Solutions to the January 2006 Topology exam

Problem 1

(1) Suppose that $\bigcap_{n=1}^{\infty} X_n = \emptyset$. Then the ascending sequence of subsets $X - X_1 \subset X - X_2 \subset \cdots X - X_n \subset \cdots$ forms an open covering of X. Since X is compact, $X - X_n = X$, or $X_n = \emptyset$, for some $n > 0$.

(2) $\bigcap_{n=1}^{\infty}[n, \infty) = \emptyset$ in $\mathbf{R}$ or $\bigcap_{n=1}^{\infty}[\sqrt{2} - 1/n, \sqrt{2} + 1/n] = \emptyset$ in $\mathbf{R} - \{\sqrt{2}\}$.

Problem 2

(1) $\Longrightarrow$: This is immediate from the definition of subspace topology.
$\Longleftarrow$: Suppose that $X_n \cap A$ is open in X_n for all $n \geq 1$. Then also $\mathrm{Int}(X_n) \cap A$ is open in $\mathrm{Int}(X_n)$. Since $\mathrm{Int}(X_n)$ is open in X, this implies that $\mathrm{Int}(X_n) \cap A$ is open in X for all n. Thus $A = \bigcup_{n=1}^{\infty} \mathrm{Int}(X_n) \cap A$ is open in X.

(2) Apply (1) to the open set $X - A$:

$$\begin{aligned} A \text{ is closed} &\iff X - A \text{ is open} \\ &\overset{(1)}{\iff} X_n \cap (X - A) = X_n - (X_n \cap A) \text{ is open in } X_n \text{ for all } n \\ &\iff X_n \cap A \text{ is closed in } X_n \text{ for all } n \end{aligned}$$

Problem 3

(1) $\mathbf{R}P^2$.

(2) K is an abstract simplicial complex with 14 2-simplices and all their faces. There are 8 vertices, 21 edges, and 14 2-simplices so the Euler characteristic is $\chi(K) = 8 - 21 + 14 = 1 = \chi(\mathcal{P})$.

Problem 4

(1) B_2 is open in M since it is open in the open set U. It is a general fact that when Y is open in X, $\mathrm{Int}_Y(A) = \mathrm{Int}(A)$ for any $A \subset Y$ as the relatively open sets in Y are the open sets in Y; see General Topology, Chp 2, §5. In particular, $\mathrm{Int}_{B_2}(A) = \mathrm{Int}(A)$.

(2) There is a compact set K such that $B_1 \subset K \subset B_2$. Namely, take $K = \varphi^{-1}(C)$ where $C \subset \mathbf{R}^n$ is compact and $\varphi(B_1) \subset C \subset \varphi(B_2)$. Since K is closed in the Hausdorff space M and $A \subset K$, also $\overline{A} \subset K$. Therefore $\overline{A} \subset K \subset B_2$. Now $\mathrm{Cl}_{B_2}(A) = B_2 \cap \overline{A} = \overline{A}$.

(3) $\partial_{B_2}(A) = \mathrm{Cl}_{B_2}(A) - \mathrm{Int}_{B_2}(A) = \mathrm{Cl}(A) - \mathrm{Int}(A) = \partial(A)$.