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Munkres §4

Ex. 4.2. We assume that there exists a set $\mathbf{R}$ equipped with two binary operations, $+$ and $\cdot$, and a linear order $<$ such that

(1) $(\mathbf{R}, +, \cdot)$ is a field.
(2) $x < y \Rightarrow x + z < y + z$ and $0 < x, 0 < y \Rightarrow 0 < xy$
(3) $(\mathbf{R}, <)$ is a linear continuum

Using these axioms we can establish all the usual rules of artihmetic.

(c): $\Rightarrow$: Assume that $x > 0$. Adding $-x$ to this gives $0 > -x$. .
$\Leftarrow$: Assume that $-x < 0$. Adding x to this gives $0 < x$.

(g): Since $0 \neq 1$ in a field, we have either $0 < 1$ or $1 < 0$ by Comparability. We rule out the latter possibility. If $1 < 0$, then $-1 > 0$ so also $1 = (-1) \cdot (-1) > 0$, a contradiction. Thus we have $0 < 1$ and then also $-1 < 0$ by point (c).

Ex. 4.3 (Morten Poulsen).

(a). Let $\mathcal{A}$ be a collection of inductive sets. Since $1 \in A$ for all $A \in \mathcal{A}$, it follows that $1 \in \cap_{A\in\mathcal{A}} A$. Let $a \in \cap_{A\in\mathcal{A}} A$. Since A is inductive for all $A \in \mathcal{A}$, it follows that $a + 1 \in A$ for all $A \in \mathcal{A}$, hence $a + 1 \in \cap_{A\in\mathcal{A}} A$. So $\cap_{A\in\mathcal{A}} A$ is inductive.

(b). By definition $\mathbf{Z}_+ = \cap_{A\in\mathcal{A}} A$, where $\mathcal{A}$ is the collection of all inductive subsets of the real numbers.

Proof of (1): The set $\mathbf{Z}_+$ is inductive by (a).

Proof of (2): Suppose $A \subset \mathbf{Z}_+$ is inductive. Since A inductive, it follows by the definition of $\mathbf{Z}_+$ that $\mathbf{Z}_+ \subset A$, i.e. $A = \mathbf{Z}_+$.

Ex. 4.4 (Morten Poulsen).

(a). Let A be the set of $n \in \mathbf{Z}_+$ for which the statement holds.

The set A is inductive: It is clear that $1 \in A$, since the only nonempty subset of $\{1\}$ is $\{1\}$. Suppose $n \in A$. Let B be a nonempty subset of $\{1, \dots, n+1\}$. If $n + 1 \in B$ then $n + 1$ is the largest element in B. If $n + 1 \notin B$ then the set $B \cap \{1, \dots, n\}$ contains a largest element, since $n \in A$.

So $A \subset \mathbf{Z}_+$ is inductive, by the principle of induction, it follows that $A = \mathbf{Z}_+$, as desired.

(b). Consider!

Ex. 4.5 (Morten Poulsen).

(a). Let $a \in \mathbf{Z}_+$. Let

$$X = \{\, x \in \mathbf{R} \mid a + x \in \mathbf{Z}_+ \,\} .$$

The set X is inductive: $1 \in X$, since $a \in \mathbf{Z}_+$ and $\mathbf{Z}_+$ inductive. Suppose $x \in X$. Since $a + (x + 1) = (a + x) + 1$, $a + x \in \mathbf{Z}_+$ and $\mathbf{Z}_+$ inductive, it follows that $x + 1 \in X$.

By ex. 4.3(a) it follows that $X \cap \mathbf{Z}_+ \subset \mathbf{Z}_+$ is inductive. By the principle of induction, it follows that $X \cap \mathbf{Z}_+ = \mathbf{Z}_+$, which proves (a).

(b). Let $a \in \mathbf{Z}_+$. Let

$$X = \{\, x \in \mathbf{R} \mid ax \in \mathbf{Z}_+ \,\} .$$

The set X is inductive: $1 \in X$, since $a1 = a \in \mathbf{Z}_+$. Suppose $x \in X$. Since $a(x + 1) = ax + a$ and $ax, a \in \mathbf{Z}_+$, it follows by (a) that $x + 1 \in X$.

As above, it follows that $X \cap \mathbf{Z}_+ = \mathbf{Z}_+$, which proves (b).

(c). Let

$$X = \{\, x \in \mathbf{R} \,|\, x - 1 \in \mathbf{Z}_+ \cup \{0\} \,\} \,.$$

The set X is inductive: $1 \in X$, since $1 - 1 = 0 \in \mathbf{Z}_+ \cup \{0\}$. Suppose $x \in X$. Note that $(x+1) - 1 = (x-1) + 1$. If $x - 1 = 0$ then $(x-1) + 1 = 1 \in \mathbf{Z}_+ \cup \{0\}$. If $x - 1 \in \mathbf{Z}_+$ then, since $\mathbf{Z}_+$ is inductive, $(x-1) + 1 \in Z_+ \subset \mathbf{Z}_+ \cup \{0\}$. So $x + 1 \in X$.

As above, it follows that $X \cap \mathbf{Z}_+ = \mathbf{Z}_+$, which proves (c).

(d). Let $c \in \mathbf{Z} = \mathbf{Z}_- \cup \{0\} \cup \mathbf{Z}_+$, where $\mathbf{Z}_-$ is negatives of the elements of $\mathbf{Z}_+$. First we prove the result for $d = 1$:

(i) $c + 1 \in \mathbf{Z}$: If $c \in \mathbf{Z}_+$ the result follows from (a). It is clear if $c = 0$. If $c \in \mathbf{Z}_-$ then $c + 1 = -(-c - 1)$, since $-c \in \mathbf{Z}_+$, it follows from (c) that $-c - 1 \in \mathbf{Z}_+ \cup \{0\}$, hence $c + 1 \in \mathbf{Z}$.

(ii) $c - 1 \in \mathbf{Z}$: If $c \in \mathbf{Z}_+$ the result follows from (c). It is clear if $c = 0$. If $c \in \mathbf{Z}_-$ then $c - 1 = -(-c + 1)$, since $-c \in \mathbf{Z}_+$, it follows from (a) or by the inductivity of $\mathbf{Z}_+$ that $-c + 1 \in \mathbf{Z}_+$, hence $c - 1 \in \mathbf{Z}$.

Next we prove the result for $d \in \mathbf{Z}_+$: Let

$$X = \{\, x \in \mathbf{R} \,|\, c + x \in \mathbf{Z} \,\}$$

and

$$Y = \{\, y \in \mathbf{R} \,|\, c - y \in \mathbf{Z} \,\} \,.$$

The set X is inductive: $1 \in X$, c.f. (i). Suppose $x \in X$. Since $c + (x+1) = (c+x) + 1$, $c + x \in \mathbf{Z}$ and (i), it follows that $x + 1 \in X$.

The set Y is inductive: $1 \in X$, c.f. (ii). Suppose $y \in Y$. Since $c - (y+1) = (c-y) - 1$, $c - y \in \mathbf{Z}$ and (ii), it follows that $y + 1 \in X$.

As above, it follows that $X \cap \mathbf{Z}_+ = \mathbf{Z}_+$ and $Y \cap \mathbf{Z}_+ = \mathbf{Z}_+$. This proves the result for $d \in \mathbf{Z}_+$. The result is clear if $d = 0$. The case $d \in \mathbf{Z}_-$ is now easy: Since $c + d = c - (-d)$ and $-d \in \mathbf{Z}_+$, it follows that $c + d \in \mathbf{Z}$. Since $c - d = c + (-d)$ and $-d \in \mathbf{Z}_+$, it follows that $c - d \in \mathbf{Z}$.

(e). Let $c \in \mathbf{Z}$. Let

$$X = \{\, x \in \mathbf{R} \,|\, cx \in \mathbf{Z} \,\} \,.$$

The set X is inductive: $1 \in X$, since $c1 = c \in \mathbf{Z}$. Suppose $x \in X$. Since $c(x+1) = cx + c$ and $cx \in \mathbf{Z}$, it follows, by (d), that $x + 1 \in X$.

As above, it follows that $X \cap \mathbf{Z}_+ = \mathbf{Z}_+$. Thus the result is proved for $d \in \mathbf{Z}_+$ and is clear if $d = 0$. Since $cd = (-c)(-d)$, $-c \in \mathbf{Z}$, hence if $d \in \mathbf{Z}_-$ then $-d \in \mathbf{Z}_+$, this proves the case $d \in \mathbf{Z}_-$.

References