JMM

Solutions to the January 2007 Topology exam

Problem 1

Assume that X is compact. Then the closed subspace C is also compact. So C is a discrete compact space. Then C is finite.

Problem 2

Suppose that $A \cap B$ is open for all $B \in \mathcal{B}$. Let $a \in A$. Choose a neighborhood U of a and finitely many sets $B_1, \ldots, B_m \in \mathcal{B}$ such that $x \in U \subset B_1 \cup \cdots \cup B_m$. Since the sets in $\mathcal{B}$ are closed we can assume that $a \in B_i$ for all i where $1 \leq i \leq m$. (If $a \notin B_i$ for some i then replace U by $U - B_i$ which is still an open neighborhood of a.) Since $A \cap B_i$ is open in B_i and $a \in A \cap B_i$ there is an open neighborhood U_i of x such that $x \in U_i \cap B_i \subset A \cap B_i$. Now $U_1 \cap \cdots \cap U_m \cap U$ is an open neighborhood of a and

$$\begin{aligned} U_1 \cap \cdots \cap U_m \cap U &\subset (U_1 \cap \cdots \cap U_m) \cap (B_1 \cup \cdots \cup B_m) \\ &\subset (U_1 \cap B_1) \cup \cdots \cup (U_m \cap B_m) \subset (A \cap B_1) \cup \cdots \cup (A \cap B_m) \\ &= A \cap (B_1 \cup \cdots \cup B_m) = A \end{aligned}$$

This shows that A is open in X.

Problem 3

(1) K_v is subcomplex because it is a subset of K that is closed under taking nonempty subsets.
(2) The realization of any subcomplex of K is a closed subset of $|K|$. In particular, $|K_v| \subset |K|$ is closed and the complement $\mathrm{st}(v) = |K| - |K_v|$ is open.
(3) Observe that $|K_v| = \{t \in |K| \mid t(v) = 0\}$ and $\mathrm{st}(v) = \{t \in |K| \mid t(v) > 0\}$. We shall prove that the open star $\mathrm{st}(v)$ is star-shaped. Let x be a point in $\mathrm{st}(v)$. Let $\tau \in K$ be a simplex such that x lies in $|\tau|$. Then $v \in \tau$, for otherwise $\tau \in K_v$ and $x \in |\tau| \subset |K_v|$. Now

$$[0,1] \ni \lambda \to \lambda v + (1-\lambda)x \in |\tau|$$

is a continuous path in $|\tau|$ and $|K|$ from v to x. This path actually runs in $\mathrm{st}(v)$ for $(\lambda v + (1 - \lambda)x)(v) = \lambda + (1-\lambda)x(v) > 0$ for all $\lambda \in [0,1]$ because this is a path from $x(v)$ to 1. This shows that $\mathrm{st}(v)$ is a union of closed intervals emanating from v and therefore $\mathrm{st}(v)$ is path-connected.
(4) Let $V \subset |K|$ be an open neighborhood of v. Put $V^n = V \cap |K^n|$. We shall recursively define a path-connected neighborhood U of v such that $U \subset V$. Let $U^0 = \{v\}$. Suppose that we have defined $U^0 \subset \cdots \subset U^{n-1}$ where each U^k is path-connected and $\overline{U^k} \subset V^k$. For each n-simplex $\sigma \in K$ with v as a vertex there is path-connected neighborhood U_σ of v in $|\sigma|$ such that $U_\sigma \cap \partial|\sigma| = U^{n-1} \cap |\sigma|$ and $\overline{U_\sigma} \subset V \cap |\sigma|$. Let U^n be the union of all the U_σ. Then U^n is open in $|K^n|$ since its intersection with each n-simplex is open and path-connected as a union of path-connected spaces with a point in common. Finally, $U = \bigcup U^n$ is an open path-connected neighborhood of v contained in V. This shows that $|K|$ is locally path-connected at all its vertices. But any point of $|K|$ is the vertex of some subdivision of K.

Problem 4

The surface representation

$$\begin{aligned} \langle a,b,c,d | abdc^{-1}, a^{-1}cd^{-1}b\rangle = \langle a,b,c,d | abdc^{-1}, cd^{-1}ba^{-1}\rangle = \langle a,b | abba^{-1}\rangle \\ = \langle a,b | bba^{-1}a\rangle = \langle b | bb\rangle \end{aligned}$$

represents the projective plane $\mathbf{R}P^2$.