e) Full insurance with fixed amount deductible.

f) Full insurance with proportional deductible.

**Exercise 3.2**

Let $Y$ be a Pareto distributed random variable with cumulative distribution

$$G(y) = 1 - \left( \frac{c}{y} \right)^{\alpha}, \quad (y > c; \alpha, c > 0)$$

This distribution is often used for claim amounts. Find the distribution of $bY$ ($b > 0$) and the conditional distribution of $Y$ given that $Y > d \geq c$. Comment on the Pareto distribution as a claim amount distribution.

**Chapter 4**

**Premium principles**

4A. Let $X$ be the total (random) monetary amount of some economic risk; we shall briefly call $X$ a risk. (In this book the word “risk” will be used in several different meanings, but we hope that this will not be too much confusing.) By such a risk we shall always mean a non-negative random variable; risks taking negative values do not seem realistic in non-life insurance. In this chapter we also assume that all risks have finite mean.

By the *pure premium* of the risk $X$ we shall mean the mean of $X$. In practice, one would add a risk loading to the pure premium, and then one gets the *net premium*. When administration costs are added to the net premium, we get the *gross premium*. In this book we shall for simplicity make the (very unrealistic) assumption that there are no administration costs. For the rest of this chapter “premium” will mean “net premium”.

By a *premium principle* we mean a rule $H$ that to any risk $X$ assigns a non-negative net premium $H(X)$. We shall assume that the premium principle is a function of the distribution of $X$. Thus the premium $H(X)$ is non-random, and if the risks $X$ and $Y$ are identically distributed, then $H(X) = H(Y)$.

We introduce the three most common premium principles:

i) the *expected value principle*

$$H_1(X) = (1 + a) E X; \quad (a > 0)$$

ii) the *standard deviation principle*

$$H_2(X) = E X + b\sqrt{\text{Var} X}; \quad (b > 0)$$

iii) the *variance principle*

$$H_3(X) = E X + c \text{Var} X; \quad (c > 0)$$
The expected value principle is the simplest one; here one needs only one parameter of the distribution of the risk, namely the mean. This implies that all risks with the same mean have the same premium. Intuitively one would say that a risk with a large dispersion is more dangerous than a risk with the same mean and a small dispersion and hence should have a higher premium. Thus it seems that some measure of dispersion should be included in the premium formula. This idea is taken care of by the two other principles. In particular we see that in the degenerate case Var $X = 0$ the risk loading is equal to zero. This seems reasonable; when there is no uncertainty, there is no need for a risk loading.

As an illustration, we consider for $r \geq 1$ the risk $X_r$ with distribution given by

$$\Pr (X_r = r) = \frac{1}{r} = 1 - \Pr (X_r = 0). \quad (r \geq 1)$$

We have $E X_r = 1$, and under the expected value principle we obtain

$$H_1 (X_r) = 1 + a$$

independent of $r$. We see that if $r = 1$, then the insurance company will obtain a certain gain $a$. When $r$ increases, then the probability that the company will gain the whole premium, will increase, but the amount that the company might have to pay, will also increase. Is it reasonable that the premium should not be affected by this? As Var $X_r = r - 1$, we obtain

$$H_2 (X_r) = 1 + b \sqrt{r - 1}; \quad H_3 (X_r) = 1 + c (r - 1),$$

and we see that under these two premium principles the premium increases with $r$; the premium goes to infinity when $r$ goes to infinity. If we consider the risk loading as a measure of riskiness, then we can say that under the standard deviation principle and the variance principle $X_r$ becomes more risky when $r$ increases.

4B. In this subsection we shall look at an undesirable property of the variance principle. Let $n$ be an integer greater than one, and suppose that $0 < \text{Var} \ X < \infty$. Then we have

$$H_3 \left( \frac{X}{n} \right) = \frac{1}{n} E X + \frac{c}{n^2} \text{Var} X = \frac{1}{n} H_3 (X) - \frac{c}{n} \left( 1 - \frac{1}{n} \right) \text{Var} X < \frac{1}{n} H_3 (X),$$

and thus

$$n H_3 \left( \frac{X}{n} \right) < H_3 (X).$$

This means that it will be profitable for the risk-holder to split the risk into $n$ equal proportional policies instead of insuring it in one policy. As

$$\lim_{n \to \infty} H_3 \left( \frac{X}{n} \right) = E X,$$

we see that the risk-holder can make the risk loading of the insurance company as small as he wants to (and, of course, he wants to make the total premium small).

We easily see that under the expected value principle and the standard deviation principle the premium of $X$ is $n$ times the premium of $X/n$.

4C. The previous subsection motivates the question: What properties do we want a premium principle $H$ to satisfy? In this subsection we shall discuss some properties that might seem desirable.

Property 1. For any risks $X$ and $Y$ one should have

$$H (X + Y) \leq H (X) + H (Y).$$

This property means that it should not be profitable for the risk-holder to split a risk into several policies. In the previous subsection we showed that this property is not satisfied by the variance principle.

Property 2. For any risks $X$ and $Y$ such that $\Pr (X \leq Y)$, one should have

$$H (X) \leq H (Y).$$

This property means that when one insurance alternative gives a more extensive cover than another alternative, then the latter alternative should not have a higher premium than the former one.

Property 3. For any risk $X$ one should have

$$H (X) \geq E X.$$

This property says that the net premium should not be lower than the pure premium, that is, the risk loading should be non-negative. To show that this is a reasonable requirement, let us assume that we have a portfolio of $n$ independent and identically distributed risks $X_1, \ldots, X_n$, and that $H (X) < E X$. Then, by the law of large numbers,

$$\lim_{n \to \infty} \Pr \left( \sum_{i=1}^n X_i > n H (X) \right) = 1.$$
Thus, when the size of the portfolio goes to infinity, the probability of deficit goes to one. On the other hand, if \( H(X) > E(X) \), then

\[
Pr \left( \sum_{i=1}^{n} X_i > nH(X) \right) = 0,
\]

that is, the probability of deficit goes to zero.

Property 3 will not always be satisfied in practice. Sometimes market conditions make it desirable or necessary to let an insurance class make a deficit for a while and subsidise it with surplus from other classes of insurance or other sources of profit. However in some countries, e.g. Norway, this would in principle be illegal; the Norwegian Insurance Company Act says that the premium should be reasonable compared to the risk taken over by the insurance company.

**Property 4.** For any risk \( X \) one should have

\[
Pr (X < H(X)) < 1.
\]

For the risk-holder this property is obvious; if the risk is almost surely less than the premium, he will not insure. Thus, if the insurance company really wants to cover the risk, it has to make the premium at least so low that Property 4 is satisfied.

Property 4 is often replaced with the weaker condition:

**Property 4'. For any risk \( X \) satisfying \( Pr(X \leq m) = 1 \) for some constant \( m \), one should have \( H(X) \leq m \).**

**Theorem 4.1.** If a premium principle satisfies Property 2 and the net premium is equal to the pure premium for all constant risks, then Property 4' is satisfied.

**Proof.** Let \( H \) be a premium principle that satisfies the assumptions of the theorem, and let \( X \) be a risk such that \( Pr(X \leq m) = 1 \) for some constant \( m \). Then

\[
H(X) \leq H(m) = m,
\]

and thus Property 4' is satisfied. Q.E.D.

4D. In this subsection we are going to see which of the four properties are satisfied by the three premium principles presented in subsection 4A.

i) **The expected value principle.**

It is clearly seen that Properties 1, 2, and 3 are satisfied. Property 4' is not satisfied; for a constant risk \( m > 0 \) we have \( H(m) = (1 + a)m > m \) and \( Pr(m < H(m)) = 1 \).

ii) **The standard deviation principle.**

Property 1 is equivalent to

\[
\sqrt{\text{Var}(X + Y)} \leq \sqrt{\text{Var}X + \sqrt{\text{Var}Y}},
\]

which is satisfied as

\[
\sqrt{\text{Var}(X + Y)} = \sqrt{\text{Var}X + \text{Var}Y + 2 \text{Cov}(X,Y)} \leq \sqrt{\text{Var}X + \text{Var}Y} = \sqrt{\text{Var}X + \sqrt{\text{Var}Y}}.
\]

For \( 0 \leq p \leq 1 \), let \( X_p \) denote a risk with distribution given by

\[
Pr(X_p = m) = p = 1 - Pr(X_p = 0) \quad (m > 0)
\]

We have

\[
EX_p = mp; \quad \text{Var}X_p = m^2p(1 - p),
\]

and thus

\[
f(p) = H_2(X_p) = m \left( p + b\sqrt{p(1 - p)} \right).
\]

Differentiation gives

\[
f'(p) = m \left( 1 + \frac{b(1 - 2p)}{2\sqrt{p(1 - p)}} \right).
\]

As

\[
\lim_{p \rightarrow 1} f'(p) = -\infty,
\]

there must exist a \( p < 1 \) for which \( H_2(X_p) > H_2(X_1) = m \). Thus, as \( Pr(X_p \leq m) = 1 \), Property 4' is not satisfied. As under the standard deviation principle the net premium is equal to the pure premium for all constant risks, Theorem 4.1 gives that Property 2 is not satisfied either.

Property 3 is trivially satisfied.

iii) **The variance principle.**

In subsection 4B we have already shown that Property 1 is not satisfied.
To show that Property 2 is not satisfied, we consider the same example as when discussing these properties for the standard deviation principle. We have

\[ H_3(X) = mp[1 + cn(1 - p)], \]

and we get that \( H_3(X) > m \) for \( m > (cp)^{-1} \). Thus Property 4’ is not satisfied. As under the variance principle the net premium is equal to the pure premium, Theorem 41 gives that Property 2 is not satisfied either.

Property 3 is trivially satisfied. We summarise the results of this subsection in the following table.

<table>
<thead>
<tr>
<th>Principle</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4 (4’)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expected value</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>Variance</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
</tbody>
</table>

4E. This really looks bad. None of the three most common premium principles satisfy all the desirable properties of subsection 4C, and none of them satisfy Property 4’! However, in practice it is not necessarily so bad after all. Firstly, to show that a property is not satisfied, we constructed some counter-example that was not always too realistic. Secondly, insurance people are human beings, some of them possessing a good portion of common sense. When they see a risk \( X \) satisfying \( \Pr(X < H(X)) = 1 \), they probably exclaim, “This is unfair; we cannot charge that premium.” And then they offer insurance cover for a premium \( P < H(X) \) satisfying \( \Pr(X < P) < 1 \).

Furthermore, the properties of subsection 4C were stated under the tacit assumption that the insurance company is willing to cover any risk offered to it for the premium assigned by a given premium principle. However, one aspect that we have not touched, is the possibility that the insurance company can refuse an arrangement. This aspect may be important when discussing Property 1. One could assume that an insurance company using a premium principle \( H \) not satisfying Property 1, would refuse to cover both of the two risks \( X \) and \( Y \) with separate premiums if \( H(X) + H(Y) < H(X + Y) \). With such a strategy a premium principle would not need to satisfy Property 1.

4F. Other aspects of premium calculation will be discussed later. An extensive treatment of premium principles is given in Goovaerts, De Vylder, & Haesendonck (1984); in particular they describe the Orič principle, which satisfies Properties 1, 2, 3, and 4’.

### Exercises

#### Exercise 4.1
Check whether Properties 1, 2, 3, and 4’ of subsection 4C are satisfied by the following premium principles:

a) The exponential principle

\[ H(X) = \frac{1}{a} \ln \text{E} e^{aX}. \quad (a > 0) \]

b) The Esscher principle

\[ H(X) = \frac{\text{E} X e^{aX}}{\text{E} e^{aX}}. \quad (a > 0) \]

c) The quantile principle

\[ H(X) = \min \{ h : \Pr(X > h) \leq a \}. \quad (a > 0) \]

#### Exercise 4.2
Find the net premium for a risk \( X \) which is Gamma distributed with density

\[ f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad (x > 0; \alpha, \beta > 0) \]

when using the following premium principles:

a) The variance principle.

b) The exponential principle.

c) The Esscher principle.

#### Exercise 4.3
Let \( X \) be a risk with sum insured \( S \). We assume that the risk is not underinsured, and that there can occur at most one claim during the insurance period. Let \( Y \) be the claim amount if the claim occurs. Thus \( X = Y \) if a claim occurs, and \( X = 0 \) otherwise. Let \( p \) be the probability that a claim occurs. We introduce the claim ratio

\[ Z = \frac{Y}{S}. \]

21
Under normal conditions the claim ratio will be between zero and one, and it is often assumed to be Beta distributed with density

\[
g(z) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha) \Gamma(\beta)} z^{\alpha - 1} (1 - z)^{\beta - 1}. \quad (0 < z < 1; \; \alpha, \beta > 0)
\]

This assumption is also made in the present exercise.

a) Describe the distribution of \( X \).

Find expressions for the net premium of \( X \) when it is calculated by:

b) The expected value principle.

c) The variance principle.

d) The standard deviation principle.

Exercise 4.4

Let \( X \) be a risk. We assume that there can occur at most one claim during the insurance period. Let \( Y \) be the claim amount if a claim occurs, and let \( p \) be the probability that a claim occurs. We assume that \( Y \) is Pareto distributed with cumulative distribution

\[
G(y) = 1 - \left( \frac{c}{y} \right)^{\alpha}. \quad (y > c; \; \alpha, c > 0)
\]

Find expressions for the net premium of \( X \) when it is calculated by:

a) The expected value principle.

b) The variance principle.

c) The standard deviation principle.

d) The quantile principle.

Exercise 4.5

Assume that two insurers \( A \) and \( B \) offer to insure the risk \( X \) fully or partially by proportional insurance. The two insurers apply the same premium principle, but \( A \) applies a higher value for the safety loading parameter than \( B \). How should the risk holder allocate the risk between the two insurers when the premium is calculated according to:

a) The expected value principle?

b) The standard deviation principle?

c) The variance principle?

Exercise 4.6

Show that the variance principle is additive for independent risks, that is, the premium for the sum of independent risks is equal to the sum of the premiums for each risk. Does this additivity property still hold if we assume that the risks are conditionally independent given a random variable \( \Theta \)? Consider in particular the case where the risks are conditionally identically distributed given \( \Theta \). Comment.

Exercise 4.7

In the literature on premium principles, several other properties of such principles are discussed in addition to those properties discussed in this book. One of them is iterativity. We say that a premium principle is iterative if

\[
H(H(X|\Theta)) = H(X).
\]

a) Give a verbal explanation of this property.

b) Check if the iterativity property is satisfied by the expected value principle and the variance principle. Comment.

Iterativity of premium principles is discussed more thoroughly in Goovaerts, De Vylder, & Haezendonck (1984).