JMM

Solutions to Topology Exam
april 2007

Problem 1

(1) $C^\circ = \emptyset$, so C is not open.
(2) $\overline{C} = C \cup (\mathbf{R} \times \{0\})$, so C is not closed.
(3) Note that $C_r = C \cap (\{r\} \times [0,r])$ is a closed discrete infinite subspace of C for any $r > 0$. C_r is closed in C because it is the intersection of C and a closed subset of $\mathbf{R}^2$.

Let now $U \subset C$ be a neighborhood of $(0,0)$. Since U contains a closed discrete infinite subspace, namely C_r for r sufficiently small, U is not contained in any compact subspace of C. Thus C is not locally compact at $(0,0)$.

Problem 2

(1) Clear. To see that $x \sim x$ for any x in X, choose $A \in \mathcal{A}$ so that $x \in A$. Then $x \in A$, $A \cap A \neq \emptyset$, $x \in A$, so $x \sim x$. In fact, all points of A are equivalent to x.
(2) Suppose that there exist $A_0, \ldots, A_k \in \mathcal{A}$ such that $x \in A_0$, $y \in A_k$, and $A_{i-1} \cap A_i \neq \emptyset$ for all $1 \leq i \leq k$. The union $A_0 \cup A_1$ is connected since A_0 and A_0 are connected and $A_0 \cap A_1 \neq \emptyset$. Repeating this argument finitely many times, we see that $A_0 \cup \cdots \cup A_k$ is connected.
(3) Let $E(x) \subset X$ be the equivalence class containing $x \in X$. Since all points in any $A \in \mathcal{A}$ are equivalent, $E(x) = \bigcup_{A \ni y \in E(x)} A$ is a union of sets from $\mathcal{A}$ and therefore open. $E(x)$ is also connected for, by (2), it is a union of connected sets with a point, x, in common.
(4) Since the equivalence classe are connected, the components are unions of equivalence classes; since the equivalence classes are open, each component cannot consist of more than one equivalence class.

Problem 3

(1) T is the ASC generated by $\{\{4,2,3\},\{1,5,3\},\{1,2,6\},\{1,5,6\},\{4,2,6\},\{4,5,3\}\}$. This is a triangulation of $\partial\Delta^2 \times [0,1]$ where $\Delta^2 \times \{0\} = 1,2,3$ and $\Delta^2 \times \{1\} = 4,5,5$.
(2) Remove one 2-simplex, assume it is $\{1,2,3\}$, from K_1 and one 2-simplex from K_2, assume it is $\{4,5,6\}$, and add the six 2-simplexes of T.
(3) $\chi(K) = \chi(K_2) - 1 + \chi(K_2) - 1 + 6 - 6 = \chi(K_2) + \chi(K_2) - 2$ since we removed two 2-simplexes and added six new 2-simplexes and six new 1-simplexes.