Nonlinear Time Series Modeling

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References:

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1. Introduction

Why nonlinear time series models?

$\equiv$

What are the limitations of linear time series models?

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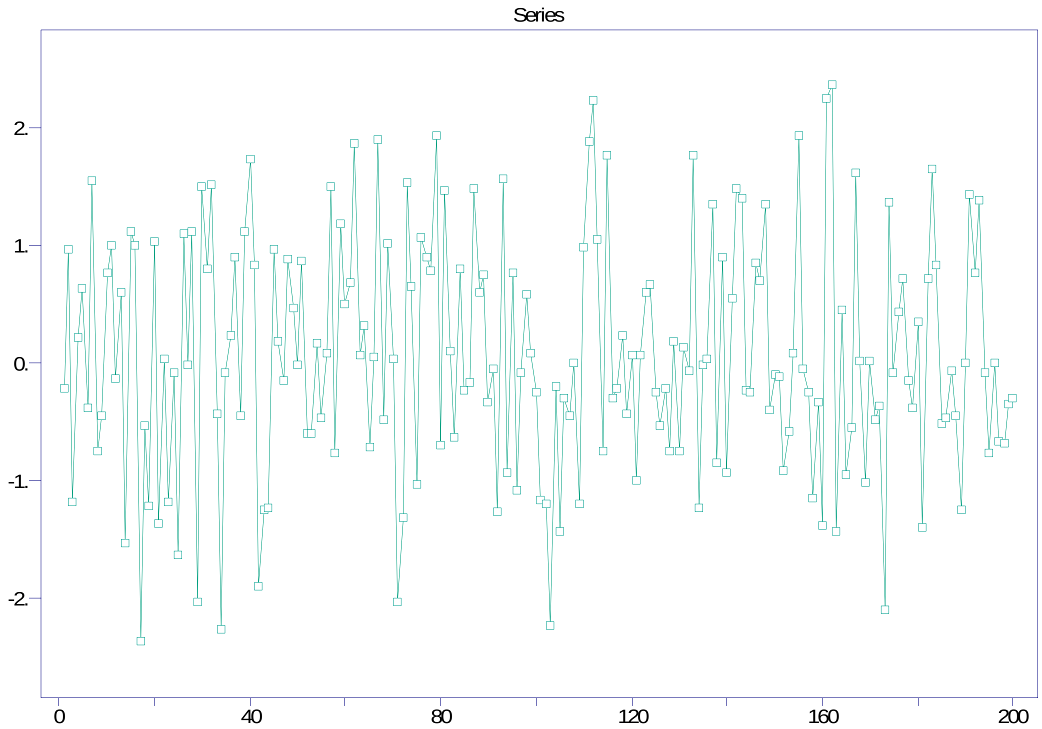
What key features in data cannot be captured by linear time series models?

What diagnostic tools (visual or statistical) suggest incompatibility of a linear model with the data?

Example: $Z_1, \ldots, Z_n \sim \text{IID}(0,\sigma^2)$



Sample autocorrelation function (ACF):

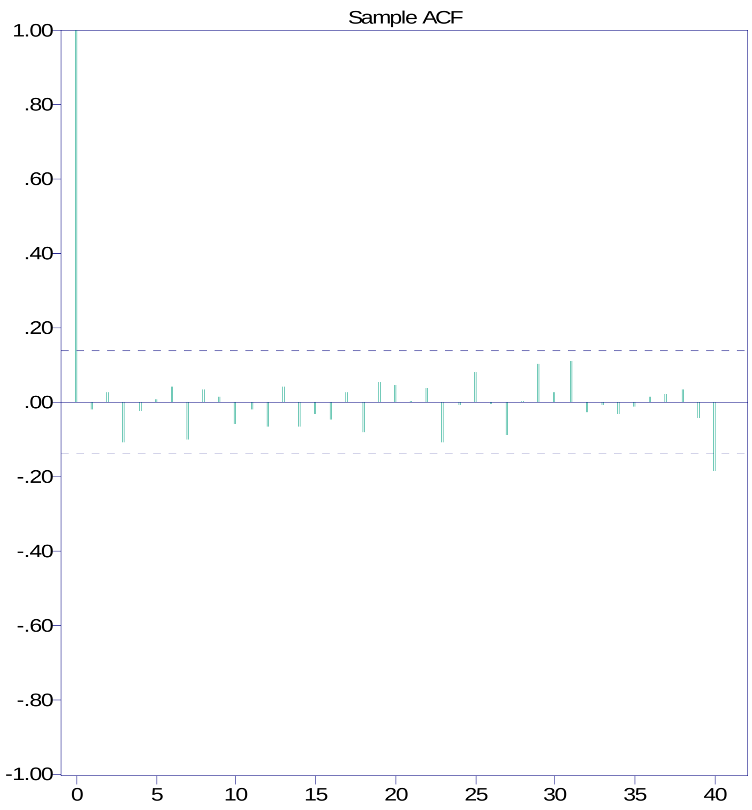
$$\hat{\rho}_Z(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)} \quad \text{where} \quad \hat{\gamma}(h) = n^{-1}\sum_{t=1}^{n-|h|}(Z_t - \bar{Z})(Z_{t+|h|} - \bar{Z})$$

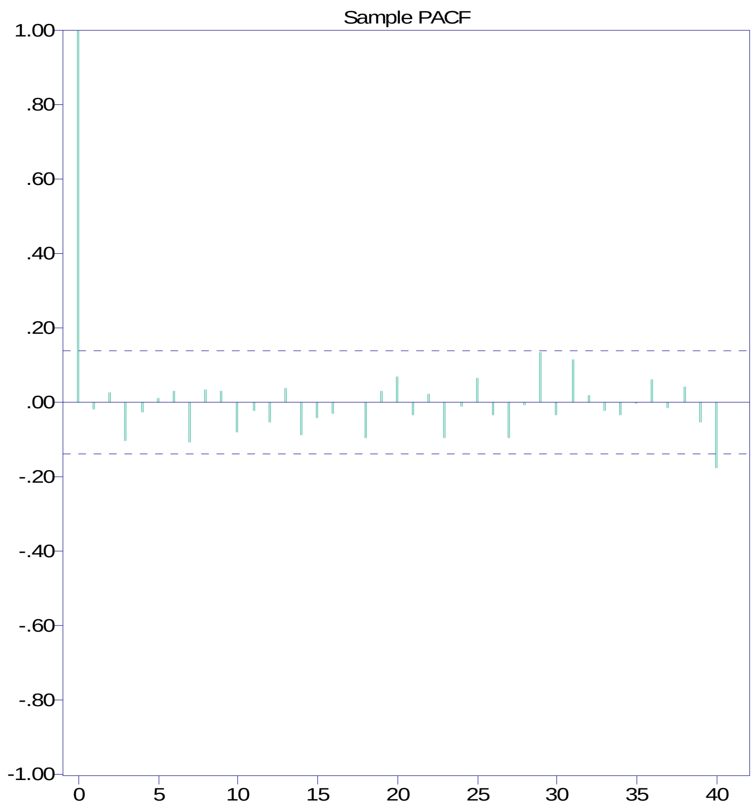
is the sample autocovariance function (ACVF).

Theorem. If $\{Z_t\} \sim \text{IID}(0,\sigma^2)$, then

$$(\hat{\rho}_Z(1),\ldots,\hat{\rho}_Z(h))' \text{ is approximately IID } N(0,1/n).$$

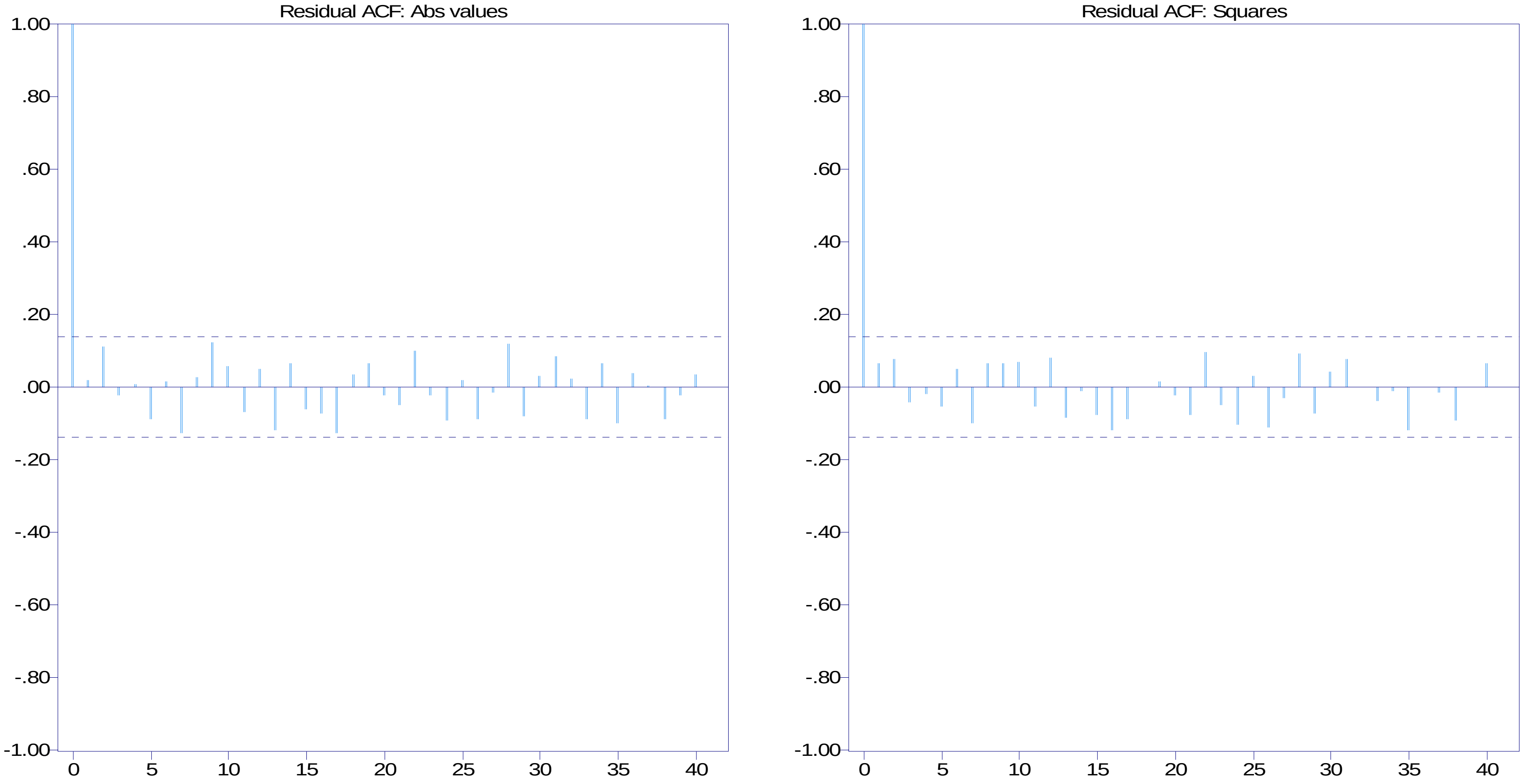
Proof: (see problem 6.24 TSTM)



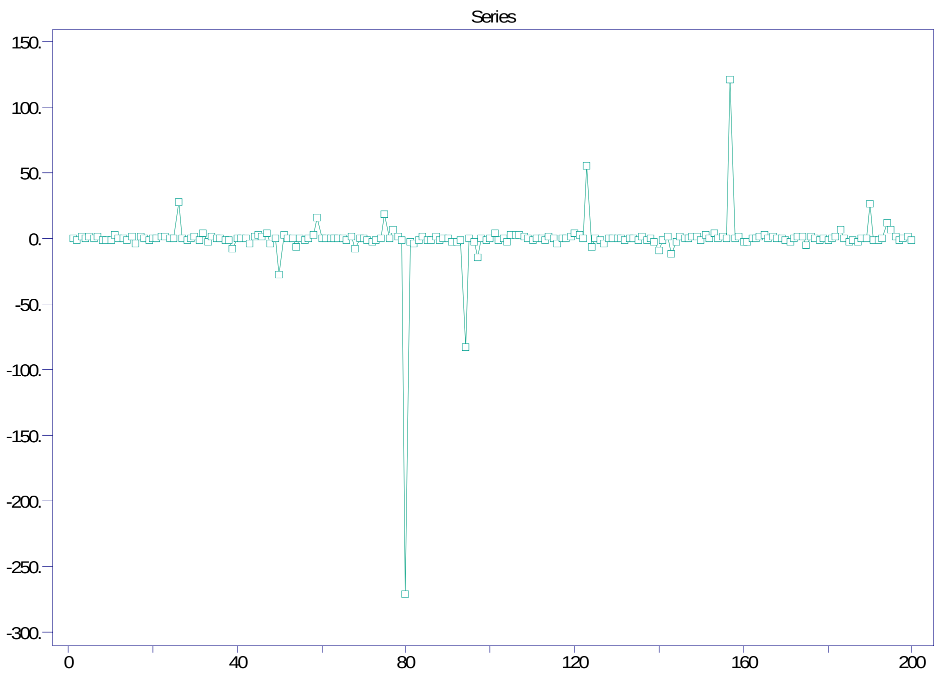


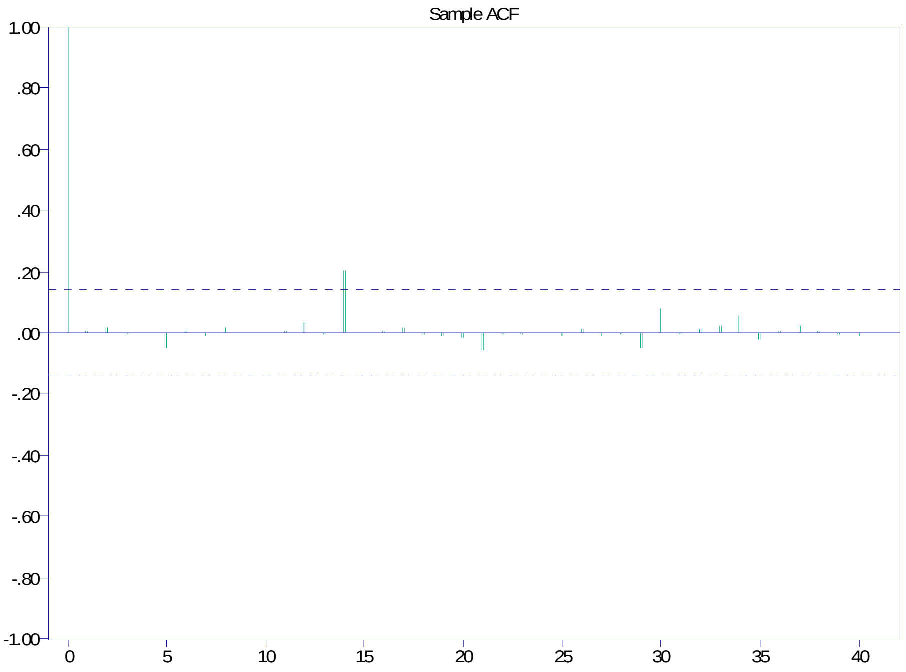
Cor. If $\{Z_t\} \sim \text{IID}(0,\sigma^2)$ and $E|Z_1|^4 < \infty$, then

$$(\hat{\rho}_{Z^2}(1),\ldots,\hat{\rho}_{Z^2}(h))'$$ is approximately IID N(0,1/n).



What if $E|Z_1|^2 = \infty$? For example, suppose $\{Z_t\}$~ IID Cauchy.



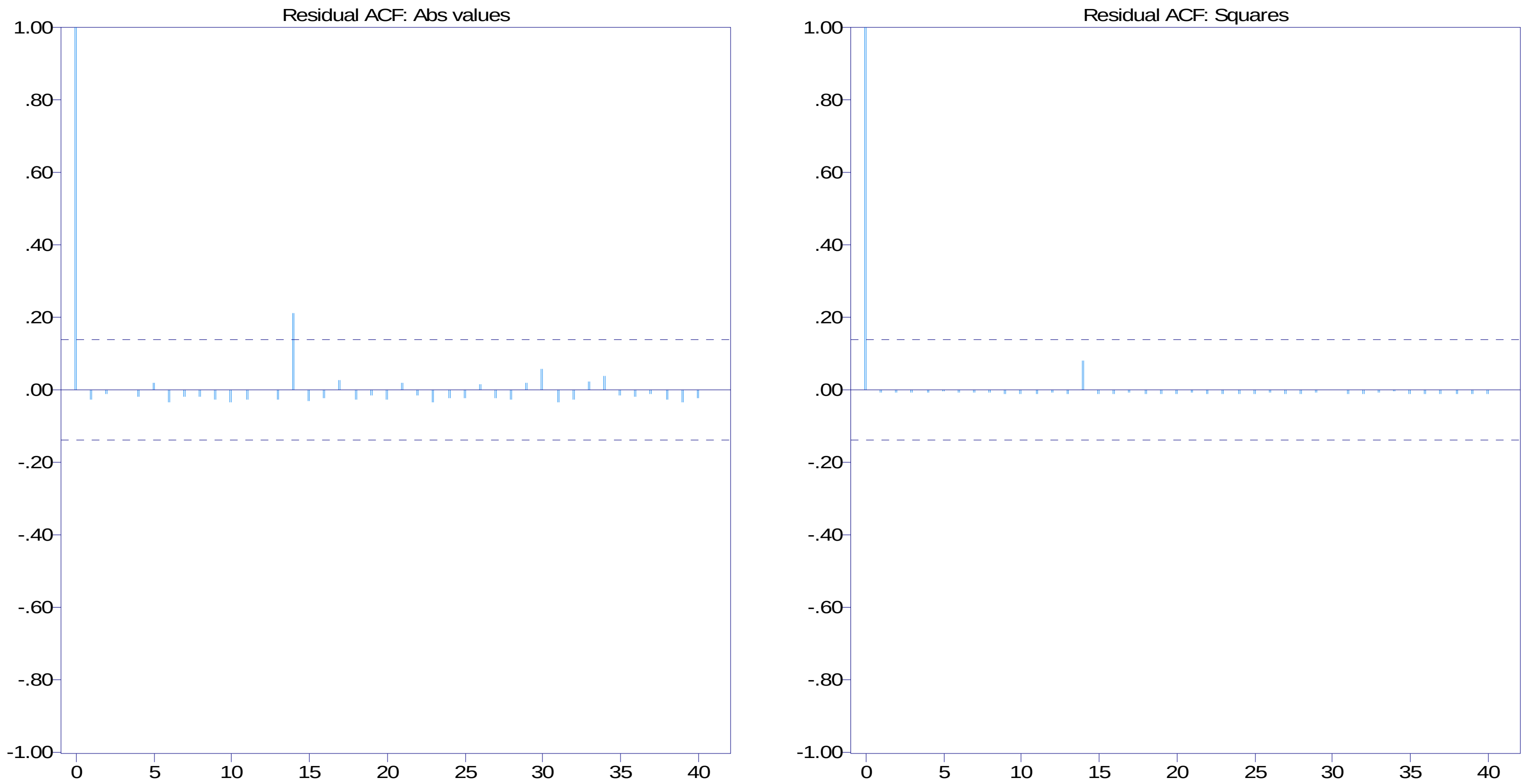


Result (see TSTM 13.3): If $\{Z_t\}$~ IID Cauchy, then

$$\frac{n}{\ln n}\hat{\rho}_Z(h) \Rightarrow \frac{S_1}{S_{.5}},$$

S_1 and $S_{.5}$ are independent stable random variables.

How about the ACF of the squares?



Result: If $\{Z_t\}$~ IID Cauchy, then

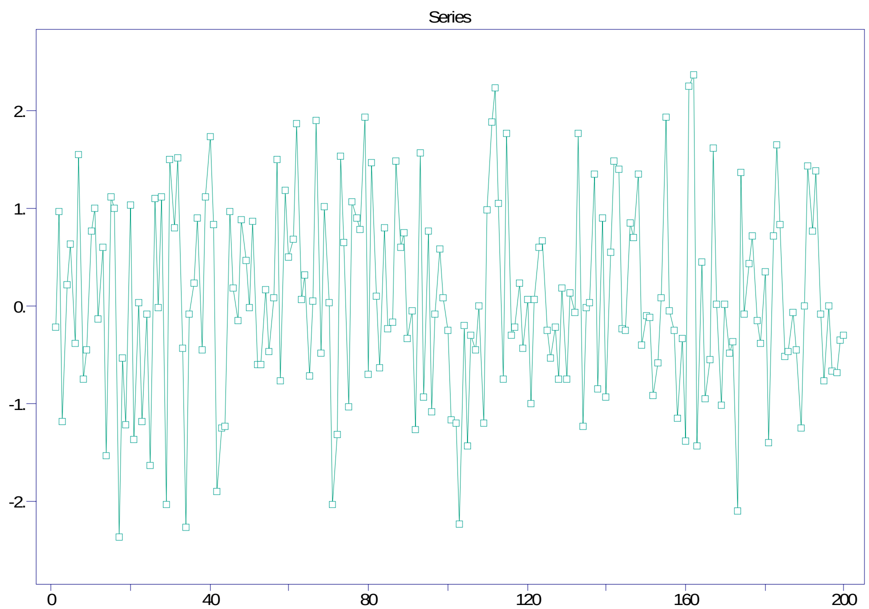
$$\left(\frac{n}{\ln n}\right)^2 \hat{\rho}_{Z^2}(h) \Rightarrow \frac{S_{1/2}}{S_{.25}},$$

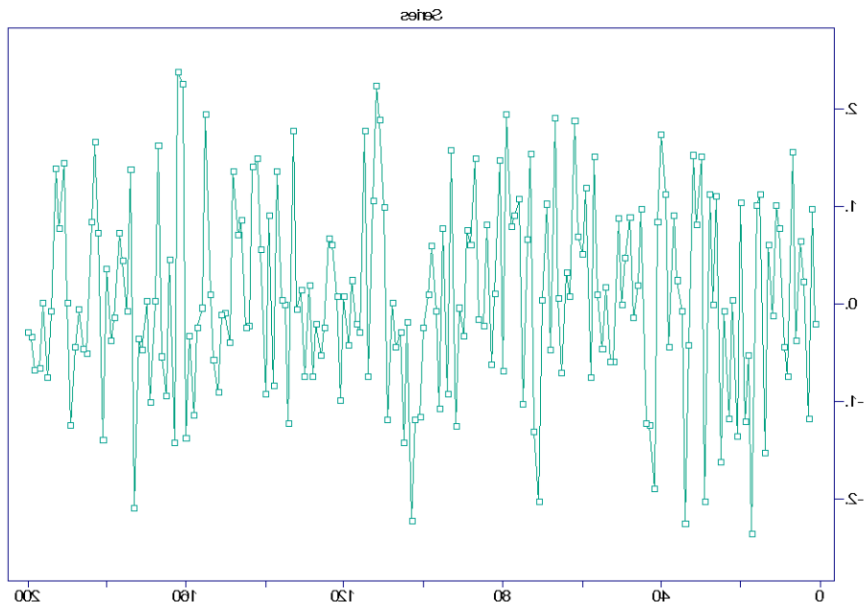
$S_{.5}$ and $S_{.25}$ are independent stable random variables.

Reversibility. The stationary sequence of random variables $\{X_t\}$ is *time-reversible* if $(X_1, \ldots, X_n) =_d (X_n, \ldots, X_1)$.

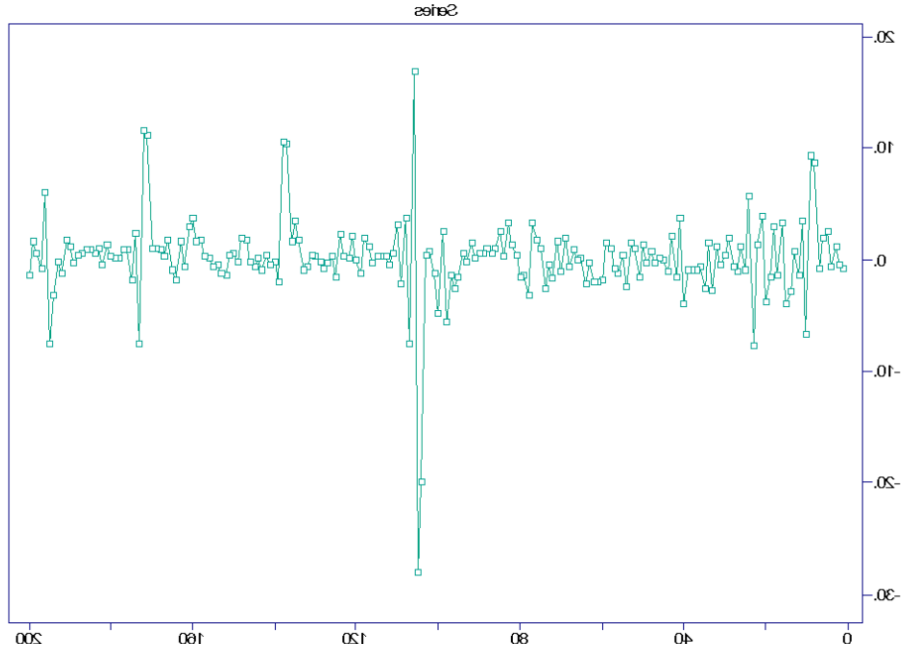
Result: IID sequences $\{Z_t\}$ *are* time-reversible.

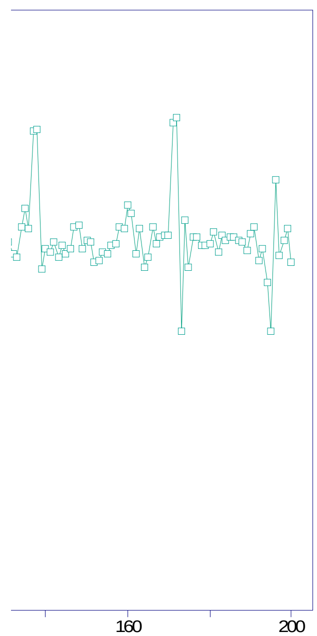
Application: If plot of time series does not look time- reversible, then it *cannot* be modeled as an IID sequence. Use the “flip and compare” inspection test!

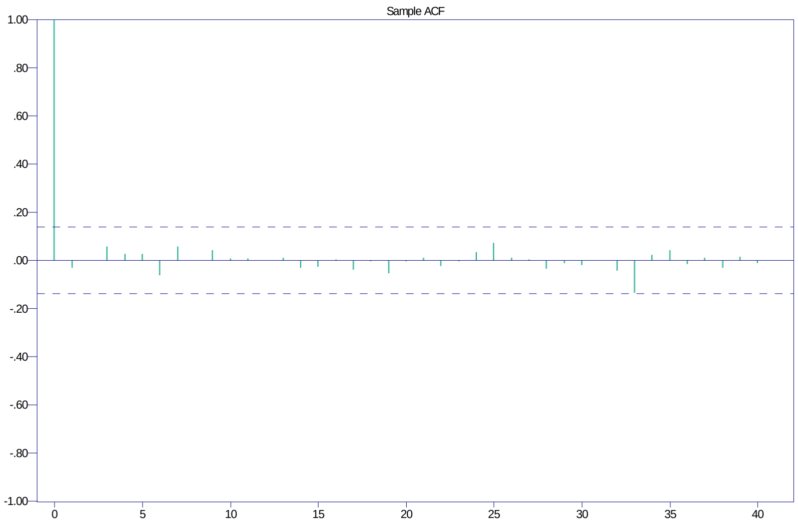


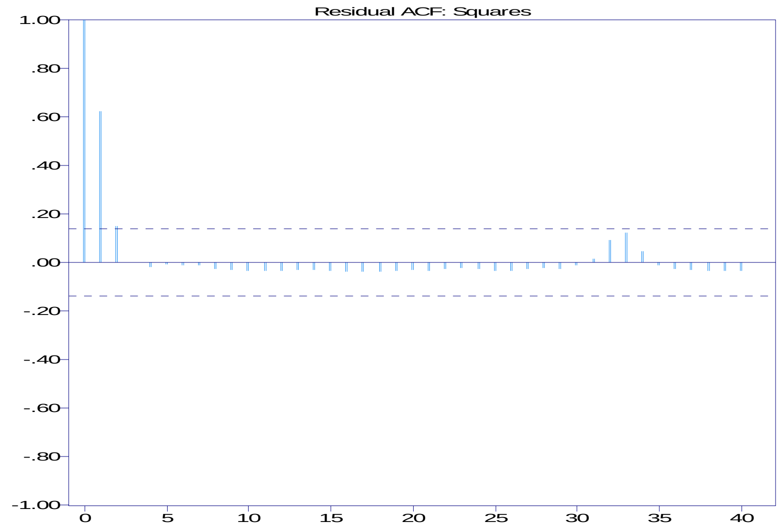


Reversibility. *Does the following series look time-reversible?*



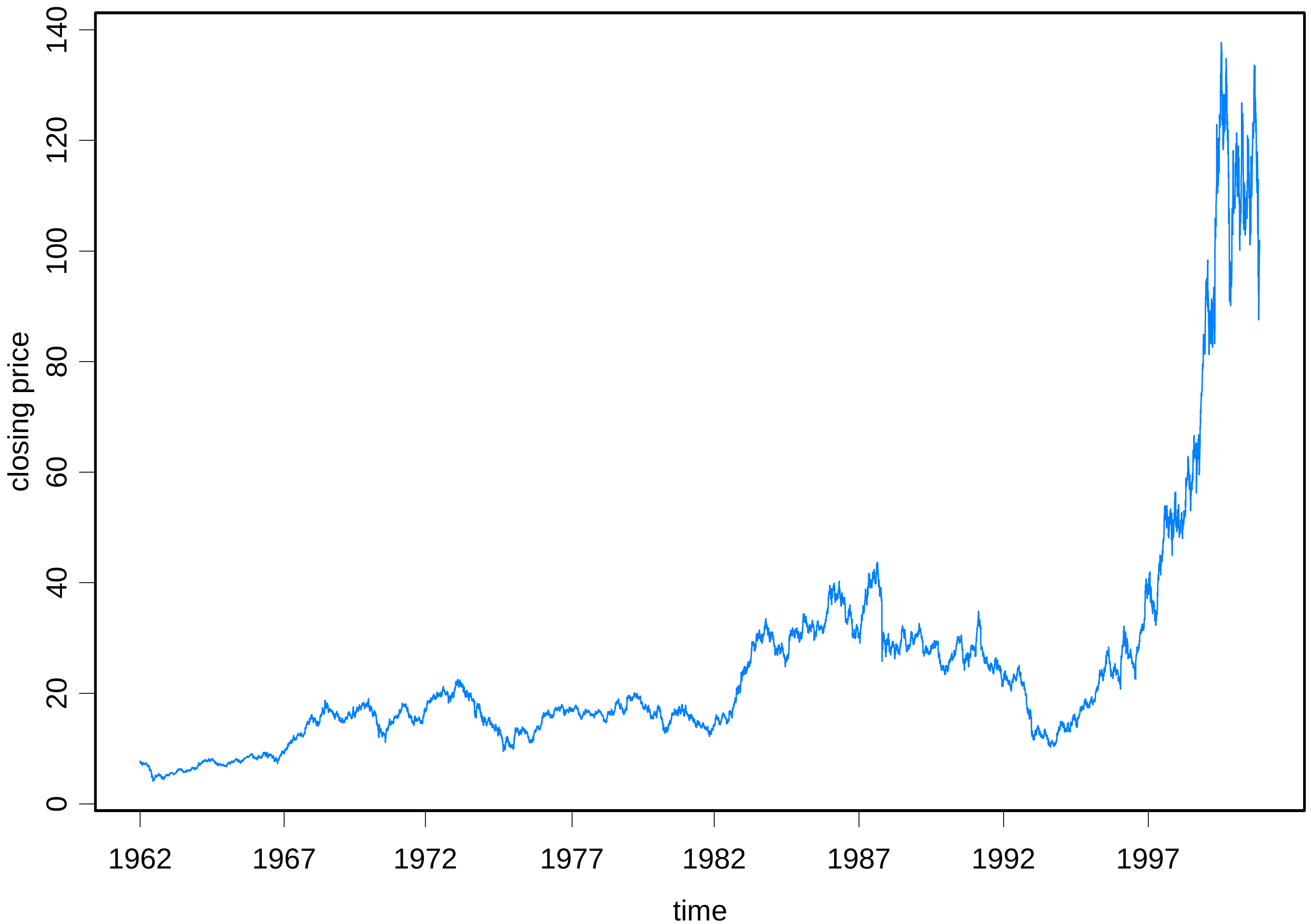




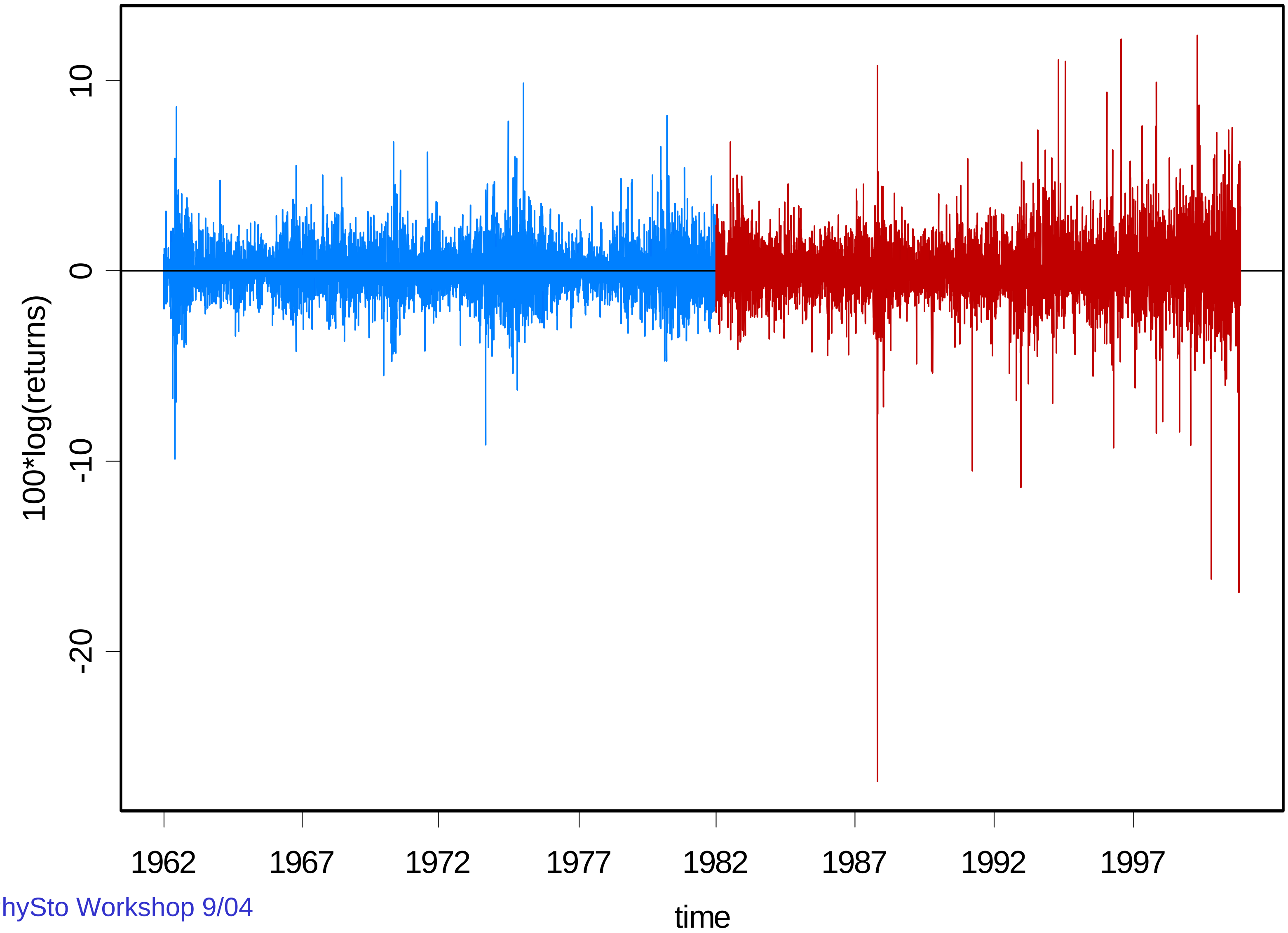


2. Examples

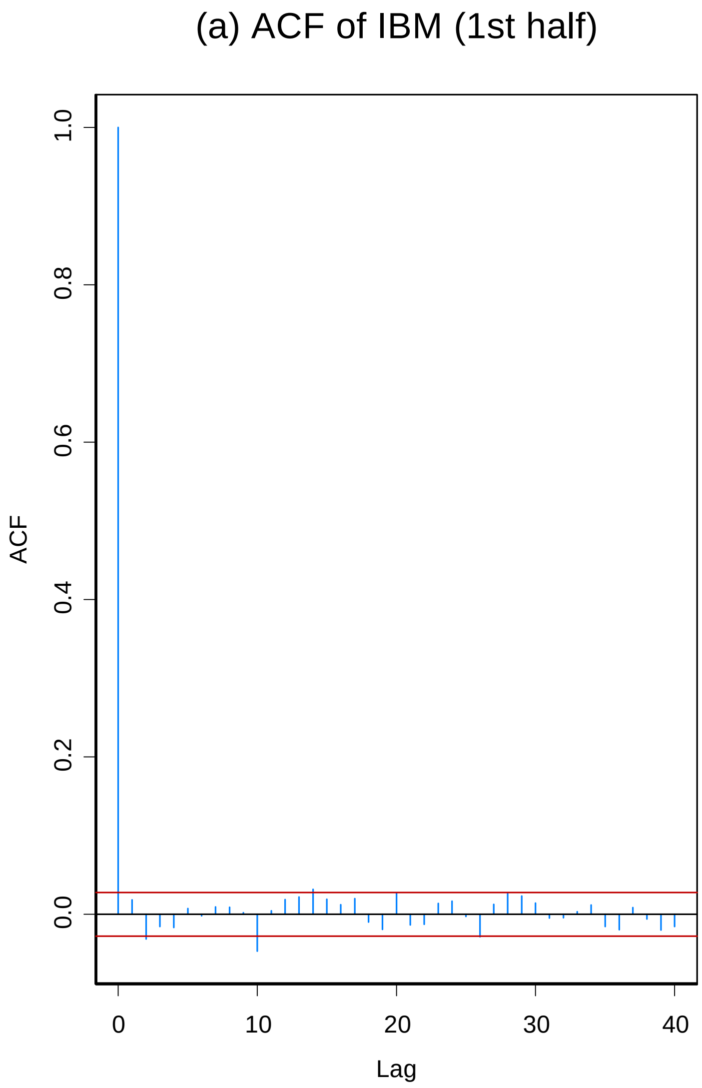
Closing Price for IBM 1/2/62-11/3/00

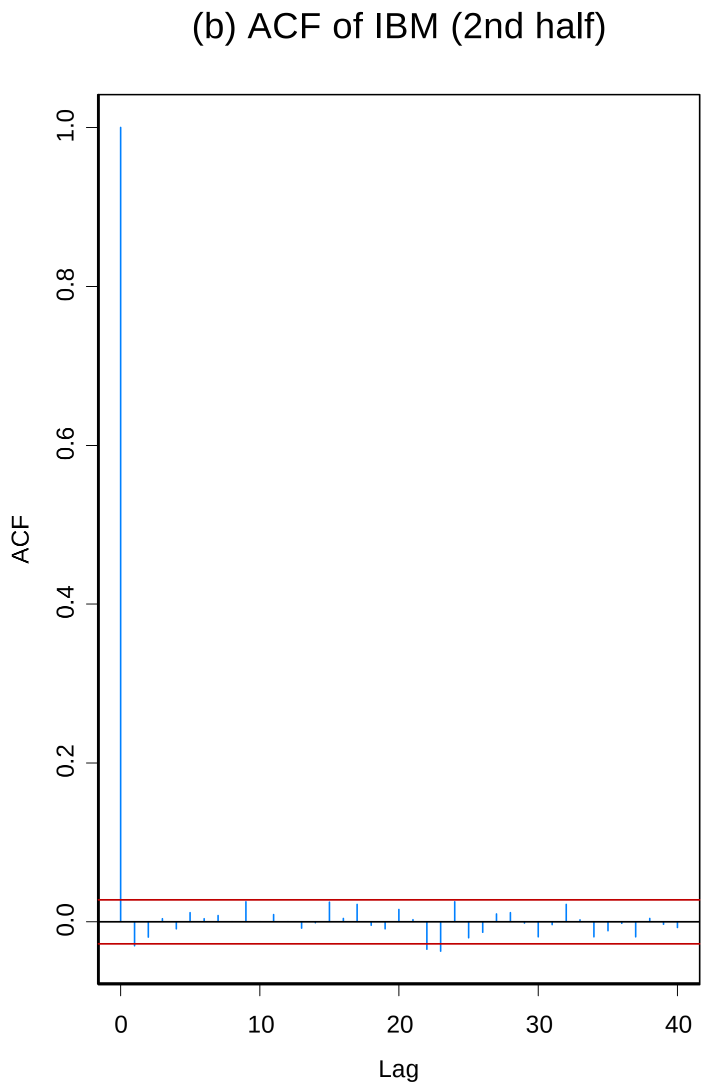


Log returns for IBM 1/3/62-11/3/00 (blue=1961-1981)



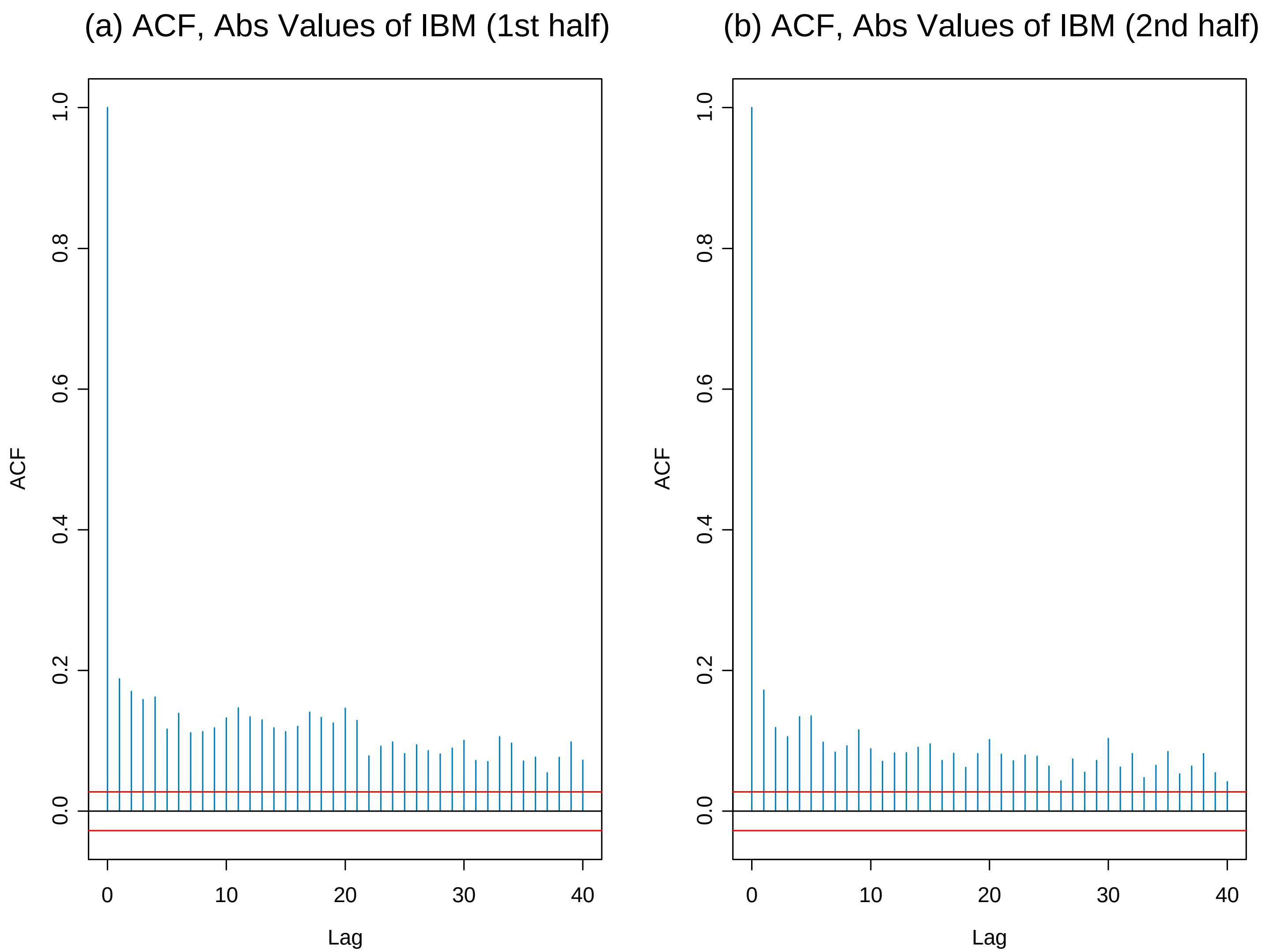
Sample ACF IBM (a) 1962-1981, (b) 1982-2000





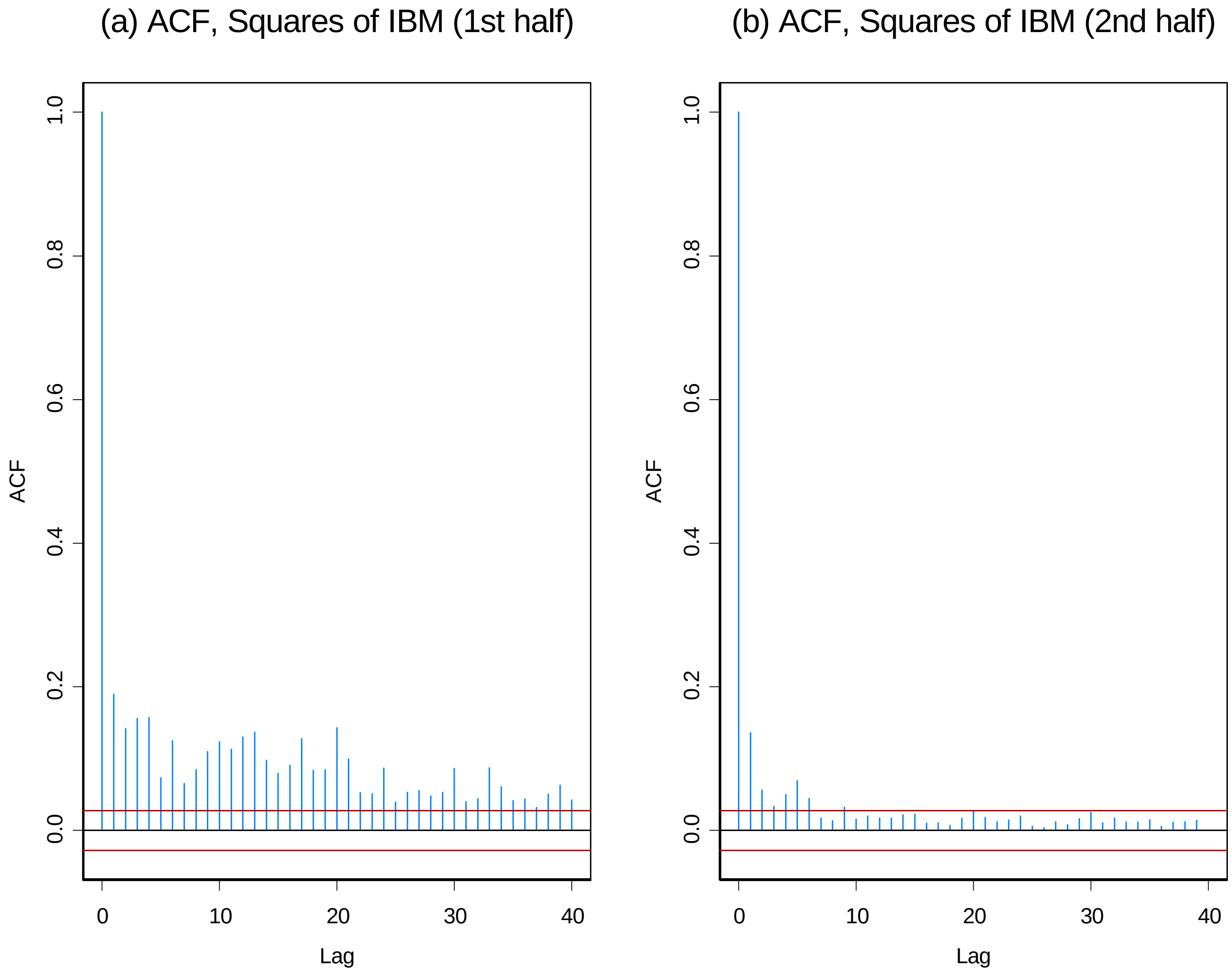
Remark: Both halves look like white noise?

Sample ACF of abs values for IBM (a) 1961-1981, (b) 1982-2000



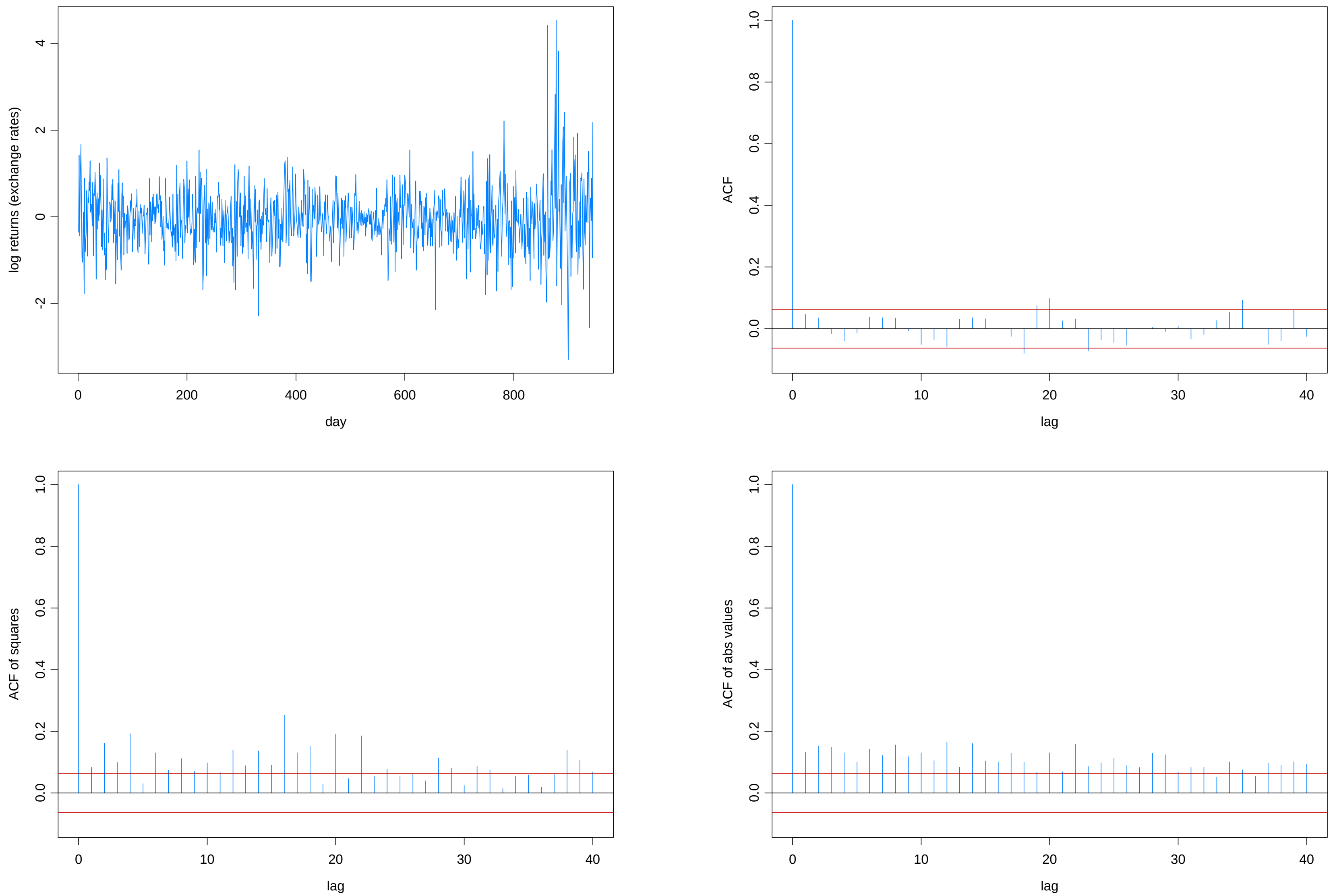
Remark: Series are not independent white noise?

ACF of squares for IBM (a) 1961-1981, (b) 1982-2000

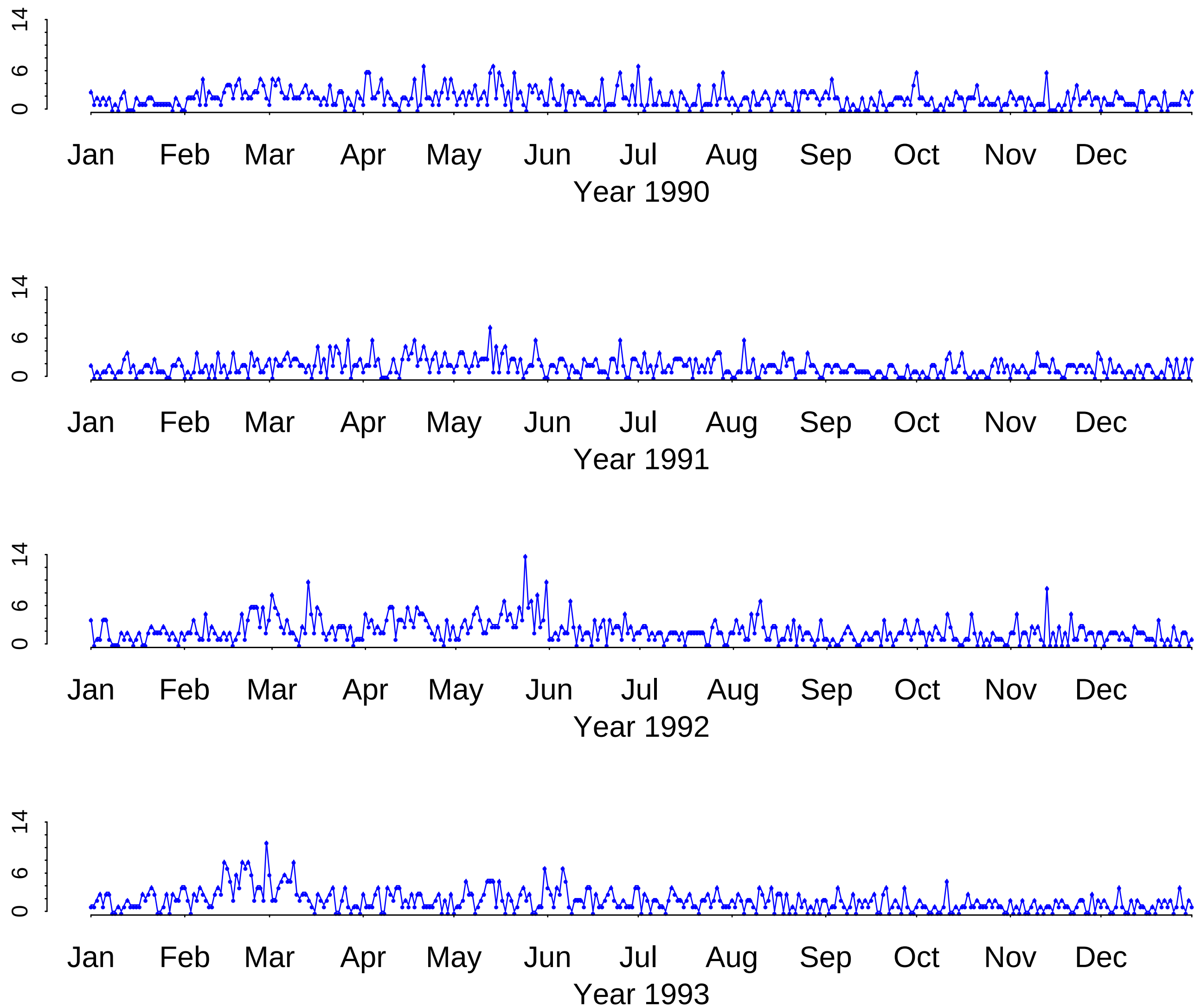


<u>Remark:</u> Series are not independent white noise? Try *GARCH* or a *stochastic volatility* model.

Example: Pound-Dollar Exchange Rates
(Oct 1, 1981 – Jun 28, 1985; Koopman website)



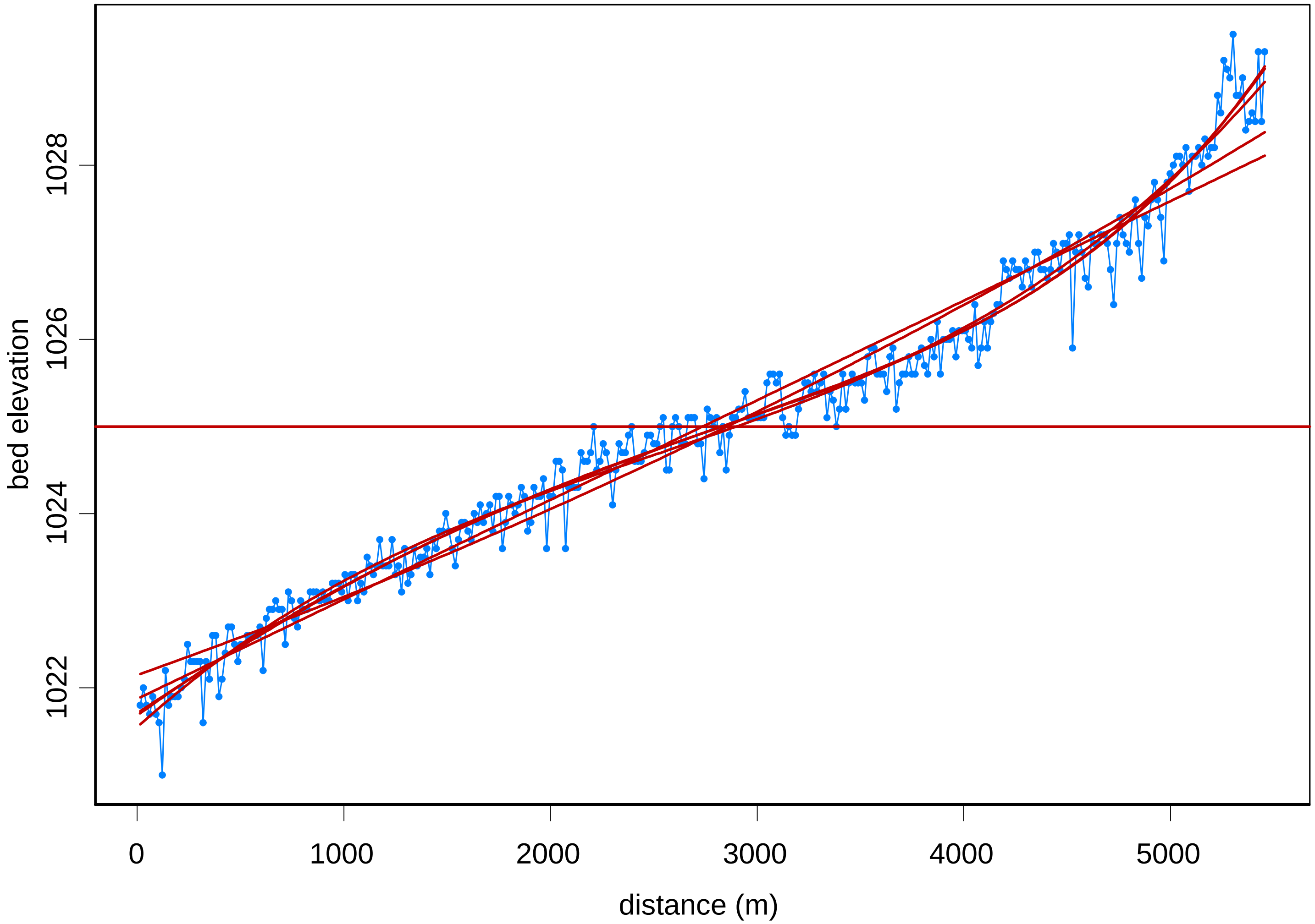
Example: Daily Asthma Presentations (1990:1993)



Remark: Usually marginal distribution of a linear process is continuous.

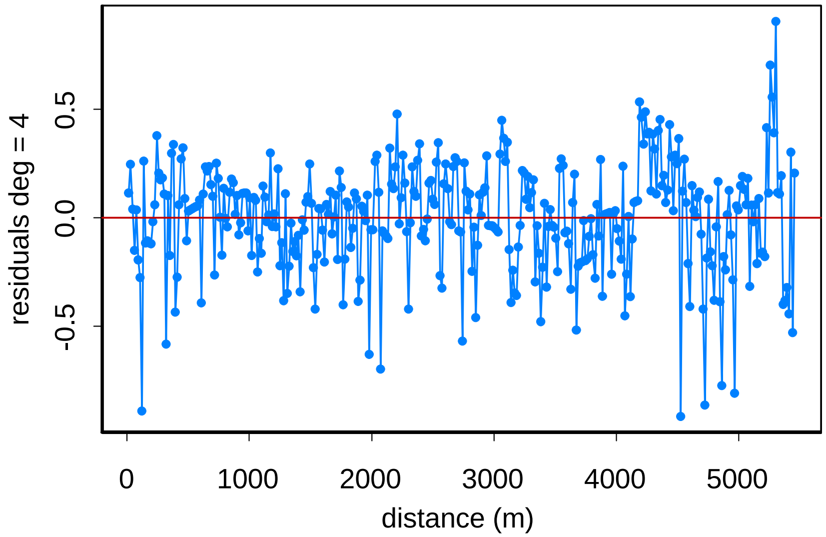
Muddy Creek- tributary to Sun River in Central Montana

Muddy Creek: surveyed every 15.24 meters, total of 5456m; 358 measurements



Degree	AIC_c
0	1455
1	294.3
2	251.3
3	47.1
4	34.0
5	35.5

Muddy Creek: residuals from poly(d=4) fit



Minimum AIC_c ARMA model:
ARMA(1,1)

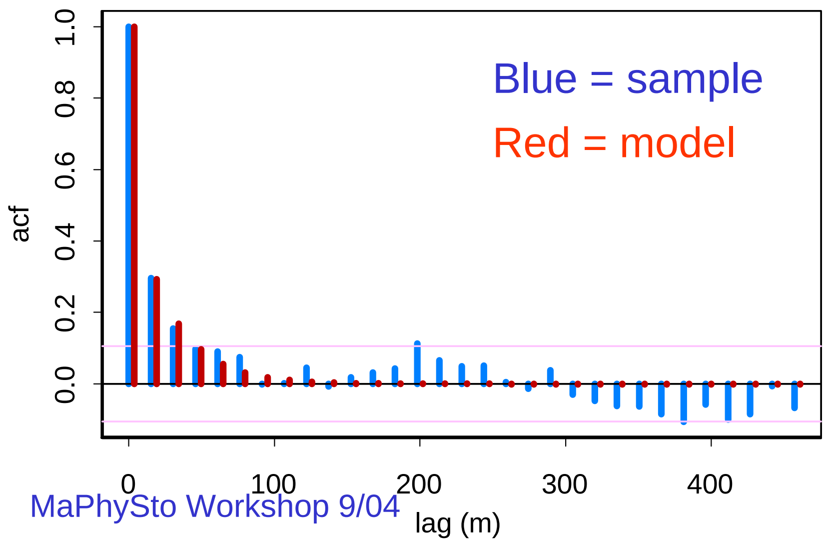
$Y_t = .574\ Y_{t-1} + \varepsilon_t - .311\ \varepsilon_{t-1}$,
$\{\varepsilon_t\} \sim WN(0,.0564)$

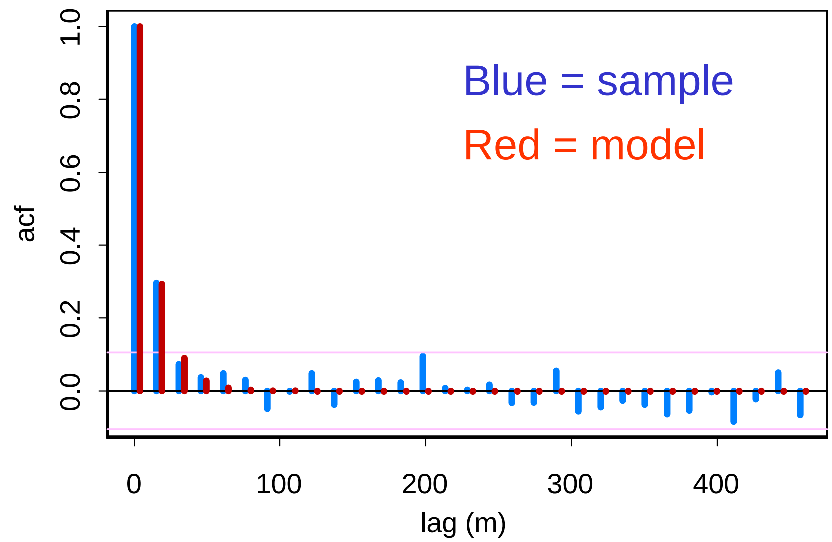
Noncausal ARMA(1,1) model:

$Y_t = 1.743\ Y_{t-1} + \varepsilon_t - .311\ \varepsilon_{t-1}$

Some theory:

- LS estimates of trend parameters are *asymptotically efficient*.
- LS estimates are *asymptotically indep* of cov parameter estimates.





Muddy Creek (cont)

Summary of models fitted to Muddy Creek bed elevation:

Degree	AIC_c	ARMA	AIC_c
0	1455	(1,2)	59.67
1	294.3	(2,1)	26.98
2	251.3	(2,1)	26.30
3	47.1	(1,1)	7.12
4	**34.0**	**(1,1)**	**2.78**
5	35.5	(1,1)	4.68

Example: NEE=Net Ecosystem Exchange in Harvard Forest

- About half of the CO2 emitted by humans accumulates in the atomosphere
- Other half is absorbed by “sink” processes on land and in the oceans

$$NEE = (Rh + Ra) - GPP \quad \text{(carbon flux)}$$

GPP = Gross Primary Production (photosysynthesis)

Rh = Heterotrophic (microbial) respiration

Ra = autotrophic (plant) respiration.

The NEE data from the Harvard Forest consists of hourly measurements. We will aggregate over the day and consider daily data from Jan 1, 1992 to Dec 31, 2001.

Go to ITSM Demo

3. Linear Processes

3.1 Preliminaries

Def: The stochastic process $\{X_t\ , t=0, \pm 1, \pm 2, \ldots\}$ defined on a probability space is called a *discrete-time time series*.

Def: $\{X_t\}$ is *stationary* or *weakly stationary* if

i. $E|X_t|^2 < \infty$, for all t.

ii. $EX_t = m$, for all t.

iii. $\text{Cov}(X_t, X_{t+h})=\gamma(h)$ depends on h only.

Def: $\{X_t\}$ is *strictly stationary* if $(X_1, \ldots, X_n) =_d (X_{1+h}, \ldots, X_{n+h})$ for all $n \geq 1$ and $h=0, \pm 1, \pm 2, \ldots$

Remarks:

i. SS + ($E|X_t|^2 < \infty$) $\Rightarrow$ weak stationarity

ii. WS $\not\Rightarrow$ SS (think of an example)

iii. WS + Gaussian $\Rightarrow$ SS (why?)

3.1 Preliminaries (cont)

Def: $\{X_t\}$ is a *Gaussian time series* if

$$(X_m, \ldots, X_n) \text{ is multivariate normal}$$

for all integers $m < n$, i.e., all finite dimensional distributions are normal.

Remark: A Gaussian time series is completely determined by the mean function and covariance functions,

$$m(t) = EX_t \quad \text{and} \quad \gamma(s,t) = \text{Cov}(X_s, X_t).$$

If follows that a Gaussian TS is stationary (SS or WS) if and only if

$$m(t) = m \quad \text{and} \quad \gamma(s,t) = \gamma(t\text{-}s) \text{ depends only on the time lag } t\text{-}s.$$

3.1 Preliminaries (cont)

Def: $\{X_t\}$ is a *linear time series* with mean 0 if

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j},$$

where $\{Z_t\} \sim \mathrm{WN}(0,\sigma^2)$ and $\sum_{j=-\infty}^{\infty} \psi_j^2 < \infty.$

Important remark: As a reminder WN means uncorrelated random variables and not necessarily independent noise nor independent Gaussian noise.

Proposition: A linear TS is stationary with

i. $EX_t = 0$, for all t.

ii. $\gamma(h) = \sigma^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h}$ and $\rho(h) = \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h} / \sum_{j=-\infty}^{\infty} \psi_j^2$

If $\{Z_t\} \sim \mathrm{IID}(0,\sigma^2)$, then the linear TS is strictly stationary.

Is the converse to the previous proposition true? That is, are all stationary processes linear?

Answer: Almost.

3.2 Wold Decomposition (TSTM Section 5.7)

Example: Set

$$X_t = A\cos(\omega t) + B\sin(\omega t),\ \omega\in(0,\pi),$$

where $A,B \sim \mathrm{WN}(0,\sigma^2)$. Then $\{X_t\}$ is stationary since

- $E\,X_t = 0$,
- $\gamma(h) = \sigma^2\cos(\omega h)$

Def: Let $\tilde{P}_n(\cdot)$ be the *best linear predictor operator* onto the linear span of the observations $X_n, X_{n-1}, \ldots$.

For this example,

$$\tilde{P}_{n-1}(X_n) = X_n.$$

Such processes with this property are called *deterministic.*

3.2 Wold Decomposition (cont)

The Wold Decomposition. If $\{X_t\}$ is a nondeterministic stationary time series with mean zero, then

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j} + V_t,$$

where

i. $\psi_0 = 1$, $\Sigma\, \psi_j^2 < \infty$.

ii. $\{Z_t\} \sim \text{WN}(0,\sigma^2)$

iii. $\text{Cov}(Z_s,V_t) = 0$ for all s and t

iv. $\tilde{P}_t(Z_t) = Z_t$ for all t.

v. $\tilde{P}_s(V_t) = V_t$ for all s and t.

vi. $\{V_t\}$ is deterministic.

The sequences $\{Z_t\}$, $\{V_t\}$, and $\{\psi_t\}$ are unique and can be written as

$$Z_t = X_t - \tilde{P}_{t-1}(X_t), \quad \psi_j = E(X_t Z_{t-j}) / E(Z_t^2), \quad V_t = X_t - \sum_{j=0}^{\infty} \psi_j Z_{t-j}.$$

3.2 Wold Decomposition (cont)

Remark. For many time series (in particular for all ARMA processes) the deterministic component V_t is 0 for all t and the series is then said to be purely nondeterministic.

Example. Let

$X_t = U_t + Y$, where $\{U_t\} \sim \mathrm{WN}(0,\sigma^2)$ and is independent of $Y \sim (0,\tau^2)$. Then, in this case, $Z_t = U_t$ and $V_t = Y$ (see TSTM, problem 5.24).

Remarks:

- If $\{X_t\}$ is purely nondeterministic, then $\{X_t\}$ is a linear process.
- Spectral distribution for nondeterministic processes has the form $F_X = F_U + F_V$, where $U_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$ which has spectral density

$$f(\lambda) = \frac{\sigma^2}{2\pi} \left| \sum_{j=0}^{\infty} \psi_j e^{ij\lambda} \right|^2 = \frac{\sigma^2}{2\pi} \left| \psi(e^{i\lambda}) \right|^2$$

3.2 Wold Decomposition (cont)

- If $\sigma^2 = E(X_t - \tilde{P}_{t-1}(X_t))^2 > 0$, then

  $F_X = F_U + F_V$,

  is the Lebesque decomposition of the spectral distribution function; F_U is the *absolutely continuous part* and F_V is the *singular part*.

Example. Let

$X_t = U_t + Y$, where $\{U_t\} \sim WN(0,\sigma^2)$ and is independent of $Y \sim (0,\tau^2)$. Then

$$F_X(d\lambda) = \frac{\sigma^2}{2\pi}(d\lambda) + \tau^2\delta_0(d\lambda)$$

Kolmogorov's Formula.

$$\sigma^2 = 2\pi \exp\{(2\pi)^{-1}\int_{-\pi}^{\pi} \ln f(\lambda)d\lambda\}, \text{ where } \sigma^2 = E(X_t - \tilde{P}_{t-1}(X_t))^2.$$

Clearly $\sigma^2 > 0 \text{ iff } \int_{-\pi}^{\pi} \ln f(\lambda)d\lambda > -\infty.$

3.2 Wold Decomposition (cont)

Example (TSTM problem 5.23).

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}, \quad \{Z_t\} \sim WN(0,\tau^2), \quad \psi_j = \left(\frac{1}{\pi}\right)\left(\frac{\sin j}{j}\right).$$

This process has a spectral density function but is deterministic!!

Example (see TSTM problem 5.20). Let

$$X_t = \varepsilon_t - 2\varepsilon_{t-1}, \quad \{\varepsilon_t\} \sim WN(0,\tau^2),$$

and set

$$Z_t = (1 - .5B)^{-1} X_t$$

$$= \sum_{j=0}^{\infty} .5^j X_{t-j} = \varepsilon_t - 2\varepsilon_{t-1} + .5(\varepsilon_{t-1} - 2\varepsilon_{t-2}) + .5^2(\varepsilon_{t-2} - \varepsilon_{t-3}) + \cdots$$

$$= \varepsilon_t - 3\sum_{j=1}^{\infty} .5^j \varepsilon_{t-j}$$

It follows that $\{Z_t\} \sim WN(0,\sigma^2)$ and $X_t = Z_t - .5Z_{t-1}$ is the WD for $\{X_t\}$.

a) If $\{\varepsilon_t\}$~IID N(0,σ^2), is $\{Z_t\}$ IID? Answer?

b) If $\{\varepsilon_t\}$~IID(0,σ^2), is $\{Z_t\}$ IID? Answer?

3.2 Wold Decomposition (cont)

Remark: In this last example, the process $\{Z_t\}$ is called an *allpass model of order* 1. More on this type of process later.

Go to ITSM Demo

3.3 Reversibility

Recall that the stationary time series $\{X_t\}$ is *time-reversible* if

$$(X_1, \ldots, X_n) =_d (X_n, \ldots, X_1) \text{ for all } n.$$

3.3 Reversibility

Recall that the stationary time series $\{X_t\}$ is *time-reversible* if

$$(X_1, \ldots, X_n) =_d (X_n, \ldots, X_1) \quad \text{for all } n.$$

Theorem (Breidt & Davis 1991). Consider the linear time series $\{X_t\}$

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}, \quad \{Z_t\} \sim IID,$$

where $\psi(z) \neq \pm z^r \psi(z^{-1})$ for any integer r. Assume either

(a) Z_0 has mean 0 and finite variance and $\{X_t\}$ has a spectral density positive almost everywhere.

or

(b) $1/\psi(z)=\pi(z)=\Sigma_j \pi_j z^j$, the series converging absolutely in some annulus D containing the unit circle and

$$\pi(B)X_t = \Sigma_j \pi_j X_{t-j} = Z_t.$$

Then $\{X_t\}$ is time-reversible if and only if Z_0 is Gaussian.

3.3 Reversibility (cont)

Remark: The condition $\psi(z) \neq \pm z^r \psi(z^{-1})$ on the filter precludes the filter from being symmetric about one of the coefficients. In this case, the time series would be time-reversible for non-Gaussian noise. For example, consider the series

$$X_t = Z_t - .5Z_{t-1} + Z_{t-2}, \quad \{Z_t\} \sim IID$$

Here $\psi(z)=1 - .5z + z^2 = z^2 (1 - .5 z^{-1} + z^2)= z^2 \psi(z^{-1})$ and the series is time-reversible.

Proof of Theorem: Clearly any stationary Gaussian time series is time-reversible (why?). So suppose Z_0 is nonGaussian and assume (a). If $\{X_t\}$ time-reversible, then

$$Z_t = \frac{1}{\psi(B)} X_t =_d \frac{1}{\psi(B^{-1})} X_t = \frac{\psi(B)}{\psi(B^{-1})} Z_t = \sum_{j=-\infty}^{\infty} a_j Z_{t-j}.$$

3.3 Reversibility (cont)

The first equality takes a bit of argument and relies on the spectral representation of $\{X_t\}$ given by

$$X_t = \int_{(-\pi,\pi]} e^{it\lambda} dZ(\lambda),$$

where $Z(\lambda)$ is a process of orthogonal increments (see TSTM, Chapter 4). It follows, by the assumptions on the spectral density of $\{X_t\}$ that

$$\frac{1}{\psi(B^{\pm})} X_t = \int_{(-\pi,\pi]} \frac{1}{\psi(e^{\mp i\lambda})} e^{it\lambda} dZ(\lambda),$$

is well defined. So

$$Z_t =_d \frac{\psi(B)}{\psi(B^{-1})} Z_t = \sum_{j=-\infty}^{\infty} a_j Z_{t-j}.$$

and, by the assumption on $\psi(z)$, the rhs is a non-trivial sum. Note that

$$\sum_{j=-\infty}^{\infty} a_j^2 = 1$$ Why?

The above relation is a characterization of a Gaussian distribution (see Kagan, Linnik, and Rao (1973).) □

3.3 Reversibility (cont)

Example: Recall for the example

$$X_t = \varepsilon_t - 2\varepsilon_{t-1}, \quad \{\varepsilon_t\} \sim IID(0, \tau^2),$$

and non-normal, the Wold decomposition is given by

$$X_t = Z_t - .5Z_{t-1},$$

where

$$Z_t = \varepsilon_t - 3\sum_{j=1}^{\infty} .5^j \varepsilon_{t-j}.$$

By previous result, $\{Z_t\}$ cannot be time-reversible and hence is *not IID*.

Remark: This theorem can be used to show *identifiability* of the parameters and noise sequence for an ARMA process.

3.4 Identifiability

Motivating example: The *invertible* MA(1) process

$$X_t = Z_t + \theta Z_{t-1}, \quad \{Z_t\} \sim IID(0,\sigma^2), \quad |\theta| < 1,$$

has a *non-invertible* MA(1) representation,

$$X_t = \varepsilon_t + \theta^{-1}\varepsilon_{t-1}, \quad \{\varepsilon_t\} \sim WN(0,\theta^2\sigma^2), \quad |\theta| < 1.$$

Question: Can the $\{\varepsilon_t\}$ also be IID?

Answer: Only if the Z_t are Gaussian.

If the Z_t are Gaussian, then there is an *identifiability* problem,

$$(\theta,\sigma^2) \leftrightarrow (\theta^{-1},\theta^2\sigma^2), \qquad |\theta| < 1,$$

give the same model.

3.4 Identifiability (cont)

For ARMA processes $\{X_t\}$ satisfying the recursions,

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \cdots \theta_q Z_{t-q}, \quad \{Z_t\} \sim IID(0, \sigma^2),$$
$$\phi(B) X_t = \theta(B) Z_t$$

casuality and invertibility are typically assumed, i.e.,

$$\phi(z) \neq 0 \text{ and } \theta(z) \neq 0 \text{ for} \,|\, \mathrm{z} \,|\leq 1.$$

By flipping roots of the AR and MA polynomials from outside the unit circle to inside the unit circle, there are approximately 2^{p+q} *equivalent ARMA representations* of X_t driven with noise that is white (not IID). For each of these *equivalent* representations, the noise is only IID in the Gaussian case.

Bottom line: For nonGaussian ARMA, there is a distinction between causal and noncausal; and invertible and non-invertible models.

3.4 Identifiability (cont)

Theorem (Cheng 1992): Suppose the linear time series

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}, \quad \{Z_t\} \sim IID(0, \sigma^2), \sum_j \psi_j^2 < \infty,$$

has a positive spectral density a.e. and can also be represented as

$$X_t = \sum_{j=-\infty}^{\infty} \eta_j Y_{t-j}, \quad \{Y_t\} \sim IID(0, \tau^2), \sum_j \eta_j^2 < \infty.$$

Then if $\{X_t\}$ is nonGaussian, it follows that

$$Y_t = cZ_{t-t_0}, \quad \eta_j = \frac{1}{c}\psi_{j+t_0},$$

for some positive constant c.

Proof of Theorem: As in the proof of the reversibility result, we can write

$$Z_t = \frac{1}{\psi(B)} X_t = \frac{\eta(B)}{\psi(B)} Y_t = \sum_{j=-\infty}^{\infty} a_j Y_{t-j} \text{ and } Y_t = \sum_{j=-\infty}^{\infty} b_j Z_{t-j}$$

3.4 Identifiability (cont)

Now let $\{Y(s,t)\}$ ~IID, $Y(s,t) =_d Y_1$ and set

$$U_t = \sum_{s=-\infty}^{\infty} a_s Y(s,t).$$

Clearly, $\{U_t\}$ is IID with same distribution as Z_1. Consequently,

$$Y_1 =_d \sum_{t=-\infty}^{\infty} b_t U_t = \sum_{t=-\infty}^{\infty}\sum_{s=-\infty}^{\infty} b_t a_s Y(s,t).$$

Since

$$\sum_{t=-\infty}^{\infty}\sum_{s=-\infty}^{\infty} b_t^2 a_s^2 = 1,$$

Which by applying Theorems 5.6.1 and 3.3.1 in Kagan, Linnik, and Rao (1973), the sum above is trivial, i.e., there exists integers m and n such that a_m and b_n are the only two nonzero coefficients. It follows that

$$Y_t = b_n Z_{t-n}, \quad \eta_j = \frac{1}{b_n}\psi_{j+n}.$$

3.5 Linear Tests

Cumulants and Polyspectra. We cannot base tests for linearity on second moments. A direct approach is to consider moments of higher order and corresponding generalizations of spectral analysis.

Suppose that $\{X_t\}$ satisfies $\sup_t E|X_t|^k < \infty$ for some $k \geq 3$ and

$$E(X_{t_0} X_{t_1} \cdots X_{t_j}) = E(X_{t_0+h} X_{t_1+h} \cdots X_{t_j+h})$$

for all $t_0, t_1, \ldots, t_j$, $h=0$, and $j=0, \ldots, k-1$.

k^{th} order cumulant. Coefficient, $C_k(r_1, \ldots, r_{k-1})$, of $i^k z_1 z_2 \ldots z_k$ in the Taylor series expansion about $(0,0,\ldots,0)$ of

$$\chi(z_1, \ldots, z_k) = \ln E \exp(iz_1 X_t + iz_2 X_{t+r_1} + \cdots + iz_k X_{t+r_{k-1}})$$

3.5 Linear Tests (cont)

3[rd] order cumulant.

$$C_3(r,s) = E\big((X_t - \mu)(X_{t+r} - \mu)(X_{t+s} - \mu)\big)$$

If

$$\sum_r \sum_s |C_3(r,s)| < \infty$$

then we define the *bispectral density* or (*3rd – order polyspectral density*) To be the Fourier transform,

$$f_3(\omega_1, \omega_2) = \frac{1}{(2\pi)^2} \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} C_3(r,s) e^{-ir\omega_1 - is\omega_2},$$

$-\pi \le \omega_1, \omega_2 \le \pi.$

3.5 Linear Tests (cont)

k^{th} - order polyspectral density.

Provided

$$\sum_{r_1}\sum_{r_2}\cdots\sum_{r_{k-1}}|C_k(r_1,\ldots,r_{k-1})|<\infty,$$

$$f_k(\omega_1,\ldots,\omega_{k-1}):=$$

$$\frac{1}{(2\pi)^{k-1}}\sum_{r_1=-\infty}^{\infty}\sum_{r_2=-\infty}^{\infty}\cdots\sum_{r_{k-1}=-\infty}^{\infty}C_k(r_1,\ldots,r_{k-1})e^{-ir_1\omega_1-\cdots-ir_{k-1}\omega_{k-1}},$$

$-\pi \le \omega_1, \ldots, \omega_{k-1} \le \pi$. (See Rosenblatt (1985) *Stationary Sequences and Random Fields* for more details.)

3.5 Linear Tests (cont)

Applied to a linear process. If $\{X_t\}$ has the Wold decomposition

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}, \quad \{Z_t\} \sim IID(0, \sigma^2),$$

with $E|Z_t|^3 < \infty$, $EZ_t^3 = \eta$, and $\Sigma_j |\psi_j| < \infty$, then

$$C_3(r,s) = \eta \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+r} \psi_{j+s}$$

where $\psi_j := 0$ for $j < 0$. Hence

$$f_3(\omega_1, \omega_2) = \frac{\eta}{4\pi^2} \psi(e^{i\omega_1 + i\omega_2}) \psi(e^{-i\omega_1}) \psi(e^{-i\omega_2}).$$

3.5 Linear Tests (cont)

The spectral density of $\{X_t\}$ is

$$f(\omega) = \frac{\sigma^2}{2\pi} |\psi(e^{i\omega})|^2 .$$

Hence, defining

$$\phi(\omega_1, \omega_2) = \frac{|f_3(\omega_1, \omega_2)|^2}{f(\omega_1) f(\omega_2) f(\omega_1 + \omega_2)},$$

we find that

$$\phi(\omega_1, \omega_2) = \frac{\eta^2}{2\pi\sigma^6}.$$

Testing for constancy of $\phi(\cdot)$ thus provides a test for linearity of $\{X_t\}$ (see Subba Rao and Gabr (1980)).

3.5 Linear Tests (cont)

Gaussian linear process. If $\{X_t\}$ is Gaussian, then $EZ^3=0$, and the third order cumulant is zero (why?). In fact $C_k \equiv 0$ for all $k > 2$.

It follows that $f_3(\omega_1, \omega_2) \equiv 0$ for all $\omega_1, \omega_2 \in [0,\pi]$. A test for linear Gaussianity can therefore be obtained by estimating $f_3(\omega_1,\omega_2)$ and testing the hypothesis that $f_3 \equiv 0$ (see Subba Rao and Gabr (1980)).

3.6 Prediction

Suppose $\{X_t\}$ is a purely nondeterministic process with WD given by

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}, \quad \{Z_t\} \sim WN(0, \sigma^2).$$

Then

$$Z_t = X_t - \tilde{P}_{t-1}(X_t)$$

so that

$$\tilde{P}_{t-1} X_t = \sum_{j=1}^{\infty} \psi_j Z_{t-j} \quad .$$

Question. When does the *best linear predictor* equal the *best predictor*?

That is, when does

$$\tilde{P}_{t-1} X_t = E(X_t \mid X_{t-1}, X_{t-2} \ldots) ?$$

3.6 Prediction (cont)

$$\tilde{P}_{t-1}X_t = E(X_t \mid X_{t-1}, X_{t-2} \ldots)\,?$$

Answer. Need

$$Z_t = X_t - \tilde{P}_{t-1}X_t \perp \sigma(X_{t-1}, X_{t-2}, \ldots)$$

or, equivalently,

$$E(Z_t \mid X_{t-1}, X_{t-2}, \ldots) = 0.$$

That is,

BLP = BP

if and only if $\{Z_t\}$ is a *Martingale-difference sequence*.

Def. $\{Z_t\}$ is a *Martingale-difference sequence wrt a filtration* F_t (*an increasing sequence of sigma fields*) if $E|Z_t| < \infty$ for all t and

a) Z_t *is* F_t measurable

b) $E(Z_t \mid F_{t-1})=0$ a.s.

Remarks.

1) An IID sequence with mean zero is a MG difference sequence.
2) A purely nondeterministic Gaussian process is a Gaussian linear process. This follows by the Wold decomposition and the fact that the resulting $\{Z_t\}$ sequence must be IID $N(0,\sigma^2)$.

Example (Whittle): Consider the noncausal AR(1) process given by

$$X_t = 2\, X_{t-1} + Z_t\,,$$

where $\{Z_t\}$~IID $P(Z_t = -1) = P(Z_t = 0)=.5$. Iterating backwards in time, we find that

$$\begin{aligned} X_{t-1} &= .5X_t - .5Z_t \\ &= .5^2 X_{t+1} - .5^2 Z_{t+1} - .5Z_t \\ &\vdots \\ &= .5(-Z_t - .5Z_{t+1} - .5^2 Z_{t+2} - \cdots). \end{aligned}$$

3.6 Prediction (cont)

$$X_t = .5(-Z_{t+1} - .5Z_{t+2} - .5^2 Z_{t+3} - \cdots)$$
$$= \frac{Z^*_{t+1}}{2} + \frac{Z^*_{t+2}}{2^2} + \frac{Z^*_{t+3}}{2^3} + \cdots, \qquad Z^*_{t+1} = -Z_{t+1}$$

is a binary expansion of a uniform (0,1) random variable. Notice that from X_t, we can find X_{t+1}, by lopping off the first term in the binary expansion. This operation is exactly,

$$X_{t+1} = 2\, X_t \text{ mod } 1$$
$$= \begin{cases} 2X_t, & \text{if } X_t < .5, \\ 2X_t - 1, & \text{if } X_t > .5. \end{cases}$$

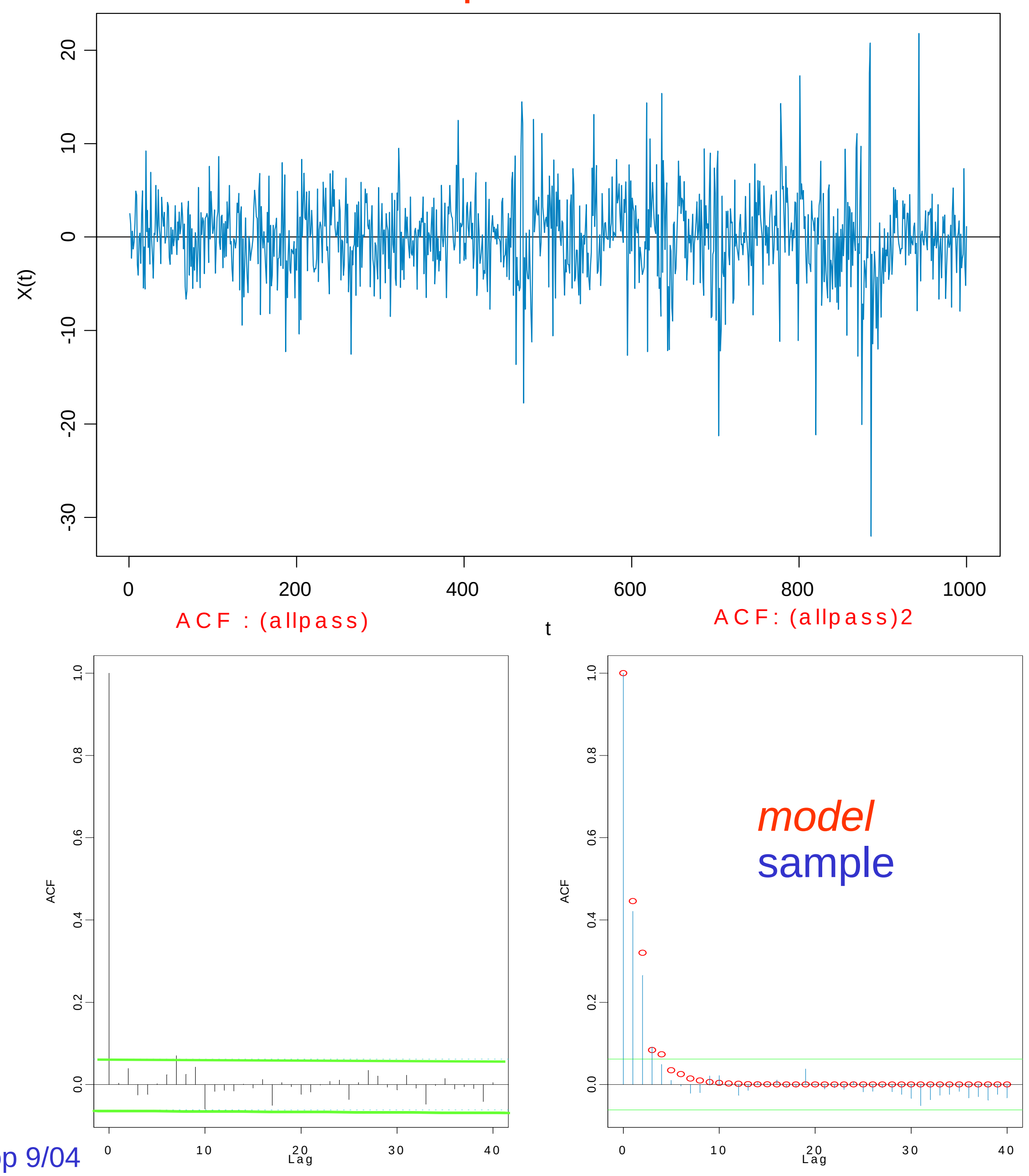
Properties:

1. $EX_t = \frac{1}{2}$.
2. $\rho_X(h) = (0.5)^{|h|}$.
3. $P(X_t|1, X_s, s < t) = \frac{1}{4} + \frac{1}{2}X_{t-1}$.
4. $E(X_t|X_s, s < t) = 2X_{t-1}(\text{mod } 1) = X_t$.
5. $X_t - \frac{1}{2} = \frac{1}{2}(X_{t-1} - \frac{1}{2}) + \epsilon_t, \quad \{\epsilon_t\} \sim \text{WN}(0, \sigma^2)$.

4. Allpass models

Realization from an all-pass model of order 2

(t3 noise)



4. Allpass models (cont)

Causal AR polynomial: $\phi(z)=1-\phi_1 z - \ldots - \phi_p z^p$, $\phi(z) \neq 0$ for $|z|\leq 1$.

Define MA polynomial:

$$\theta(z) = -z^p \phi(z^{-1})/\phi_p = -(z^p - \phi_1 z^{p-1} - \ldots - \phi_p)/\phi_p$$

$\neq 0$ for $|z|\geq 1$ (MA polynomial is non-invertible).

Model for data $\{X_t\}$: $\phi(B)X_t = \theta(B) Z_t$, $\{Z_t\} \sim$ IID (non-Gaussian)

$$B^k X_t = X_{t-k}$$

Examples:

All-pass(1): $X_t - \phi X_{t-1} = Z_t - \phi^{-1} Z_{t-1}$, $|\phi| < 1$.

All-pass(2): $X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = Z_t + \phi_1/\phi_2 Z_{t-1} - 1/\phi_2 Z_{t-2}$

Properties:

- causal, non-invertible ARMA with MA representation

$$X_t = \frac{B^p \phi(B^{-1})}{-\phi_p \phi(B)} Z_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$$

- uncorrelated (flat spectrum)

$$f_X(\omega) = \frac{\left|e^{-ip\omega}\right|^2 \left|\phi(e^{i\omega})\right|^2}{\phi_p^2 \left|\phi(e^{-i\omega})\right|^2} \frac{\sigma^2}{2\pi} = \frac{\sigma^2}{\phi_p^2 2\pi}$$

- zero mean
- data are dependent if noise is non-Gaussian

  (e.g. Breidt & Davis 1991).
- squares and absolute values are correlated.
- X_t is heavy-tailed if noise is heavy-tailed.

Estimation for All-Pass Models

☞ Second-order moment techniques do not work

- least squares
- Gaussian likelihood

☞ Higher-order cumulant methods

- Giannakis and Swami (1990)
- Chi and Kung (1995)

☞ Non-Gaussian likelihood methods

- likelihood approximation assuming known density
- quasi-likelihood

☞ Other

- LAD- least absolute deviation
- R-estimation (minimum dispersion)

4.1 Application of Allpass models

Noninvertible MA models with heavy tailed noise

$$X_t = Z_t + \theta_1 Z_{t-1} + \cdot\ \cdot\ \cdot + \theta_q Z_{t-q}\,,$$

a. $\{Z_t\}$ ~ IID nonnormal

b. $\theta(z) = 1 + \theta_1 z + \cdot\ \cdot\ \cdot + \theta_q z^q$

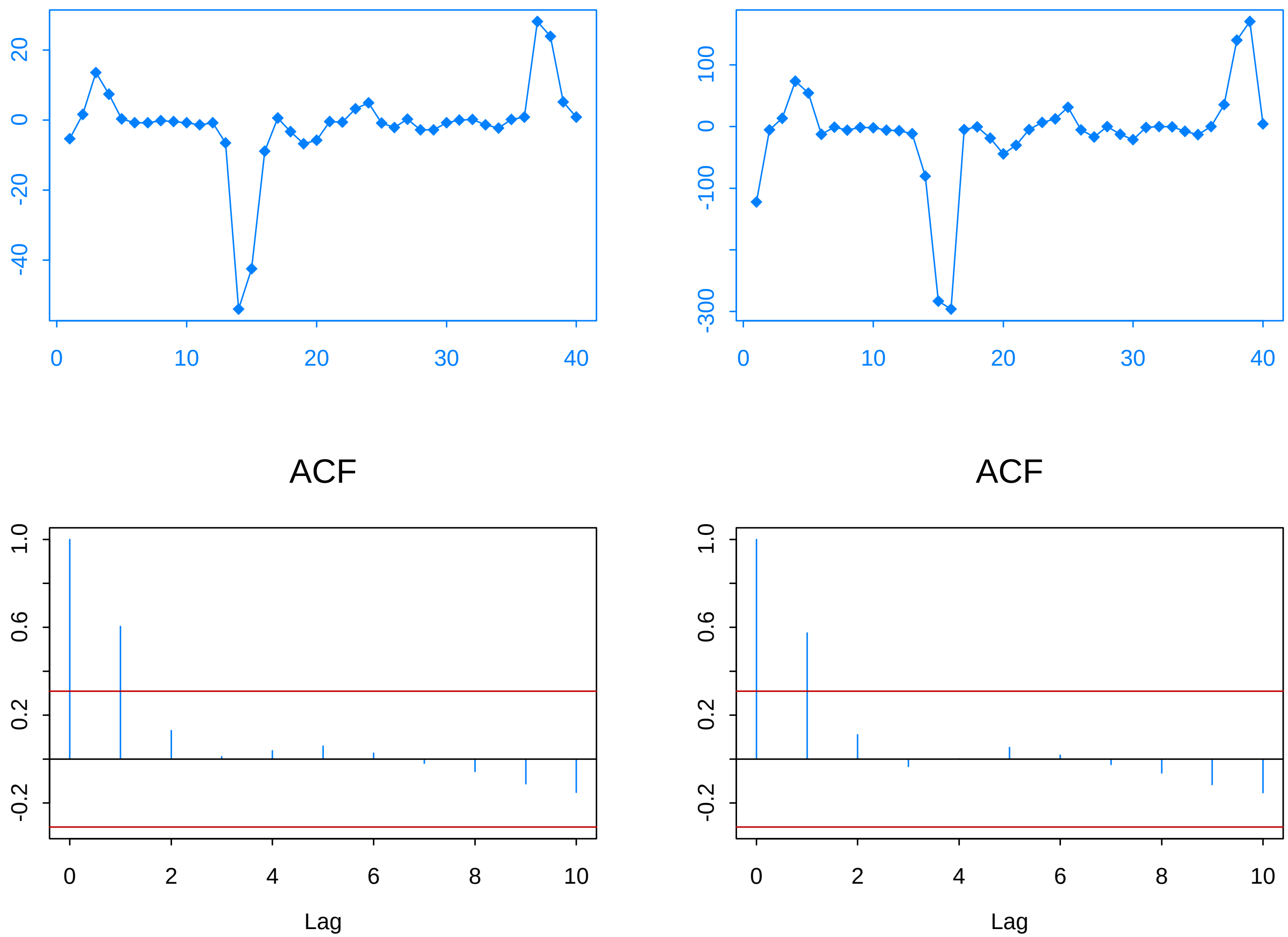
No zeros inside the unit circle $\Rightarrow$ *invertible*

Some zero(s) inside the unit circle $\Rightarrow$ *noninvertible*

Realizations of an invertible and noninvertible MA(2) processes

Model: $X_t = \theta_*(B)\, Z_t$, $\{Z_t\} \sim \text{IID}(\alpha = 1)$, where

$\theta_i(B) = (1 + 1/2B)(1 + 1/3B)$ and $\theta_{ni}(B) = (1 + 2B)(1 + 3B)$



Application of all-pass to noninvertible MA model fitting

Suppose $\{X_t\}$ follows the noninvertible MA model

$$X_t = \theta_i(B)\, \theta_{ni}(B)\, Z_t\,, \quad \{Z_t\} \sim \text{IID}.$$

Step 1: Let $\{U_t\}$ be the residuals obtained by fitting a purely invertible MA model, i.e.,

$$\begin{aligned} X_t &= \hat{\theta}(B)U_t \\ &\approx \theta_i(B)\tilde{\theta}_{ni}(B)U_t, \quad (\tilde{\theta}_{ni} \text{ is the invertible version of } \theta_{ni}). \end{aligned}$$

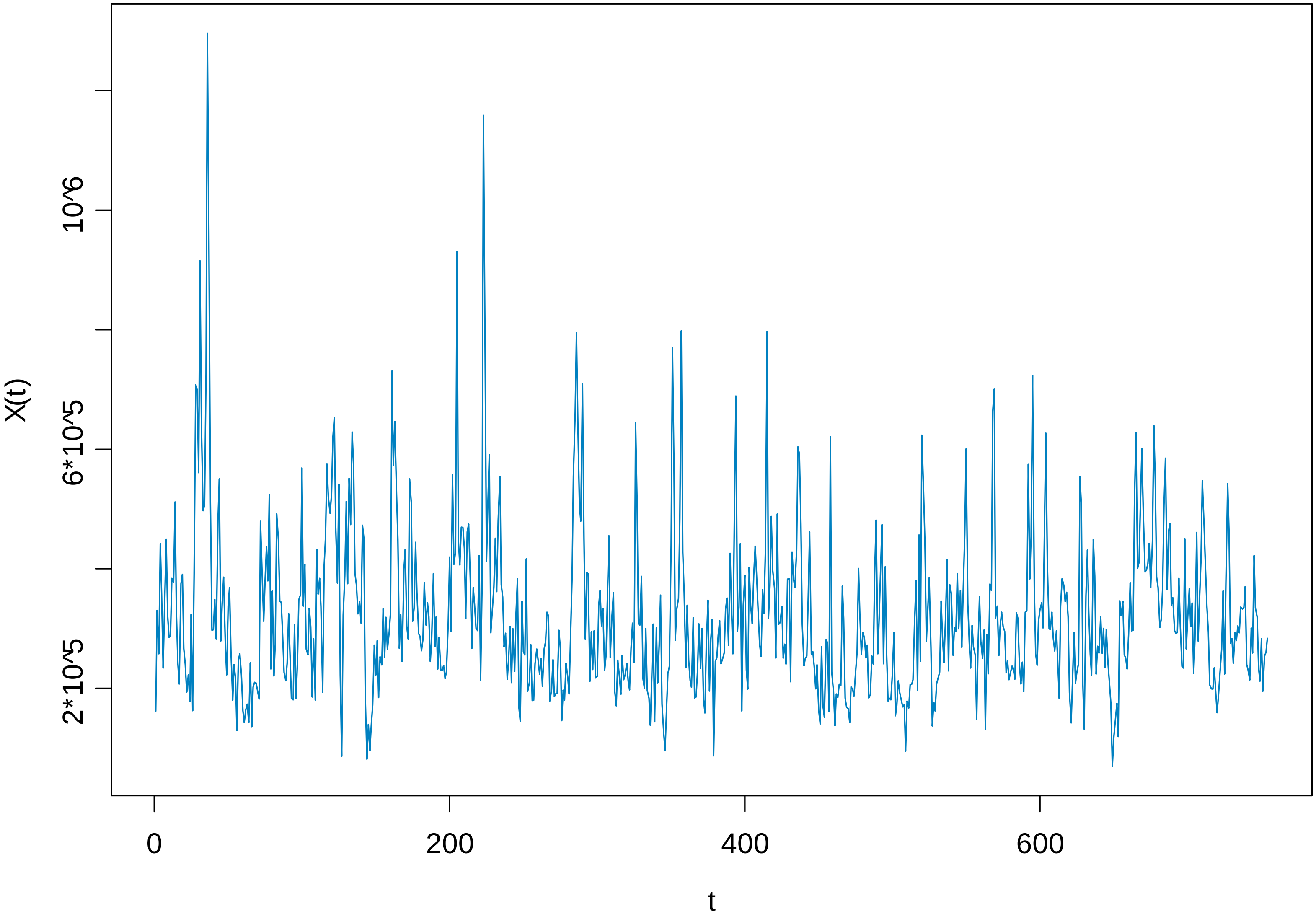
So

$$U_t \approx \frac{\theta_{ni}(B)}{\tilde{\theta}_{ni}(B)} Z_t$$

Step 2: Fit a purely causal AP model to $\{U_t\}$

$$\tilde{\theta}_{ni}(B)U_t = \theta_{ni}(B)Z_t.$$

Volumes of Microsoft (MSFT) stock traded over 755 transaction days (6/3/96 to 5/28/99)



Analysis of MSFT:

Step 1: Log(volume) follows MA(4).

$X_t = (1+.513B+.277B^2+.270B^3+.202B^4)\ U_t$ (invertible MA(4))

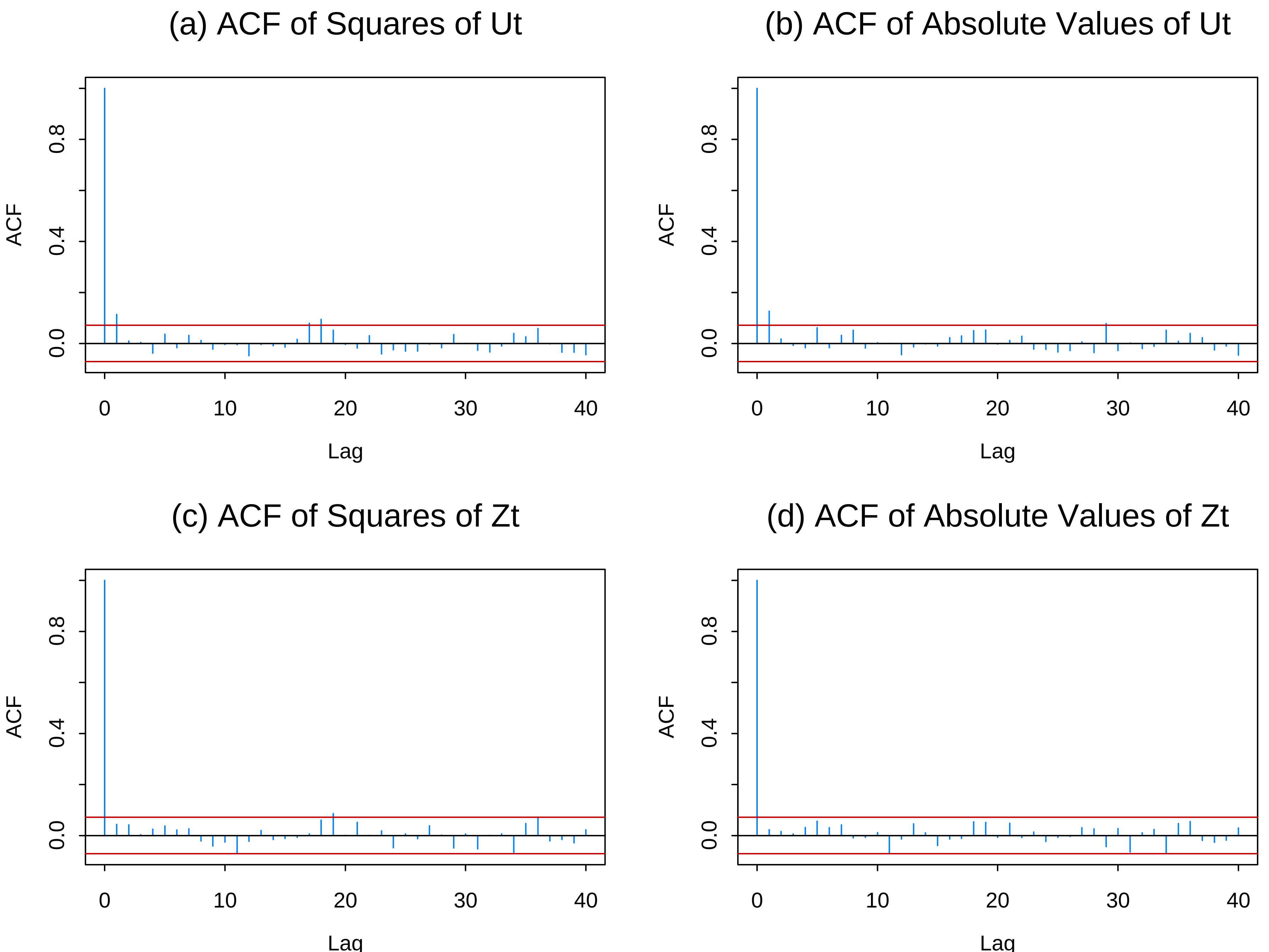
Step 2: All-pass model of order 4 fitted to $\{U_t\}$ using MLE (t-dist):

$$(1-.628B+-.229B^2+.131B^3-.202B^4)U_t$$
$$=(1-.649B+1.135B^2+3.116B^3-4.960B^4)Z_t.\ (\hat{\nu}=6.26)$$

(Model using R-estimation is nearly the same.)

Conclude that $\{X_t\}$ follows a noninvertible MA(4) which after refitting has the form:

$X_t = (1+1.34B+1.374B^2+2.54B^3+4.96B^4)\ Z_t$, $\{Z_t\}$~IID t(6.3)



(a) ACF of Squares of Ut
(b) ACF of Absolute Values of Ut
(c) ACF of Squares of Zt
(d) ACF of Absolute Values of Zt
ACF
0.0
0.4
0.8
Lag
0
10
20
30
40

Summary: Microsoft Trading Volume

☞ Two-step fit of noninvertible MA(4):

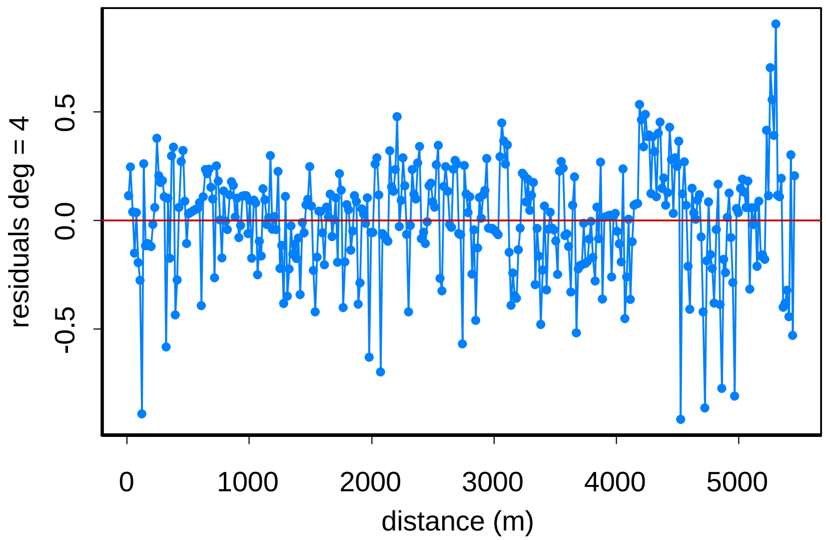
- invertible MA(4): residuals not iid
- causal AP(4); residuals iid

☞ Direct fit of purely noninvertible MA(4):

$$(1+1.34B+1.374B^2+2.54B^3+4.96B^4)$$

☞ For MCHP, invertible MA(4) fits.

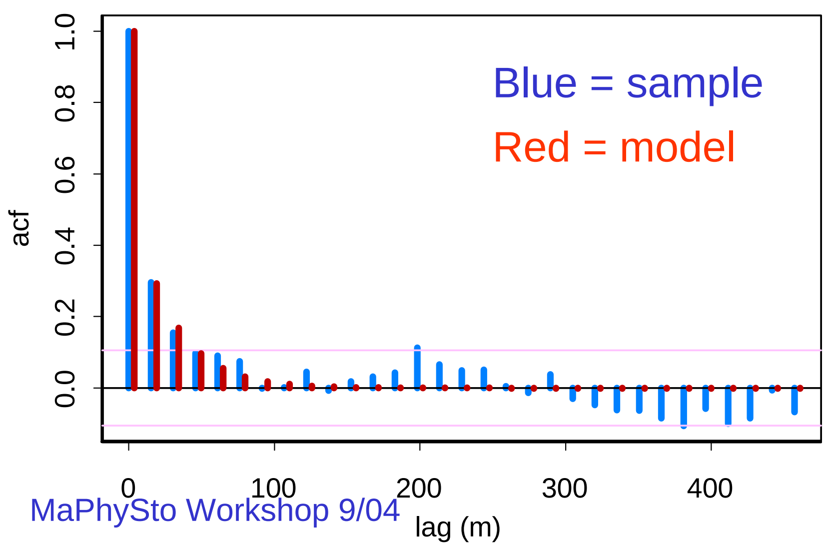
Muddy Creek: residuals from poly(d=4) fit

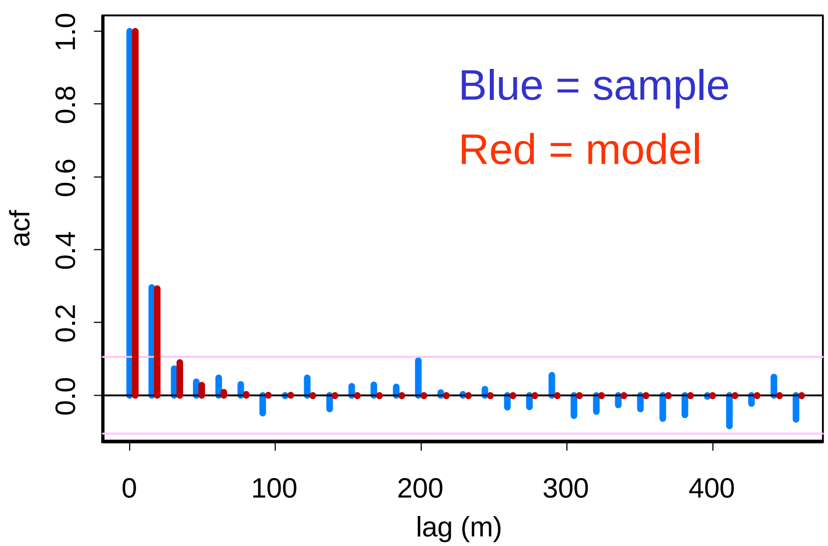


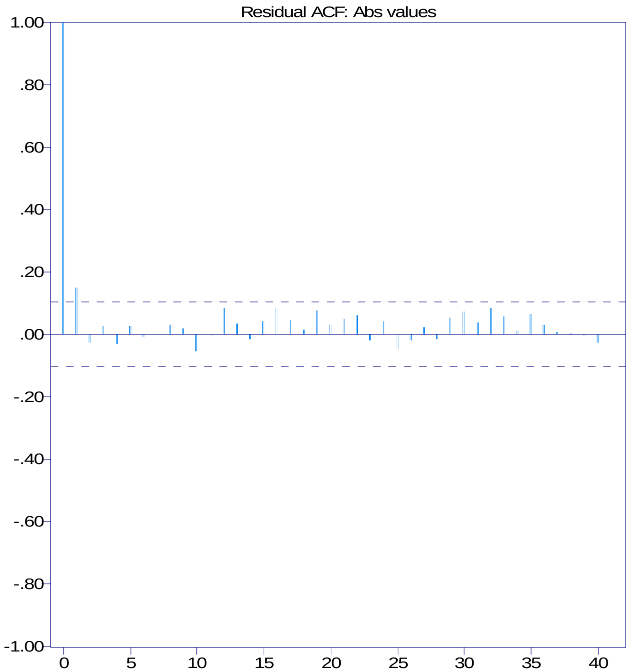
Minimum AIC_c ARMA model:

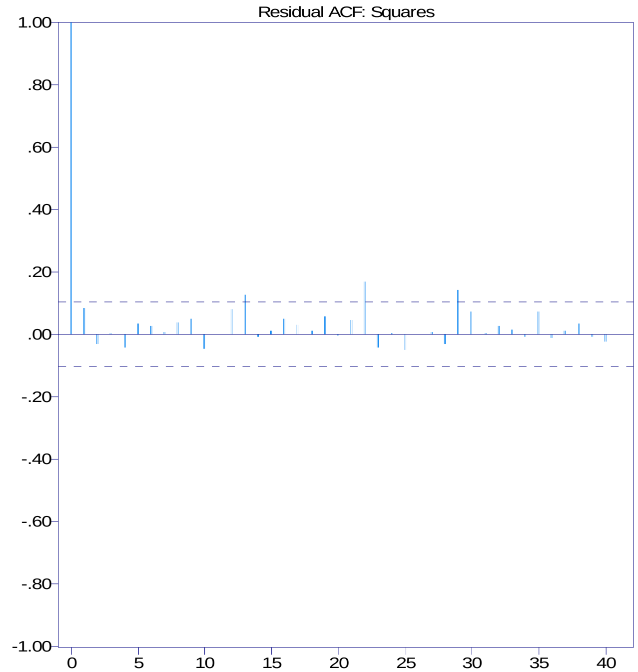
ARMA(1,1)

$Y_t = .574\ Y_{t-1} + \varepsilon_t - .311\ \varepsilon_{t-1}$, $\{\varepsilon_t\}$~WN(0,.0564)







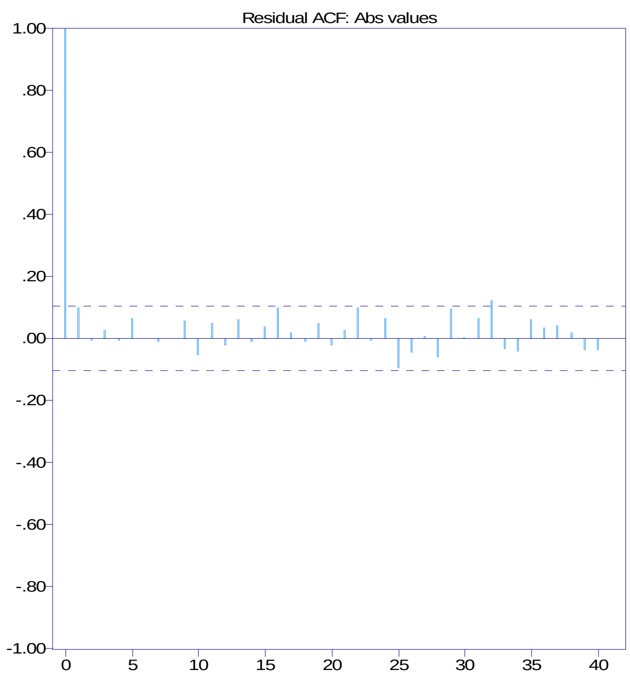


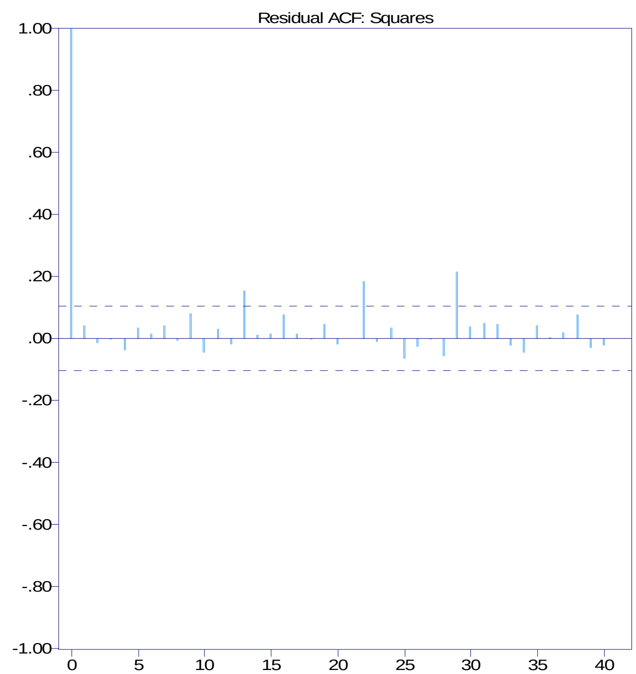
Causal ARMA(1,1) model

$Y_t = .574\ Y_{t\text{-}1}$

$+\ \varepsilon_t - .311\ \varepsilon_{t\text{-}1},$

$\{\varepsilon_t\}$~WN(0,.0564)





Noncausal ARMA(1,1) model:

$Y_t = 1.743\ Y_{t\text{-}1}$

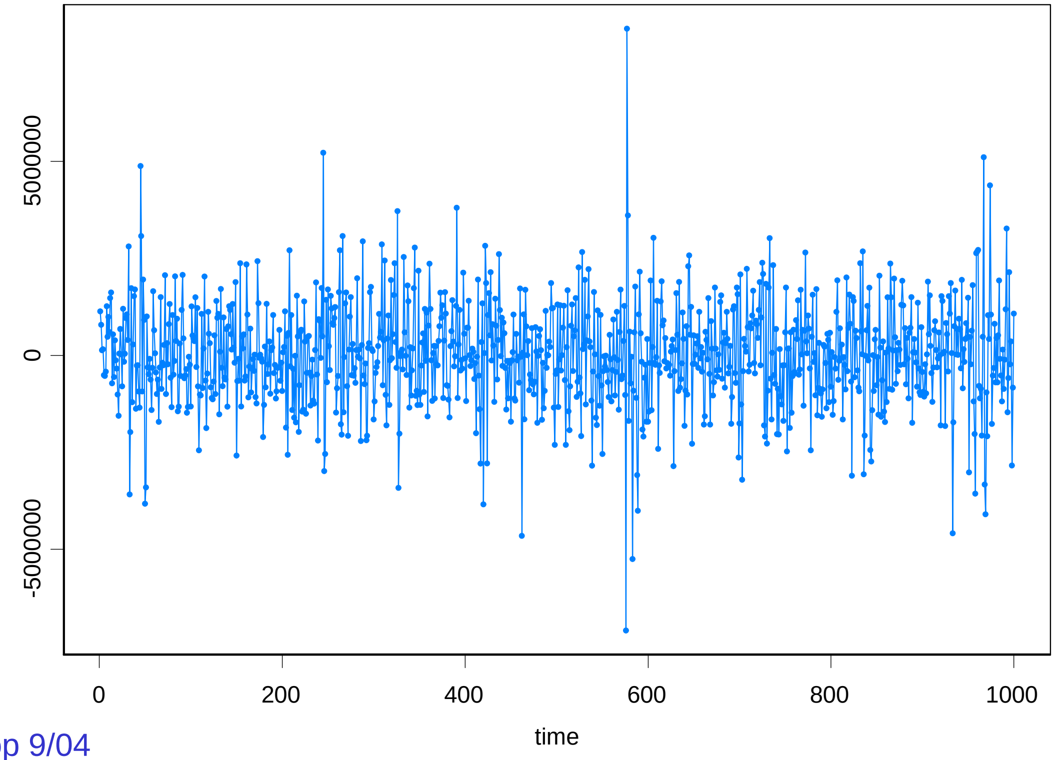
$+\ \varepsilon_t - .311\ \varepsilon_{t\text{-}1}$

Example: Seismogram Deconvolution

Simulated water gun seismogram

$$X_t = \sum_k \beta_k Z_{t-k}$$

- $\{\beta_k\}$ = wavelet sequence (Lii and Rosenblatt, 1988)
- $\{Z_t\}$ IID reflectivity sequence



Water Gun Seismogram Fit

Step 1: AICC suggests ARMA (12,13) fit

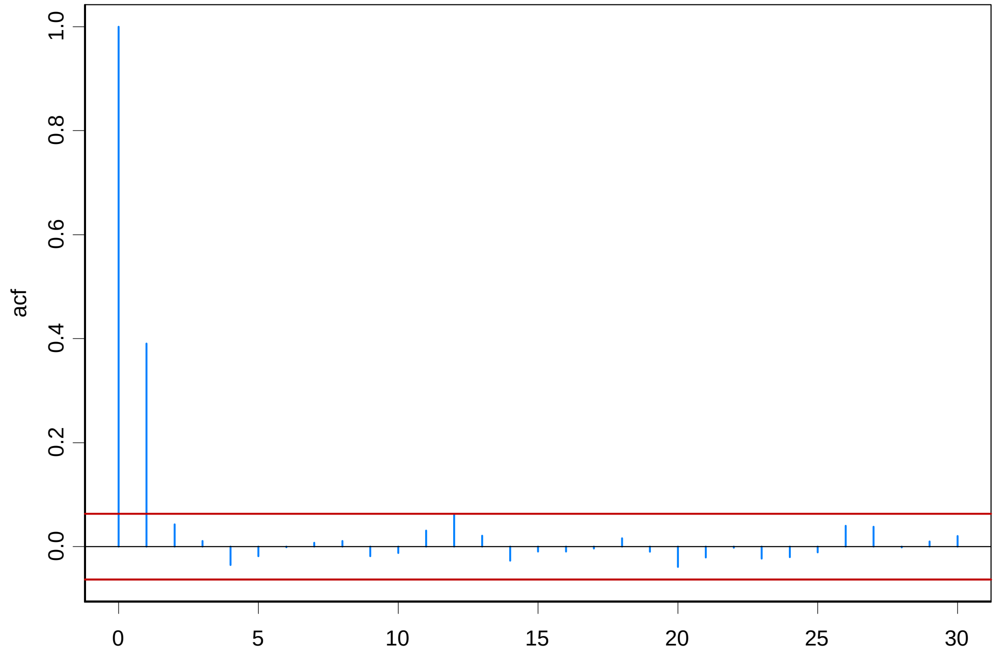
- fit invertible ARMA(12,13) via *Gaussian MLE*
- residuals $\{\hat{W}_t\}$ not IID

Step 2: fit all-pass to $\{\hat{W}_t\}$ residuals

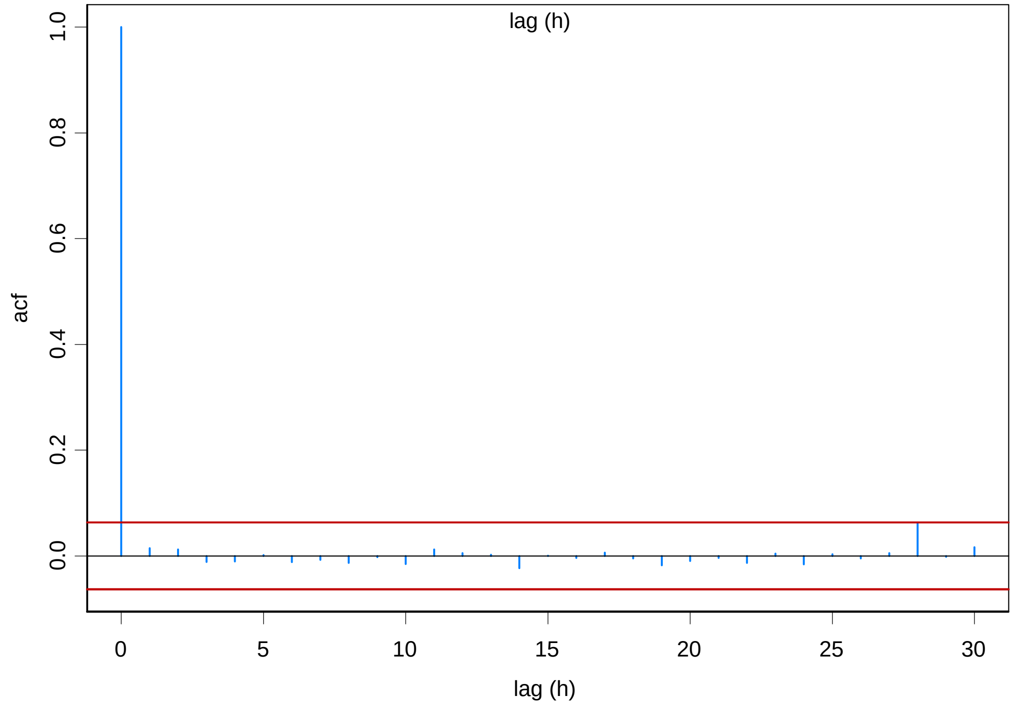
- order selected is $r = 2$.
- residuals $\{\hat{Z}_t\}$ appear IID

Step 3: Conclude that $\{X_t\}$ follows a non-invertible ARMA

ACF of W_t^2

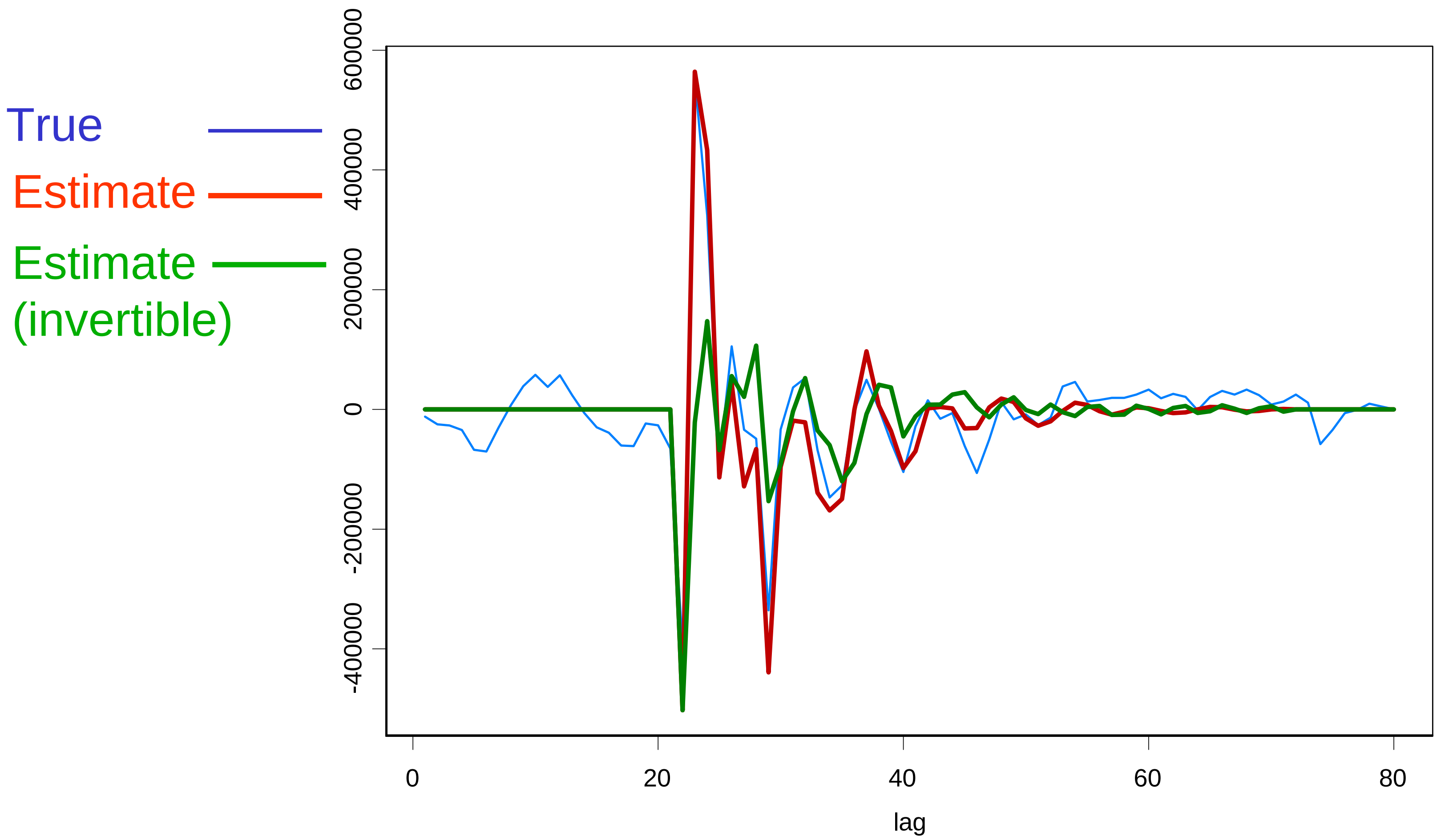


ACF of Z_t^2



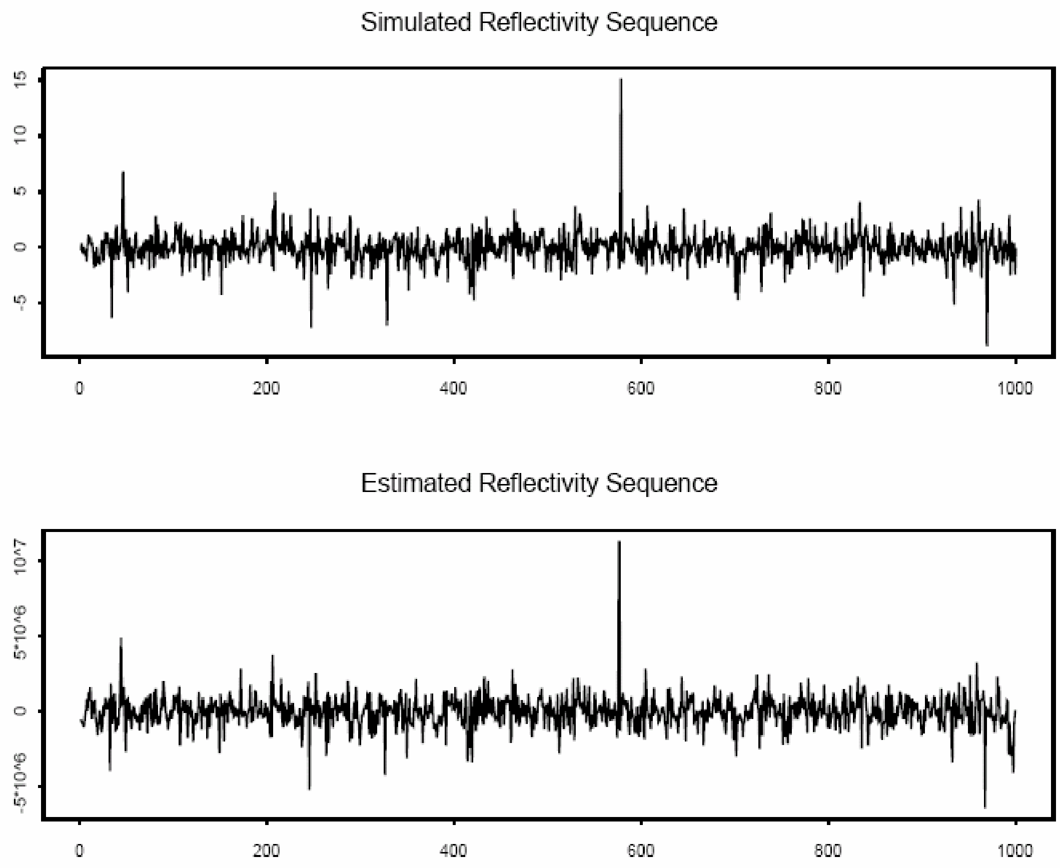
Water Gun Seismogram Fit (cont)

Recorded water gun wavelet and its estimate



Water Gun Seismogram Fit (cont)

Simulated reflectivity sequence and its estimates



4.2 Estimation for Allpass Models: approximating the likelihood

Data: $(X_1, \ldots, X_n)$

Model:
$$X_t = \phi_{01} X_{t-1} + \cdots + \phi_{0p} X_{t-p} \\ -(Z_{t-p} - \phi_{01} Z_{t-p+1} - \cdots - \phi_{0p} Z_t) / \phi_{0r}$$

where ϕ_{0r} is the last non-zero coefficient among the ϕ_{0j}'s.

Noise: $z_{t-p} = \phi_{01} z_{t-p+1} + \cdots + \phi_{0p} z_t - (X_t - \phi_{01} X_{t-1} - \cdots - \phi_{0p} X_{t-p}),$

where $z_t = Z_t / \phi_{0r}.$

More generally define,

$$z_{t-p}(\phi) = \begin{cases} 0, & \text{if } t = n+p, \ldots, n+1, \\ \phi_1 z_{t-p+1}(\phi) + \cdots + \phi_p z_t(\phi) - \phi(B) X_t, & \text{if } t = n, \ldots, p+1. \end{cases}$$

Note: $z_t(\phi_0)$ is a close approximation to z_t (initialization error)

Assume that Z_t has density function f_σ and consider the vector

$$\mathbf{z} = (\underbrace{X_{1-p},\ldots,X_0,z_{1-p}(\phi),\ldots,z_0(\phi)}, \underbrace{z_1(\phi),\ldots,z_{n-p+1}(\phi),\ldots,z_n(\phi)})'$$

↖ independent pieces ↗

Joint density of **z**:

$$h(\mathbf{z}) = h_1(X_{1-p},\ldots,X_0,z_{1-p}(\phi),\ldots,z_0(\phi))$$
$$\bullet\left(\prod_{t=1}^{n-p} f_\sigma(\phi_q z_t(\phi))\,|\,\phi_q\,|\right) h_2(z_{n-p+1}(\phi),\ldots,z_n(\phi)),$$

and hence the joint density of the data can be approximated by

$$h(\mathbf{x}) = \left(\prod_{t=1}^{n-p} f_\sigma(\phi_q z_t(\phi))\,|\,\phi_q\,|\right)$$

where $q=\max\{0 \le j \le p: \phi_j \ne 0\}$.

Log-likelihood:

$$L(\phi,\sigma) = -(n-p)\ln(\sigma/|\phi_q|) + \sum_{t=1}^{n-p} \ln f(\sigma^{-1}\phi_q z_t(\phi))$$

where $f_\sigma(z) = \sigma^{-1} f(z/\sigma)$.

Least absolute deviations: choose Laplace density

$$f(z) = \frac{1}{\sqrt{2}} \exp(-\sqrt{2}\,|z|)$$

and log-likelihood becomes

$$\text{constant} - (n-p)\ln\kappa - \sum_{t=1}^{n-p} \sqrt{2}\,|z_t(\phi)|/\kappa, \quad \kappa = \sigma/|\phi_q|$$

Concentrated Laplacian likelihood

$$l(\phi) = \text{constant} - (n-p)\ln \sum_{t=1}^{n-p} |z_t(\phi)|$$

Maximizing $l(\phi)$ is equivalent to minimizing the absolute deviations

$$m_{\text{n}}(\phi) = \sum_{t=1}^{n-p} |z_t(\phi)|.$$

Assumptions for MLE

☞ Assume $\{Z_t\}$ iid $f_\sigma(z)=\sigma^{-1}f(\sigma^{-1}z)$ with

- σ a scale parameter
- mean 0, variance σ^2
- further smoothness assumptions (integrability, symmetry, etc.) on f
- Fisher information:

$$\tilde{I} = \sigma^{-2}\int (f'(z))^2 / f(z)dz$$

Results

☞ Let $\gamma(h)$ = ACVF of AR model with AR poly $\phi_0(.)$ and

$$\Gamma_p = [\gamma(\mathrm{j-k})]_{\mathrm{j,k}=1}^{\mathrm{p}}$$

☞ $$\sqrt{n}(\hat{\phi}_{\mathrm{MLE}} - \phi_0) \xrightarrow{D} N(0, \frac{1}{2(\sigma^2\tilde{I} - 1)}\sigma^2\Gamma_p^{-1})$$

Further comments on MLE

Let $\alpha=(\phi_1, \ldots, \phi_p, \sigma/|\phi_p|, \beta_1, \ldots, \beta_q)$, where $\beta_1, \ldots, \beta_q$ are the parameters of pdf *f.*

Set

- ☞ $\hat{\mathrm{I}} = \sigma_0^{-2} \int (f'(z;\beta_0))^2 / f(z;\beta_0) dz$
- ☞ $\hat{\mathrm{K}} = \alpha_{0,p+1}^{-2} \left\{ \int z^2 (f'(z;\beta_0))^2 / f(z;\beta_0) dz - 1 \right\}$
- ☞ $\mathrm{L} = -\alpha_{0,p+1}^{-1} \int z \dfrac{f'(z;\beta_0)}{f(z;\beta_0)} \dfrac{\partial f(z;\beta_0)}{\partial \beta_0} dz$
- ☞ $\mathrm{I}_f(\beta_0) = \int \dfrac{1}{f(z;\beta_0)} \dfrac{\partial f(z;\beta_0)}{\partial \beta_0} \dfrac{\partial f^T(z;\beta_0)}{\partial \beta_0} dz$ (Fisher Information)

Under smoothness conditions on *f* wrt $\beta_1, \ldots, \beta_q$ we have

$$\sqrt{n}(\hat{\alpha}_{\mathrm{MLE}} - \alpha_0) \xrightarrow{D} N(0, \Sigma^{-1}),$$

where

$$\Sigma^{-1} = \begin{bmatrix} \frac{1}{2(\sigma_0^2 \hat{I} - 1)} \sigma^2 \Gamma_p^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (\hat{K} - L' I_f^{-1} L)^{-1} & -\hat{K}^{-1} L' (I_f - L\hat{K}^{-1} L')^{-1} \\ \mathbf{0} & -(I_f - L\hat{K}^{-1} L')^{-1} L\hat{K}^{-1} & (I_f - L\hat{K}^{-1} L')^{-1} \end{bmatrix}$$

Note: $\hat{\phi}_{\mathrm{MLE}}$ is asymptotically independent of $\hat{\alpha}_{p+1,\mathrm{MLE}}$ and $\hat{\beta}_{\mathrm{MLE}}$

Asymptotic Covariance Matrix

- For LS estimators of AR(p):

$$\sqrt{n}(\hat{\phi}_{\mathrm{LS}} - \phi_0) \xrightarrow{D} N(0, \sigma^2 \Gamma_p^{-1})$$

- For LAD estimators of AR(p):

$$\sqrt{n}(\hat{\phi}_{\mathrm{LAD}} - \phi_0) \xrightarrow{D} N(0, \frac{1}{4\sigma^2 f^2(0)} \sigma^2 \Gamma_p^{-1})$$

- For LAD estimators of AP(p):

$$\sqrt{n}(\hat{\phi}_{\mathrm{LAD}} - \phi_0) \xrightarrow{D} N(0, \frac{\mathrm{Var}(|Z_1|)}{2(2\sigma^2 f_\sigma(0) - E|Z_1|)^2} \sigma^2 \Gamma_p^{-1})$$

- For MLE estimators of AP(p):

$$\sqrt{n}(\hat{\phi}_{\mathrm{MLE}} - \phi_0) \xrightarrow{D} N(0, \frac{1}{2(\sigma^2 \hat{I} - 1)} \sigma^2 \Gamma_p^{-1})$$

Laplace: (LAD=MLE)

$$\frac{\mathrm{Var}(|Z_1|)}{2(2\sigma^2 f_\sigma(0) - E|Z_1|)^2} = \frac{1}{2} = \frac{1}{2(\sigma^2 \hat{I} - 1)}$$

Students t_ν, $\nu > 2$:

LAD: $\frac{(\nu-2)}{8\Gamma^2((\nu+1)/2)}(\pi(\nu-1)^2\Gamma^2(\nu/2) - 4(\nu-2)\Gamma^2((\nu+1)/2))$

MLE: $\frac{1}{2(\sigma^2\hat{I}-1)} = \frac{(\nu-2)(\nu+3)}{12}$

Student's t_3:

LAD: .7337

MLE: 0.5

ARE: .7337/.5=1.4674

R-Estimation:

Minimize the objective function

$$S(\phi) = \sum_{t=1}^{n-p} \varphi\left(\frac{t}{n-p+1}\right) z_{(t)}(\phi)$$

where $\{z_{(t)}(\phi)\}$ are the ordered $\{z_t(\phi)\}$, and the weight function φ satisfies:

- φ is differentiable and nondecreasing on (0,1)
- φ' is uniformly continuous
- $\varphi(x) = -\varphi(1-x)$

Remarks:

- $S(\phi) = \sum_{t=1}^{n-p} \varphi\left(\frac{R_t(\phi)}{n-p+1}\right) z_t(\phi)$
- For LAD, take $\varphi(x) = \begin{cases} -1, & 0 < x < 1/2, \\ 1, & 1/2 < x < 1. \end{cases}$

Assumptions for R-estimation

☞ Assume $\{Z_t\}$ iid with density function f (distr F)

- mean 0, variance σ^2

☞ Assume weight function φ is nondecreasing and continuously differentiable with $\varphi(x) = -\varphi(1-x)$

Results

☞ Set

$$\tilde{J} = \int_0^1 \varphi^2(s)ds, \quad \tilde{K} = \int_0^1 F^{-1}(s)\varphi(s)ds, \quad \tilde{L} = \int_0^1 f(F^{-1}(s))\varphi'(s)ds$$

☞ If $\sigma^2\tilde{L} > \tilde{K}$, then

$$\sqrt{n}(\hat{\phi}_R - \phi_0) \xrightarrow{D} N(0, \frac{\sigma^2\tilde{J} - \tilde{K}^2}{2(\sigma^2\tilde{L} - \tilde{K})^2}\sigma^2\Gamma_p^{-1})$$

<u>Further comments on R-estimation</u>

☞ $\varphi(x) = x - 1/2$ is called the Wilcoxon weight function

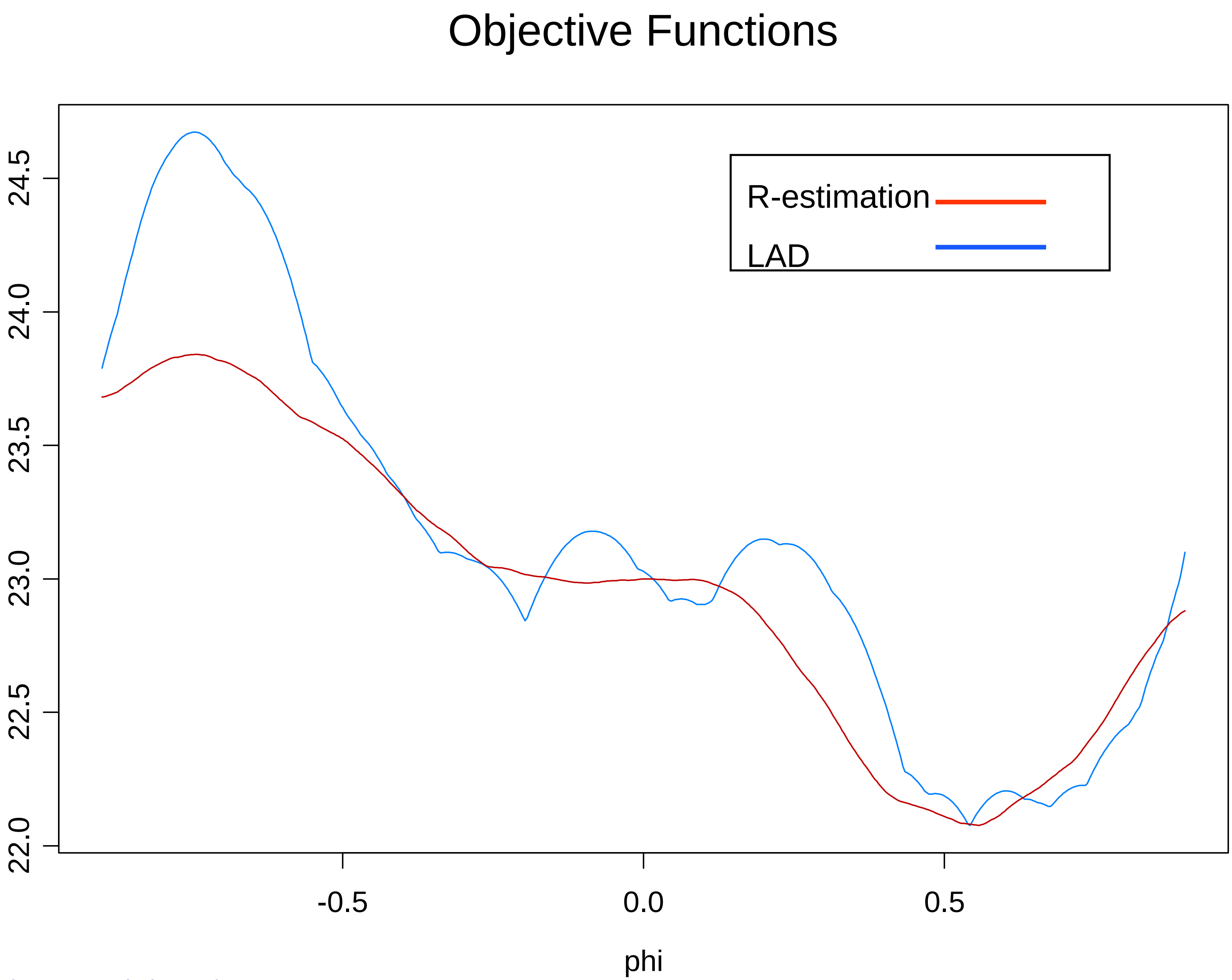
☞ By formally choosing $\varphi(x) = \begin{cases} -1, & 0 < x < 1/2, \\ 1, & 1/2 < x < 1. \end{cases}$

we obtain

$$\frac{\sigma^2 \tilde{J} - \tilde{K}^2}{2(\sigma^2 \tilde{L} - \tilde{K})^2} \sigma^2 \Gamma_p^{-1} = \frac{\mathrm{Var}(|Z_1|)}{2(2\sigma^2 f_\sigma(0) - E|Z_1|)^2} \sigma^2 \Gamma_p^{-1}.$$

That is R = LAD, asymptotically.

☞ The R-estimation objective function is smoother than the LAD-objective function and hence easier to minimize.


Objective Functions
R-estimation
LAD
24.5
24.0
23.5
23.0
22.5
22.0
-0.5
0.0
0.5
phi

Summary of asymptotics

☞ Maximum likelihood:

$$\sqrt{n}(\hat{\phi}_{\mathrm{MLE}} - \phi_0) \xrightarrow{D} N(0, \frac{1}{2(\sigma^2 \tilde{I} - 1)} \sigma^2 \Gamma_p^{-1})$$

☞ R-estimation

$$\sqrt{n}(\hat{\phi}_{\mathrm{R}} - \phi_0) \xrightarrow{D} N(0, \frac{\sigma^2 \tilde{J} - \tilde{K}^2}{2(\sigma^2 \tilde{L} - \tilde{K})^2} \sigma^2 \Gamma_p^{-1})$$

☞ Least absolute deviations:

$$\sqrt{n}(\hat{\phi}_{\mathrm{LAD}} - \phi_0) \xrightarrow{D} N(0, \frac{\mathrm{Var}(|Z_1|)}{2(2\sigma^2 f_\sigma(0) - E|Z_1|)^2} \sigma^2 \Gamma_p^{-1})$$

Laplace: (LAD=MLE)

R: $\dfrac{\sigma^2 \tilde{J} - \tilde{K}^2}{2(\sigma^2 \tilde{L} - \tilde{K})^2} = \dfrac{5}{6}$ (using $\varphi(x) = x - 1/2$, Wilcoxon)

LAD=MLE: 1/2

Students t_ν:

ν	LAD	R	MLE	LAD/R	MLE/R
3	.733	**.520**	.500	**1.411**	.962
6	6.22	**3.01**	3.00	**2.068**	.997
9	16.8	**7.15**	7.00	**2.354**	.980
12	32.6	**13.0**	12.5	**2.510**	.964
15	53.4	**20.5**	19.5	**2.607**	.952
20	99.6	**36.8**	34.5	**2.707**	.937
30	234	**83.6**	77.0	**2.810**	.921

Central Limit Theorem (R-estimation)

- Think of $\mathbf{u} = n^{1/2}(\phi - \phi_0)$ as an element of R^p
- Define

$$S_n(\mathbf{u}) = \sum_{t=1}^{n-p}\left(\varphi\left(\frac{R_t(\phi_0 + n^{-1/2}\mathbf{u})}{n-p+1}\right) z_t(\phi_0 + n^{-1/2}\mathbf{u})\right) - \sum_{t=1}^{n-p}\left(\varphi\left(\frac{R_t(\phi_0)}{n-p+1}\right) z_t(\phi_0)\right),$$

where $R_t(\phi)$ is the rank of $z_t(\phi)$ among $z_1(\phi), \ldots, z_{n-p}(\phi)$.

- Then $S_n(\mathbf{u}) \to S(\mathbf{u})$ in distribution on $C(R^p)$, where

$$S(\mathbf{u}) = |\phi_{0r}|^{-1} (\sigma^2 \tilde{L} - \tilde{K})\mathbf{u}'\sigma^{-2}\Gamma_p \mathbf{u} + \mathbf{u}'\mathbf{N}, \;\; \mathbf{N} \sim N(\mathbf{0}, 2(\sigma^2 \tilde{J} - \tilde{K}^2)\,|\phi_{0r}|^{-2}\, \sigma^{-2}\Gamma_p),$$

- Hence,

$$\begin{aligned}\arg\min S_n(\mathbf{u}) &= n^{1/2}(\hat{\phi}_R - \phi_0) \\ &\xrightarrow[D]{} \arg\min S(\mathbf{u}) \\ &= -\frac{|\phi_{0r}|}{2(\sigma^2\tilde{L} - \tilde{K})}\sigma^2\Gamma_p^{-1}\mathbf{N} \sim N\left(\mathbf{0}, \frac{\sigma^2\tilde{J} - \tilde{K}^2}{2(\sigma^2\tilde{L} - \tilde{K})^2\,|\phi_{0r}|^2}\sigma^2\Gamma_p^{-1}\right)\end{aligned}$$

Main ideas (R-estimation)

- Define

$$\tilde{S}_n(\mathbf{u}) = \sum_{t=1}^{n-p} \varphi(F_z(z_t)) z_t(\phi_0 + n^{-1/2}\mathbf{u}) - \sum_{t=1}^{n-p} \varphi(F_z(z_t)) z_t(\phi_0),$$

where F_z is the df of z_t.

- Using a Taylor series, we have

$$\tilde{S}_n(\mathbf{u}) \sim n^{-1/2}\mathbf{u}' \sum_{t=1}^{n-p} \left(\varphi(F_z(z_t)) \frac{\partial z_t(\phi_0)}{\partial \phi} \right) + 2^{-1} n^{-1} \mathbf{u}' \sum_{t=1}^{n-p} \left(\varphi(F_z(z_t)) \frac{\partial^2 z_t(\phi_0)}{\partial \phi \partial \phi'} \right) \mathbf{u}$$

$$\xrightarrow[D]{} \mathbf{u}'\mathbf{N} - \mathbf{u}'\tilde{K} \,|\, \phi_{0r} \,|^{-1} \sigma^{-2} \Gamma_p \mathbf{u}$$

- Also,

$$S_n(\mathbf{u}) - \tilde{S}_n(\mathbf{u}) = \mathbf{u}' \sigma^2 \tilde{L} \sigma^{-2} \Gamma_p \mathbf{u} / \,|\, \phi_{0r} \,| + o_p(1).$$

- Hence

$$S_n(\mathbf{u}) \xrightarrow[D]{} |\, \phi_{0r} \,|^{-1} (\sigma^2 \tilde{L} - \tilde{K}) \mathbf{u}' \sigma^{-2} \Gamma_p \mathbf{u} + \mathbf{u}'\mathbf{N}, \;\; \mathbf{N} \sim N(\mathbf{0}, 2(\sigma^2 \tilde{J} - \tilde{K}^2) \,|\, \phi_{0r} \,|^{-2} \sigma^{-2} \Gamma_p).$$

Order Selection:

Partial ACF From the previous result, if true model is of order r and fitted model is of order p > r, then

$$n^{1/2}\hat{\phi}_{p,LAD} \to N(0, \frac{\mathrm{Var}(|Z_1|)}{2(2\sigma^2 f_\sigma(0) - E|Z_1|)^2})$$

where $\hat{\phi}_{p,LAD}$ is the pth element of $\hat{\phi}_{LAD}$.

Procedure:

1. Fit high order (P-th order), obtain residuals and estimate scalar,

$$\theta^2 = \frac{\mathrm{Var}(|Z_1|)}{2(2\sigma^2 f_\sigma(0) - E|Z_1|)^2}$$

by empirical moments of residuals and density estimates.

2. Fit AP models of order p=1,2, . . . , P via LAD and obtain p-th coefficient $\hat{\phi}_{p,p}$ for each.

3. Choose model order r as the smallest order beyond which the estimated coefficients are statistically insignificant.

Note: Can replace $\hat{\phi}_{p,p}$ with $\hat{\phi}_{p,\mathrm{MLE}}$ if using MLE. In this case for p > r

$$n^{1/2}\hat{\phi}_{p,MLE} \to N(0, \frac{1}{2(\sigma^2 \hat{I} - 1)}).$$

AIC: 2*p* or not 2*p*?

- An approximately unbiased estimate of the Kullback-Leiber index of fitted to true model:

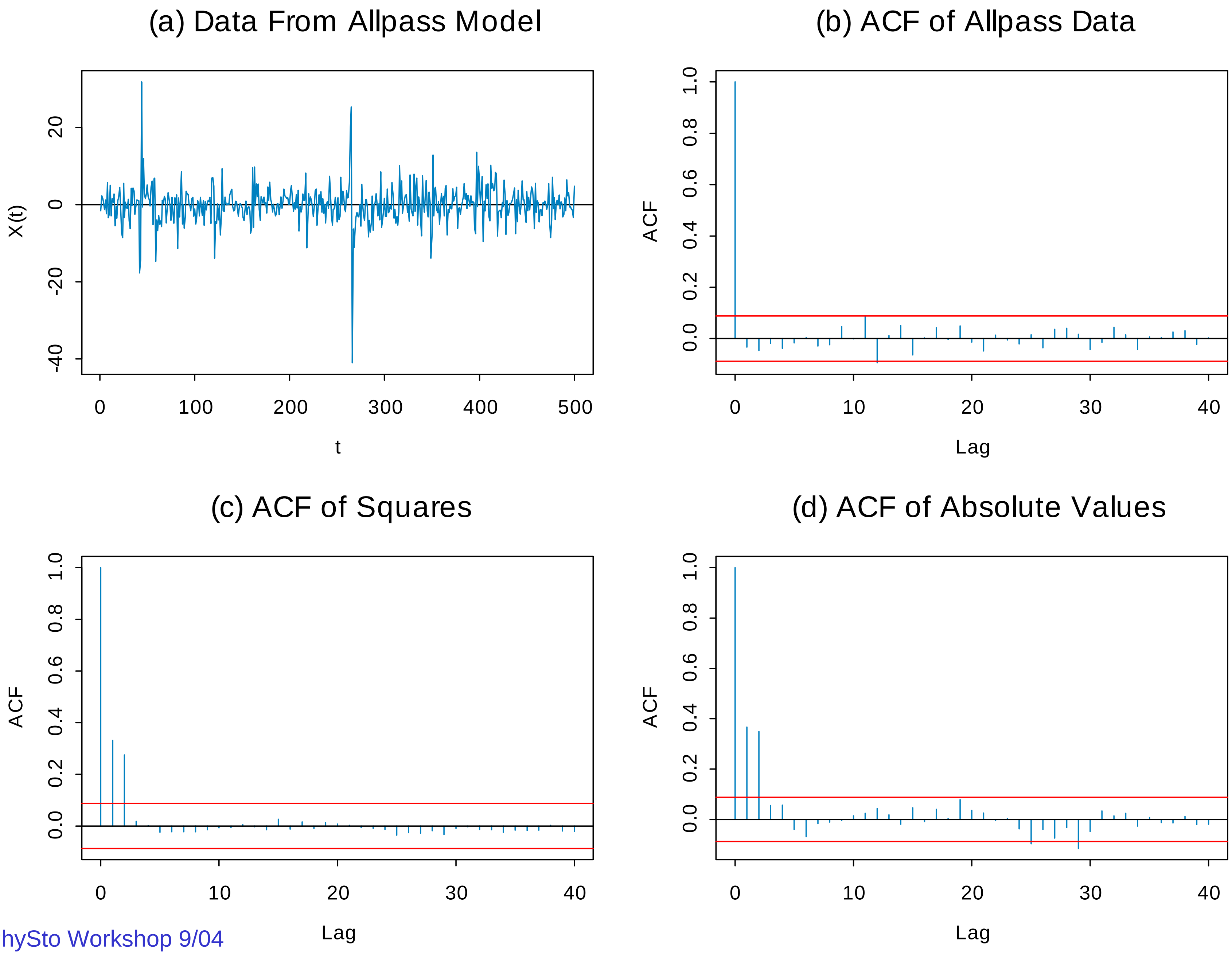
$$AIC(p) := -2L_X(\hat{\phi},\hat{\kappa}) + \frac{\mathrm{Var}(|Z_1|)}{(2\sigma^2 f_\sigma(0) - E|Z_1|)^2}\left(\frac{2\sigma^2 f_\sigma(0)}{E|Z_1|} - 1\right)p$$

- Penalty term for Laplace case:

$$\frac{\mathrm{Var}(|Z_1|)}{(2\sigma^2 f_\sigma(0) - E|Z_1|)^2}\left(\frac{2\sigma^2 f_\sigma(0)}{E|Z_1|} - 1\right)p = p$$

- Penalty term can be estimated from the data.

Sample realization of all-pass of order 2



Simulation results:

- 1000 replicates of all-pass models
- model order parameter value

model order	parameter value
1	$\phi_1 = .5$
2	$\phi_1 = .3,\ \phi_2 = .4$

- noise distribution is t with 3 d.f.
- sample sizes n=500, 5000
- estimation method is LAD

To guard against being trapped in local minima, we adopted the following strategy.

- 250 random starting values were chosen at *random*. For model of order p, k-th starting value was computed recursively as follows:

    1. Draw $\phi_{11}^{(k)}, \phi_{22}^{(k)}, \ldots, \phi_{pp}^{(k)}$ iid uniform (-1,1).
    2. For j=2, …, p, compute

$$\begin{bmatrix} \phi_{j1}^{(k)} \\ \vdots \\ \phi_{j,j-1}^{(k)} \end{bmatrix} = \begin{bmatrix} \phi_{j-1,1}^{(k)} \\ \vdots \\ \phi_{j-1,j-1}^{(k)} \end{bmatrix} - \phi_{jj}^{(k)} \begin{bmatrix} \phi_{j-1,j-1}^{(k)} \\ \vdots \\ \phi_{j-1,1}^{(k)} \end{bmatrix}$$

- Select top 10 based on minimum function evaluation.

- Run Hooke and Jeeves with each of the 10 starting values and choose best optimized value.

	Asymptotic		Empirical			
N	mean	**std dev**	mean	**std dev**	%coverage	rel eff*
500	ϕ_1=.5	**.0332**	.4979	**.0397**	94.2	11.8
5000	ϕ_1=.5	**.0105**	.4998	**.0109**	95.4	9.3

	Asymptotic		Empirical		
N	mean	**std dev**	mean	**std dev**	%coverage
500	ϕ_1=.3	**.0351**	.2990	**.0456**	92.5
	ϕ_2=.4	**.0351**	.3965	**.0447**	92.1
5000	ϕ_1=.3	**.0111**	.3003	**.0118**	95.5
	ϕ_2=.4	**.0111**	.3990	**.0117**	94.7

*Efficiency relative to maximum absolute residual kurtosis:

$$\left| \frac{1}{n-p} \sum_{t=1}^{n-p} \left(\frac{z_t(\phi)}{v_2^{1/2}} \right)^4 - 3 \right|, \qquad v_2 = \frac{1}{n-p} \sum_{t=1}^{n-p} \left(z_t(\phi) - \bar{z}(\phi) \right)^2$$

R-Estimator: Minimize the objective fcn

$$S(\phi) = \sum_{t=1}^{n-p} \left(\frac{t}{n-p+1} - \frac{1}{2} \right) z_{(t)}(\phi)$$

where $\{z_{(t)}(\phi)\}$ are the ordered $\{z_t(\phi)\}$.

		Empirical		Empirical LAD	
N		mean	**std dev**	mean	**std dev**
500	ϕ_1=.5	.4978	**.0315**	.4979	**.0397**
5000	ϕ_1=.5	.4997	**.0094**	.4998	**.0109**
500	ϕ_1=.3	.2988	**.0374**	.2990	**.0456**
	ϕ_2=.4	.3957	**.0360**	.3965	**.0447**
5000	ϕ_1=.3	.3007	**.0101**	.3003	**.0118**
	ϕ_2=.4	.3993	**.0104**	.3990	**.0117**