S_\Omega \mid x \leq \alpha\} = S_\alpha \cup \{\alpha\}$ is countable but S_Ω itself is uncountable [Lemma 10.2].

(b). For any element $\alpha \in S_\Omega$, the set $S_\alpha \cup \{\alpha\}$ is countable so its complement, $\{x \in S_\Omega \mid x > \alpha\} = (\alpha, +\infty)$, in the uncountable set S_Ω, is uncountable [Lemma 10.2, Thm 7.5].

(c). We show the stronger statement [Thm 10.3] that X_0 is not bounded from above. We do this by assuming that X_0 has an upper bound α and find a contradiction. The (non-empty) simply ordered set $(\alpha, +\infty)$ is well-ordered [p. 63], it has no largest element by (a), and each element of $(\alpha, +\infty)$, except the smallest element, has an immediate predecessor. Thus $(\alpha, +\infty)$ has the order type of $\mathbf{Z}_+$, in particular $(\alpha, +\infty)$ is countable, contradicting (b). (Let x be any element of $(\alpha, +\infty)$. Since $(\alpha, +\infty)$ does not contain an infinite descending chain [Ex 10.4], α is an iterated immediate predecessor of x and x is an iterated immediate successor of α.)

Ex. 10.7. We show the contrapositive. Let J_0 be any subset of J that is not everything. Let α be the smallest element of the complement $J - J_0$, the smallest element outside J_0. This means that $\alpha \notin J_0$ and that any element smaller than α is in J_0, i.e. $S_\alpha \subset J_0$. Thus J_0 is not inductive.

REFERENCES