

CDOs in Chains

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ABSTRACT: This paper discusses the pricing of CDOs in a Markov chain framework.

We show that in general the values of the legs satisfy systems of partial differential equations. In the special case of constant default intensities, one only needs to solve a system of ordinary differential equations, the so-called Kolmogorov differential equations.

KEYWORDS: Markov chains, credit risk, credit derivatives, contagion

JEL-CLASSIFICATION: G13, G33

1 Introduction

In recent years, the market for credit derivatives has become one of the most rapidly growing markets ever. Apart from credit default swaps (CDSs), collateralized debt obligations (CDOs) are the most important contracts. The market for CDOs exceeded \$1.5 trillion by the end of 2005. This demonstrates their importance and the need for efficient methods to price these contracts. Currently, practitioners usually apply copula approaches to capture the contagion effects within the firms of the pool. However, copulas are black boxes and practitioners start to realize some of the drawbacks of this approach. In contrast to that, Kraft and Steffensen (2006) present a Markov chain approach that is both tractable and general enough to allow the pricing of various kinds of defaultable contingent claims. This paper shows how their framework can be used efficiently to calculate the spreads of CDO tranches. In the following section, we sketch their framework and show how the tranches of a CDO can be modeled. In Section 3, we discuss the pricing of CDOs in this framework. Section 4 presents a numerical example.

2 Modeling CDOs via Markov Chains

The underlying of a CDO is a pool of I contingent claims (bonds or CDSs) issued by different firms. We assume that each underlying claim has a notional of N , which leads to a pool size of NI dollars. The default process of the underlying instruments is modeled by a Markov chain with state space $\mathcal{J} = \{0, 1, \dots, I\}$ as depicted in Figure 1. Each state label corresponds to the number of firms that have already defaulted. Therefore, the chain starts in state 0 and upon the first default jumps into state 1 and so forth. The probabilities of the consecutive defaults are modeled via the transition intensities λ^{j+1} . These intensities will be interpreted as the aggregate default intensities of the remaining firms in the pool. To keep track of the number of defaults so far, we introduce the stochastic process Z implying that $Z(t)$ denotes the number of defaults until time t (or equivalently the state in which the Markov chain is at time t). Furthermore, we introduce the $(I + 1)$ -dimensional counting process $N = (N^k)_{k \in \mathcal{J}}$, where N^k counts the number of transitions into state k , i.e.

$$N^k(t) = \#\{s \mid s \in (0, t], Z(s-) \neq k, Z(s) = k\}.$$

It is assumed that an equivalent martingale measure Q exists under which discounted asset prices are (local) martingales. We assume that there exist sufficiently regular functions λ^{jk} such that N^k admits the risk-neutral stochastic intensity process $\{\lambda^{Z(t-)k}\}_{t \in [0, T]}$, i.e.

$$M^k(t) = N^k(t) - \int_0^t \lambda^{Z(s)k}(s) ds$$

is a Q -martingale. Set $\lambda^{j*} = \sum_{k \neq j} \lambda^{jk}$. To allow for stochastic intensities, we introduce an L -dimensional process $Y = (Y_1, \dots, Y_L)$ consisting of macroeconomic state variables such as

interest rates and following the dynamics

$$dY_\iota(t) = \alpha_\iota(t)dt + \beta_\iota(t)dW(t),$$

where β is an $L \times L$ -matrix and β_ι is its ι -th row. Besides, W is an L -dimensional standard Brownian motion with uncorrelated coordinates and α_ι as well as β_ι are functions of t and $Y(t)$. The intensities and the short rate are assumed to be functions of t and Y , i.e. $\lambda^{jk}(t) = \lambda^{jk}(t, Y(t))$ and $r(t) = r(t, Y(t))$. As pointed out by Kraft and Steffensen (2006), the process Z is a time-inhomogeneous Markov chain if we condition on the history of Y . Therefore, the risk-neutral conditional transition probabilities $q^{jk}(t, u) = Q(Z(u) = k | Z(t) = j)$ satisfy the Kolmogorov backward equations¹

$$q_t^{jk}(t, u) = \lambda^{j*}(t)q^{jk}(t, u) - \sum_{\nu \neq j} \lambda^{j\nu}(t)q^{\nu k}(t, u) \quad (1)$$

with terminal conditions $q^{jk}(u, u) = \delta^{jk}$. Here q_t denotes the partial derivative with respect to t and δ^{jk} stands for the Kronecker delta, i.e. $\delta^{jk} = 1$ for $j = k$ and zero otherwise.

The tranches of the CDO are modeled by the sequence $0 = T_0 < T_1 < \dots < T_M = 1$ of attachment and detachment points. Thus the equity tranche is characterized by the interval $[T_0, T_1]$ and the mezzanine tranche by the interval $[T_1, T_2]$. The standard tranches of the North American CDX IG portfolio, for example, are described by $T_1 = 0.03$, $T_2 = 0.07$, $T_3 = 0.1$, $T_4 = 0.15$, $T_5 = 0.3$ and $T_6 = 1$. The *face value* of tranche m is $F_m := (T_m - T_{m-1})NI$. Each tranche $m = 1, \dots, M$ pays a fixed coupon as long as its notional has not been affected by subsequent defaults. Let the coupon dates be $0 = t_0 < t_1 < t_2 < \dots < t_K$ with $\Delta t = t_k - t_{k-1} = \text{const}$ for all k . We assume that the coupons have already been adjusted according to the interest rate period Δt . Quarterly paid coupons ($\Delta t = 1/4$) and an interest per annum of $s_m = 5\%$ (for tranche m), for example, yield a coupon value of $0.05/4$. However, if the tranche has already been affected by subsequent defaults, then we need to adjust the coupon payments by a *thinning factor* θ_m^j , depending on the tranche m and on the state j of the Markov chain characterizing the CDO. We denote the aggregated loss after j defaults by L^j , which is assumed to be constant for fixed j . The thinning factors (as percentages) are then defined by

$$\theta_m^j := \begin{cases} 1 & \text{if } L^j \leq NIT_{m-1}, \\ \frac{NIT_m - L^j}{F_m} & \text{if } NIT_{m-1} < L^j < NIT_m, \\ 0 & \text{otherwise} \end{cases}$$

and we set $\Delta\theta_m^j = \theta_m^{j-1} - \theta_m^j$. We illustrate the meaning of the thinning factors θ_m^j by the following example.

Example. Consider a CDO with a pool of 50 loans having notionals of 2 million dollars each. The CDO consists of three tranches (equity, mezzanine, senior) given by $T_1 = 0.03$, $T_2 = 0.06$,

¹The conditional transition probabilities also satisfy the so-called Kolmogorov forward equations. See Karlin and Taylor (1975).

$T_3 = 1$. Therefore, the tranches have face values $F_1 = 3$, $F_2 = 3$, and $F_3 = 94$ (million dollars). Assume that all recovery rates are equal to $R = 0.4$. Therefore, the thinning factors θ_m^j , $m \in \{E, M, S\}$, are given by

j	0	1	2	3	4	5	6	7	...
θ_E^j	1	$\frac{1.8}{3}$	$\frac{0.6}{3}$	0	0	0	0	0	...
θ_M^j	1	1	1	$\frac{2.4}{3}$	$\frac{1.2}{3}$	0	0	0	...
θ_S^j	1	1	1	1	1	1	$\frac{92.8}{94}$	$\frac{91.6}{94}$...

3 Pricing CDOs via Markov Chains

Using standard arguments, the value of the fee and the protection legs of the m -th tranche can be expressed as

$$V_{f,m}(t) = F_m s_m \Delta t \sum_{k=k(t)}^K \mathbb{E}_t \left[\theta_m^{Z(t_k)} e^{-\int_t^{t_k} r(u) du} \right],$$

$$V_{p,m}(t) = F_m \mathbb{E}_t \left[\int_t^{t_K} e^{-\int_t^s r(u) du} \Delta \theta_m^{Z(s-)+1} dN^{Z(s-)+1}(s) \right],$$

where $\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot | Z(t), Y(t)]$ and $k(t)$ denotes the next coupon date after time t . The fair spread $s_m^*(t)$ is chosen such that $V_{f,m}(t) = V_{p,m}(t)$. The fee leg can be interpreted as an Z -dependent coupon payment, whereas the protection leg consists of payments upon transitions of the Markov chain. Therefore, these representations can be rewritten.

Proposition 3.1 (Values of the Legs) *Assume, without loss of generality, that no default has occurred so far, i.e. $Z(t) = 0$. Then the value of the fee and protection legs are given by*

$$V_{f,m}(t) = F_m s_m \Delta t \sum_{k=k(t)}^K \sum_{i=0}^I \theta_m^i \mathbb{E}_t \left[e^{-\int_t^{t_k} r(u) du} q^{0i|Y}(t, t_k) \right], \quad (2)$$

$$V_{p,m}(t) = F_m \sum_{\nu=0}^I \Delta \theta_m^{\nu+1} \int_t^{t_K} \mathbb{E}_t \left[e^{-\int_t^s r(u) du} q^{0\nu|Y}(t, s) \lambda^{\nu\nu+1}(s) \right] ds.$$

Proof. The results follow from Proposition 3.2 in Kraft and Steffensen (2006). \square

Remarks. a) Note that most of the summands in both sums are usually zero, since most of the θ_m^j and $\Delta \theta_m^j$ are zero.

b) The transition probabilities $q^{0i|Y}$ can be found by solving the Kolmogorov equations.

For fixed $(Y(t), Z(t)) = (y, j)$, we denote the corresponding values of the legs by $\vartheta^j(t, y) = \mathbb{E}[\cdot | Y(t) = y, Z(t) = j]$ and get the following result.

Proposition 3.2 (Systems of PDEs for the Legs) *(i) The value of the time- t_k coupon payment of the fee leg satisfies the following system of partial differential equations*

$$\vartheta_t^j = r\vartheta^j - \alpha\vartheta_y^j - 0.5 \operatorname{tr}[\beta^T \vartheta_{yy}^j \beta] - \lambda^{jj+1} (\vartheta^{j+1} - \vartheta^j)$$

with terminal conditions $\vartheta^j(t_k, y) = F_m s_m \Delta t \theta_m^j$ for all $j = 0, 1, \dots, I$.

(ii) The value of the protection leg of the m -th tranche satisfies the following system of partial differential equations

$$\vartheta_t^j = r\vartheta^j - \alpha\vartheta_y^j - 0.5 \operatorname{tr}[\beta^T \vartheta_{yy}^j \beta] - \lambda^{jj+1} (F_m \Delta \theta^{j+1} + \vartheta^{j+1} - \vartheta^j)$$

with terminal conditions $\vartheta^j(T, y) = 0$ for all $j = 0, 1, \dots, I$.

Proof. These results follow from applying Theorem 3.1 in Kraft and Steffensen (2006). \square

Remark. Since both systems of partial differential equations are recursively given, one can first solve the equation for ϑ^I and then the equation for ϑ^{I-1} and so forth. Note that all solutions thus possess Feynman-Kac representations.

Of particular interest is the situation where the transition intensities are constant. This assumption implies that the transition probabilities do not depend on Y and we thus write q^{jk} (instead of $q^{jk|Y}$). In this case, the values of the legs (2) can be rewritten as follows:

$$\begin{aligned} V_{f,m}(t) &= F_m s_m \Delta t \sum_{k=k(t)}^K \sum_{i=0}^I \theta_m^i p(t, t_k) q^{0i}(t, t_k), \\ V_{p,m}(t) &= F_m \sum_{\nu=0}^I \Delta \theta_m^{\nu+1} \lambda^{\nu\nu+1} \int_t^{t_K} p(t, s) q^{0\nu}(t, s) ds, \end{aligned} \quad (3)$$

where $p(t, s)$ denotes the time- t price of a default-free zero-coupon bond with maturity s . Equations (3) possess the convenient property that the only unknown quantities are the transition probabilities q^{0i} that, however, can be calculated easily by solving the Kolmogorov equations (1), i.e. by solving a system of ordinary equations. This is a well-studied problem that, in our case, has an explicit solution. The interested reader is referred to Kraft and Steffensen (2006) who discuss this issue in great detail.

4 Numerical Example

We calibrate the transition intensities λ^{jj+1} to the spreads of the equity tranche [0%, 3%] of a CDO on the iTraxx index. We define the average intensity per non-defaulted firm in state j as $\bar{\lambda}^{jj+1} = \lambda^{jj+1}/(I - j)$. The tranche notional is 37.5 million euros and the loss given default per name is assumed to be 6 million euros. Thus, the entire tranche has been wiped out once the Markov chain has reached state 7. Consequently, only $\bar{\lambda}^{01}, \bar{\lambda}^{12}, \dots, \bar{\lambda}^{67}$ are relevant to this tranche and have to be calibrated. Table 1 contains fictitious market spreads that are similar to the spreads that were observed on the iTraxx index in September 2006. Furthermore, for simplicity, we quote the spreads as “spreads only” and not as a spread and an up-front fee (the standard convention for the iTraxx equity tranche).

maturity	market spread
5 years	10.6%
7 years	11.8%
10 years	13.1%

Table 1: The Spreads of the Equity Tranche

The calibration is done as follows. We consider $\bar{\lambda}^{01}$, $\bar{\lambda}^{34}$ and $\bar{\lambda}^{67}$ as unknowns which have to be determined by calibration. The remaining $\bar{\lambda}$ s are found by linear interpolation. The resulting average intensities $\bar{\lambda}^{jj+1}$ are plotted in Figure 2. The slope of the graph can be interpreted as the contagion that is priced into the spreads by the market. The steeper the slope, the greater is the implied contagion effect.

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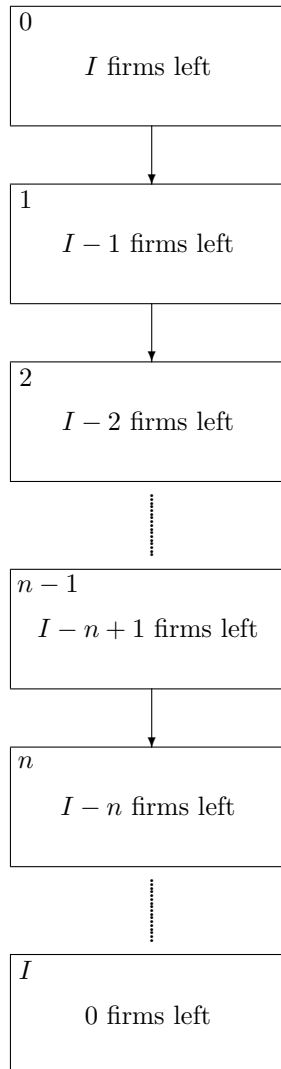


Figure 1: Model for a CDO

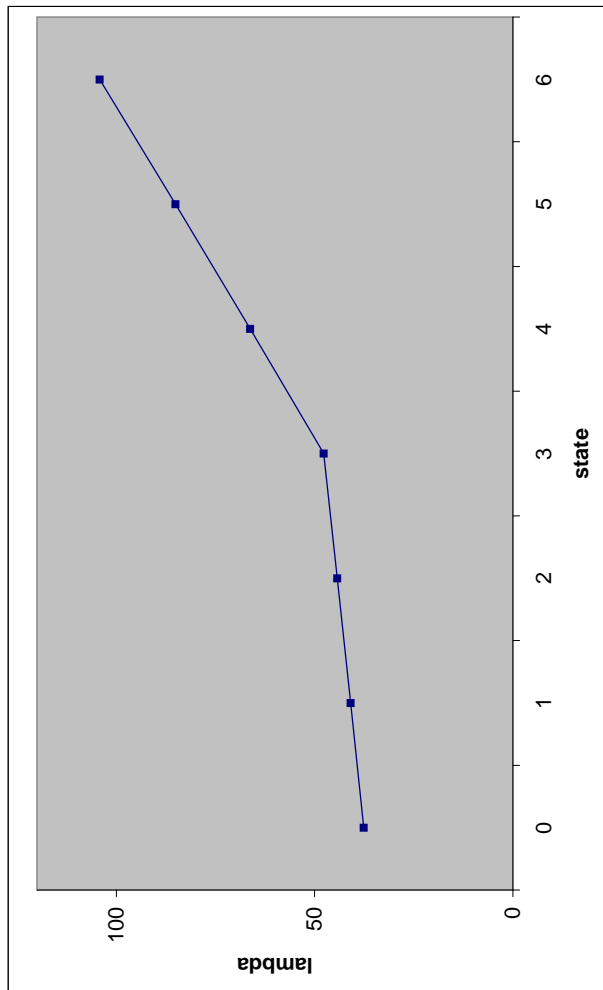


Figure 2: Calibrated Average Default Intensities