Eksamen januar 2006

Topology

Books, notes, calculators, and computers are allowed at this three hour written exam. You may write your answers in pencil.

Problem 1 (25%)

Let X be a topological space and $X_1 \supset X_2 \supset \cdots X_n \supset X_{n+1} \supset \cdots$ a descending sequence of closed *nonempty* subsets of X.

(1) Show that $\bigcap_{n=1}^{\infty} X_n \neq \emptyset$ if X is compact.

(2) Give an example where $\bigcap_{n=1}^{\infty} X_n = \emptyset$ (and X is not compact).

[Hint for (1): Consider the open subsets $X - X_n$ of X.]

Problem 2 (25%)

Let X be a topological space and $X_1 \subset X_2 \subset \cdots X_n \subset X_{n+1} \subset \cdots$ an ascending sequence of subspaces such that $X = \bigcup_{n=1}^{\infty} \operatorname{Int} X_n$. Let A be a subset of X.

(1) Show that

$$A \text{ is open in } X \iff X_n \cap A \text{ is open in } X_n \text{ for all } n \geq 1$$

(2) Show, for instance by using (1), that

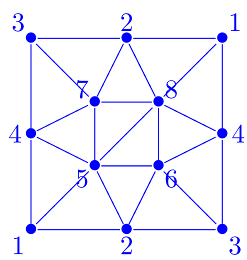
$$A \text{ is closed in } X \iff X_n \cap A \text{ is closed in } X_n \text{ for all } n \geq 1$$

Problem 3 (25%)

Let $S = |\mathcal{P}|$ be the realization of the surface representation $\mathcal{P} = \langle a, b | abab \rangle$.

(1) Which surface from the classification theorem is homeomorphic to S?

(2) Find a triangulation of S. (It suffices to find a finite abstract simplicial complex K such that $|K|$ and S are homeomorphic; you do not need to construct the homeomorphism.) See the figure below for inspiration.



Problem 4 (25%)

Let M be an n-manifold and $\varphi \colon U \to \mathbf{R}^n$ a chart defined on an open subset $U \subset M$. Let $B_1 \subset B_2 \subset U$ be the open sets such that $\varphi(B_1) = B(0,1)$ and $\varphi(B_2) = B(0,2)$ where $B(0,r) \subset \mathbf{R}^n$ is the open ball centered at $0 \in \mathbf{R}^n$ of radius $r > 0$. Let A be any subset of B_1.

(1) Show that the interior of A in M equals the interior of A in B_2.

(2) Show that the closure of A in M equals the closure of A in B_2.

(3) Show that the boundary of A in M equals the boundary of A in B_2.

(THE END)