Kramers' type law for Lévy flights*

Lévy noise induced transitions

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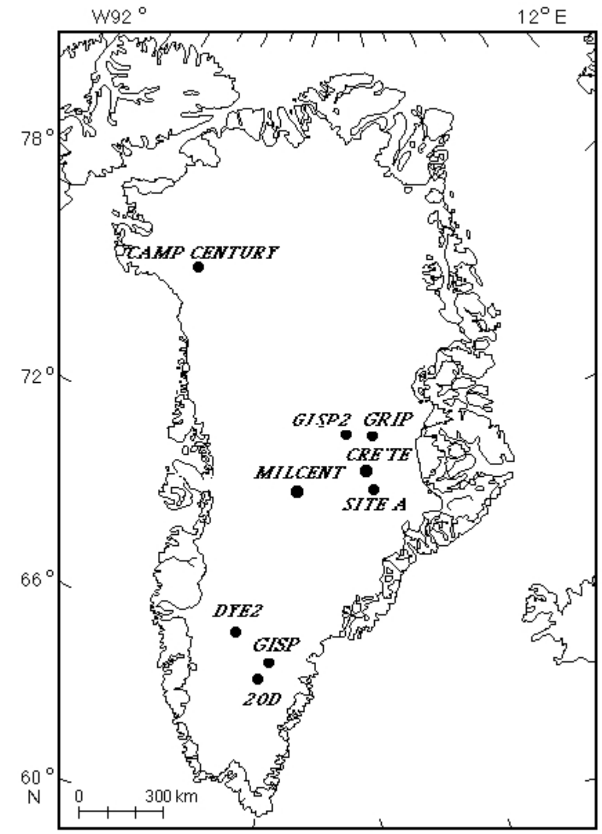
4th Conference on **Extreme Value Analysis**, Gothenburg, 2005

*Supported by the DFG Research Center **Matheon** (FZT86) in Berlin and the DFG Reaserch Project *Stochastic Dynamics of Climate States*

1. Motivation

Greenland ice-core data allows to reconstruct Earth's climate up to 200.000 years before present.

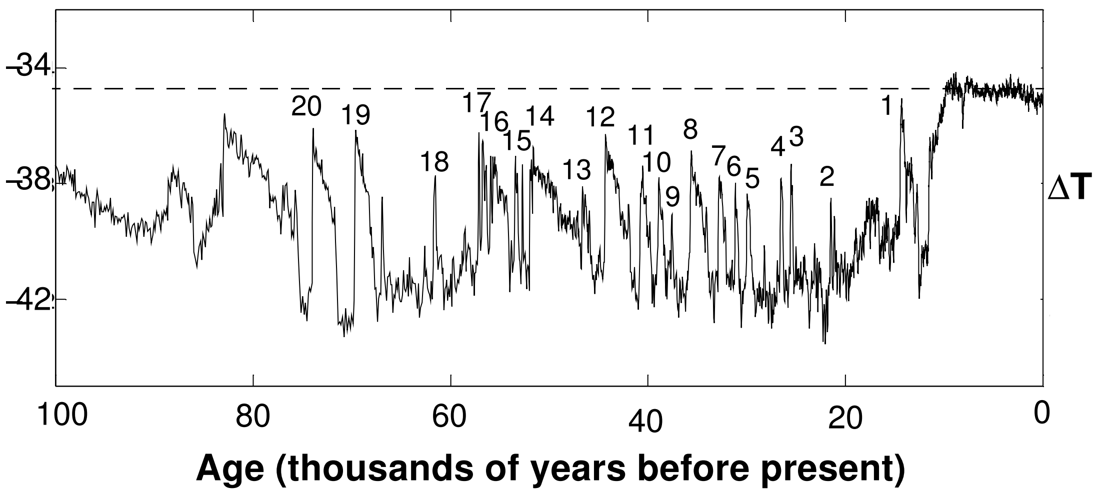
International projects:
GRIP (3028 m), GISP2 (3053.44 m), NGRIP (3084.99 m)



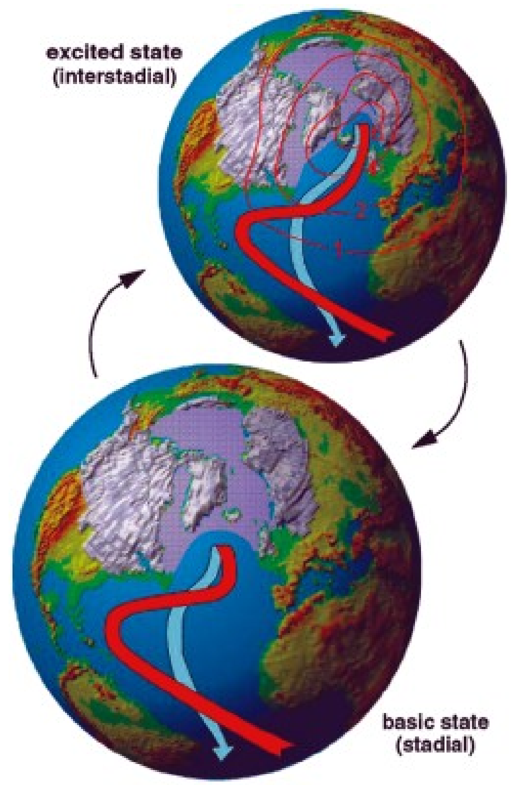
2. Paleo Proxy Data. Dansgaard-Oeschger Events

Paleo Data Proxies: Oxygen isotopes, dust, volcanic markers etc.

Global climate during the last glacial ($\sim$120 000 -10 000 b. p.) has experienced at least 20 abrupt and large-amplitude shifts (Dansgaard-Oeschger events).



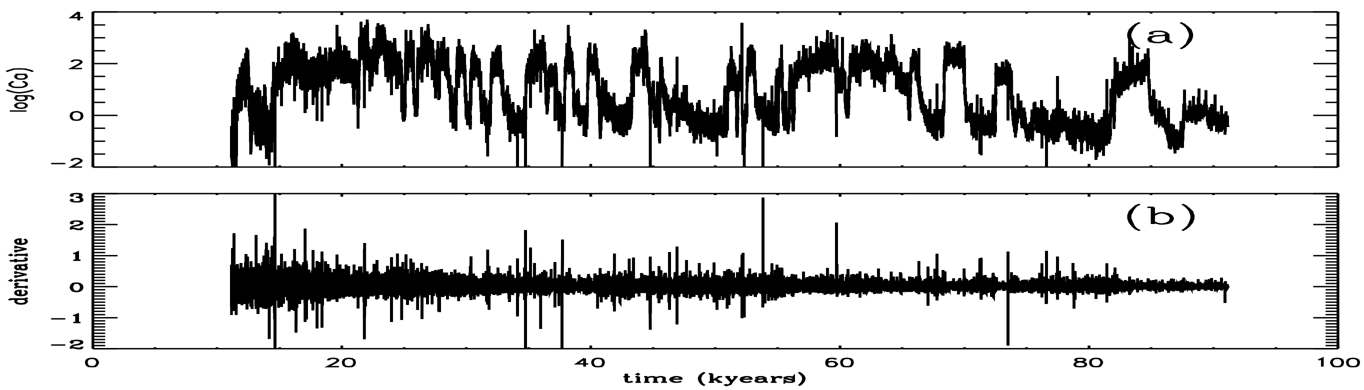
- rapid warming by 5-10 °C within at most a few decades
- plateau phase with slow cooling lasting several centuries
- rapid drop to cold stadial conditions



Simulations: Ganopolsky/Rahmstorf, Potsdam Institute for Climate Impact Research

3. Paleo Proxy Data. Detailed Look.

The calcium (Ca) signal from the GRIP ice-core:
about 80,000 data-points from 11 kyr to 91 kyr before present.



Typical interjump time: 1000 – 2000 years, mean waiting time $\sim$1470 years

What triggers the transitions?

Langevin equation for climate dynamics

$$dX_t^\varepsilon = -U'(X_t^\varepsilon)\,dt + \varepsilon\,dL_t$$

U – double-well potential, wells correspond to the climate states.

P. Ditlevsen (*Geophys. Res. Lett. 1999*): spectral analysis of the data.
Noise L has α-stable component with $\alpha \approx 1.75$.

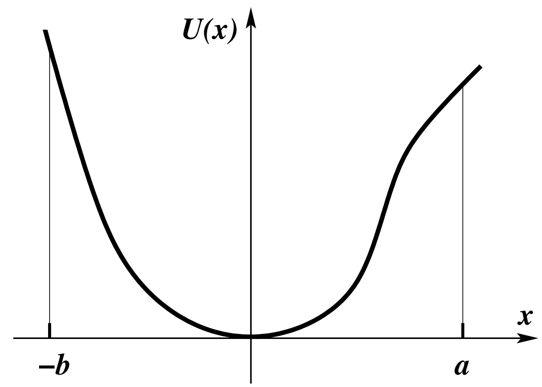
4. Object of Study. Simple System with Lévy Noise

Small noise ($\varepsilon \downarrow 0$) asymptotics of transition times for systems with Lévy noise

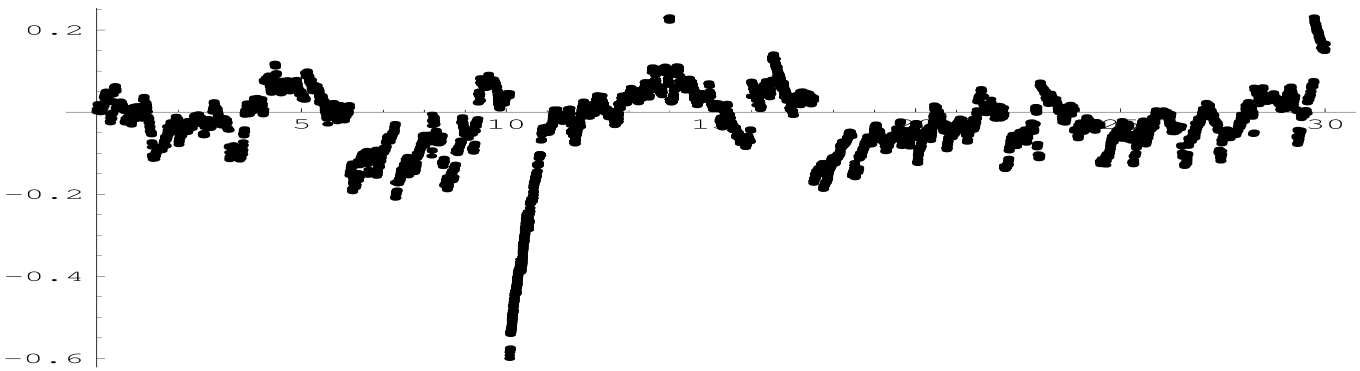
$$X_t^\varepsilon = x - \int_0^t U'(X_{s-}^\varepsilon)\, ds + \varepsilon L_t, \quad \varepsilon \downarrow 0.$$

- L — α-stable symmetric Lévy motion ($0 < \alpha < 2$) $+$ Brownian Motion

Potential $U \in \mathcal{C}^{(3)}(\mathbb{R})$:
- $U'(x)x \geq 0$
- $U'(x) = 0$ iff $x = 0$
- $U''(0) = M > 0$



$$\sigma(\varepsilon) = \inf\{t \geq 0 \,:\, X_t^\varepsilon \notin [-b, a]\}, \quad a, b < \infty \ (b = \infty)$$



Ref.: P. Imkeller, I. Pavlyukevich, **`arXiv:math.PR/0409246`** (2004)

5. The Driving Process L

L – symmetric α-stable Lévy process (plus Brownian motion).

Marginal laws determined by Lévy–Hinčin's formula

$$\mathbf{E}e^{i\lambda L_1} = \exp\left\{-\frac{d}{2}\lambda^2 + \int_{\mathbb{R}\setminus\{0\}} (e^{i\lambda y} - 1 - i\lambda y\,\mathbb{I}\{|y|\leq 1\})\frac{dy}{|y|^{1+\alpha}}\right\},$$

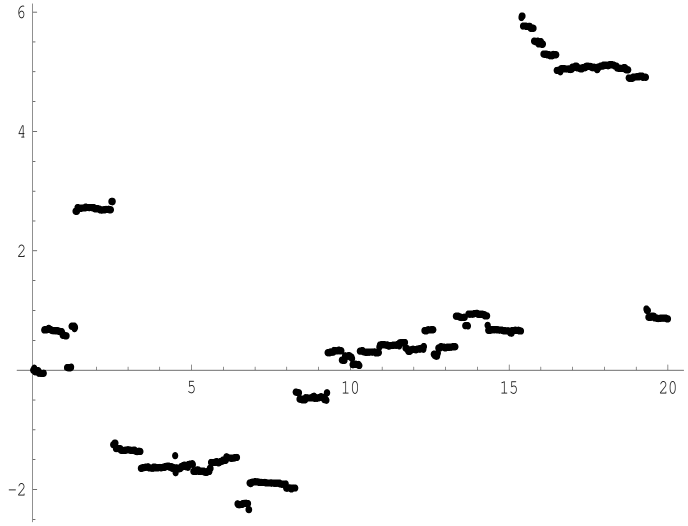
$$\alpha \in (0,2)$$

- Gaussian variance $d \geq 0$
- Lévy measure $\nu(dy) = \frac{dy}{|y|^{1+\alpha}}$, $y \neq 0$
- $\nu(\mathbb{R}) = \infty \Leftrightarrow$ countably many (small) jumps on any finite time interval,
  jump times are dense

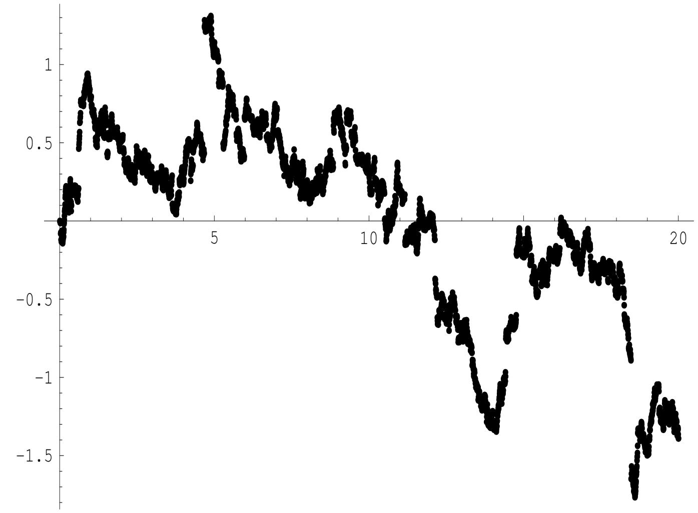
$$\mathbf{E}e^{i\lambda L_1} = \exp\left\{-\frac{d}{2}\lambda^2 - c(\alpha)|\lambda|^\alpha\right\}.$$

$$c(\alpha) \to \infty, \qquad \alpha \uparrow 2$$

6. Symmetric α–stable Lévy process L



$\alpha = 0.75$



$\alpha = 1.75$

$\alpha = 1$	Cauchy process	$\frac{1}{\pi}\frac{1}{1+x^2}$
$\alpha = 2$	Brownian motion	$\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$

In the physical literature: **Lévy Flights**

In a broader sense: **Anomalous Diffusion**

7. Exit Time. Results

Theorem 1. *There exist positive constants* ε_0, γ, δ, *and* $C > 0$ *such that for* $0 < \varepsilon \leq \varepsilon_0$ *the following asymptotics holds*

$$e^{-u(1+C\varepsilon^\delta)}(1 - C\varepsilon^\delta) \leq \mathbf{P}_x\left(\frac{\varepsilon^\alpha}{\alpha}\left[\frac{1}{a^\alpha} + \frac{1}{b^\alpha}\right]\sigma(\varepsilon) > u\right) \leq e^{-u(1-C\varepsilon^\delta)}(1 + C\varepsilon^\delta)$$

uniformly for all $x \in [-b + \varepsilon^\gamma, a - \varepsilon^\gamma]$ *and* $u \geq 0$.

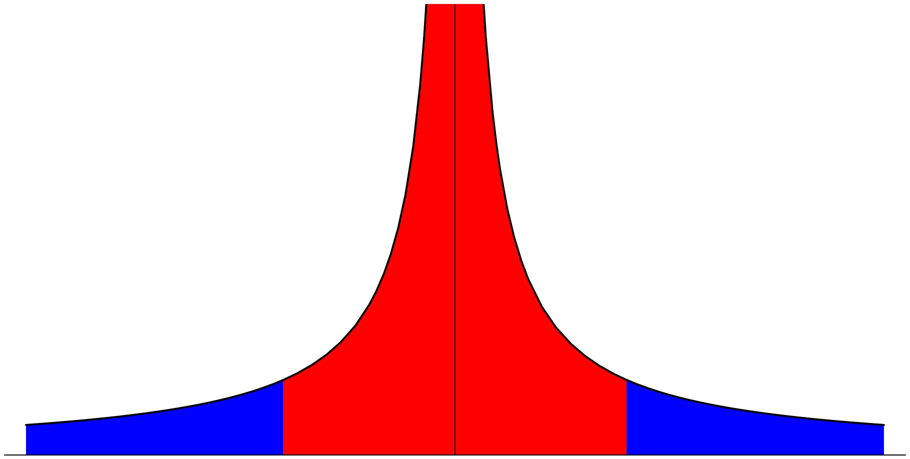
Theorem 2. *There exist positive constants* ε_0, γ *and* δ *such that for* $0 < \varepsilon \leq \varepsilon_0$ *the following asymptotics holds*

$$\mathbf{E}_x\sigma(\varepsilon) = \frac{\alpha}{\varepsilon^\alpha}\left[\frac{1}{a^\alpha} + \frac{1}{b^\alpha}\right]^{-1}(1 + \mathcal{O}(\varepsilon^\delta))$$

uniformly for all $x \in [-b + \varepsilon^\gamma, a - \varepsilon^\gamma]$.

8. Probabilistic Approach

$$L_t = \xi_t^\varepsilon + \eta_t^\varepsilon$$



$$\nu_\xi^\varepsilon(dy) = \mathbb{I}_{\{|y|\le \frac{1}{\sqrt{\varepsilon}}\}}(y)\frac{dy}{|y|^{1+\alpha}} \qquad \nu_\eta^\varepsilon(dy) = \mathbb{I}_{\{|y|> \frac{1}{\sqrt{\varepsilon}}\}}(y)\frac{dy}{|y|^{1+\alpha}}$$

$$\nu_\xi^\varepsilon(\mathbb{R}) = \infty \qquad \nu_\eta^\varepsilon(\mathbb{R}) = 2\int_{1/\sqrt{\varepsilon}}^\infty \frac{dy}{|y|^{1+\alpha}} = \frac{2}{\alpha}\varepsilon^{\alpha/2} = \beta_\varepsilon$$

$\varepsilon\xi^\varepsilon$ is a sum of BM of intensity ε and a small-jumps process, $|\Delta(\varepsilon\xi_t^\varepsilon)| \le \sqrt{\varepsilon}$.

$\varepsilon\eta^\varepsilon$ is a compound Poisson process, $|\Delta(\varepsilon\eta_t^\varepsilon)| > \sqrt{\varepsilon}$

$\varepsilon\xi^\varepsilon$ and $\varepsilon\eta^\varepsilon$ are independent.

9. The Small- and Large-Jump Parts

$0 = \tau_0, \tau_1, \tau_2, \ldots$ arrival times of η^ε

$T_k = \tau_k - \tau_{k-1}$ independent i.d. inter-arrival times

$W_k = \eta^\varepsilon_{\tau_k} - \eta^\varepsilon_{\tau_k -}$ independent i.d. jumps

$$T_k \sim \exp(\beta_\varepsilon) \qquad W_k \sim \frac{1}{\beta_\varepsilon}\nu^\varepsilon_\eta(\cdot)$$

$$\mathbf{E}T_k = \frac{1}{\beta_\varepsilon} = \frac{\alpha}{2}\frac{1}{\varepsilon^{\alpha/2}} \qquad \mathbf{P}(W_k \le x) = \frac{1}{\beta_\varepsilon}\int_{-\infty}^{x} \mathbb{I}_{\{|y|>\frac{1}{\sqrt{\varepsilon}}\}}(y)\frac{dy}{|y|^{1+\alpha}}$$

Between the big jumps X^ε is driven by $\varepsilon\xi^\varepsilon$:

$$X^\varepsilon_t = x - \int_0^t U'(X^\varepsilon_{s-})\,ds + \varepsilon\xi^\varepsilon_t, \qquad t \in [0, \tau_1),$$

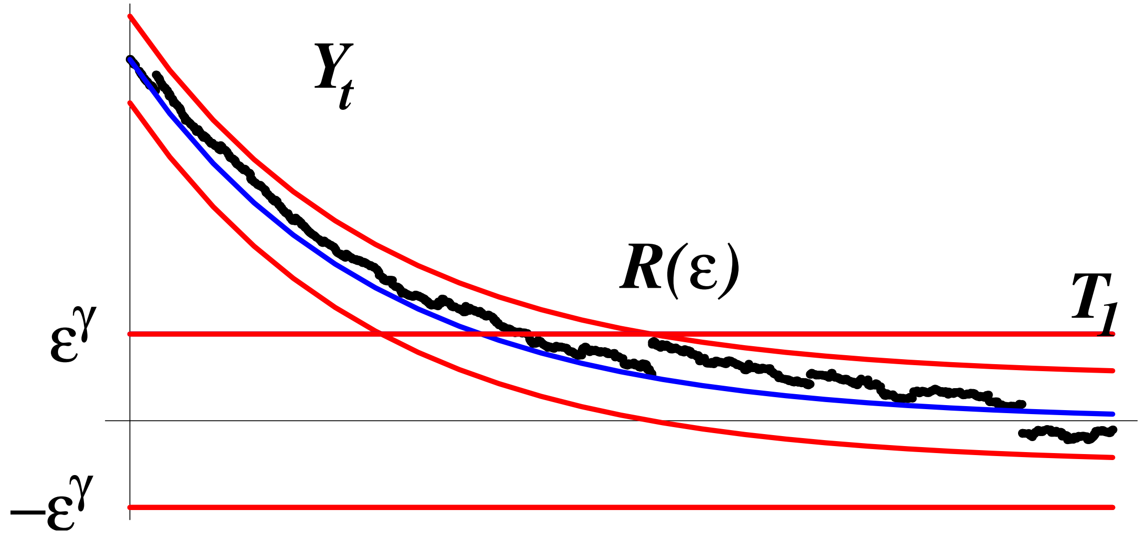
$$Y_t = x - \int_0^t U'(Y^\varepsilon_s)\,ds.$$

On inter-jump intervals X^ε is Y perturbed by $\varepsilon\xi^\varepsilon$.

10. Behaviour on the Intervals between Big Jumps

$$\dot{Y}_t = -U'(Y_t)$$

$$T_1 \sim \exp(\beta_\varepsilon)$$



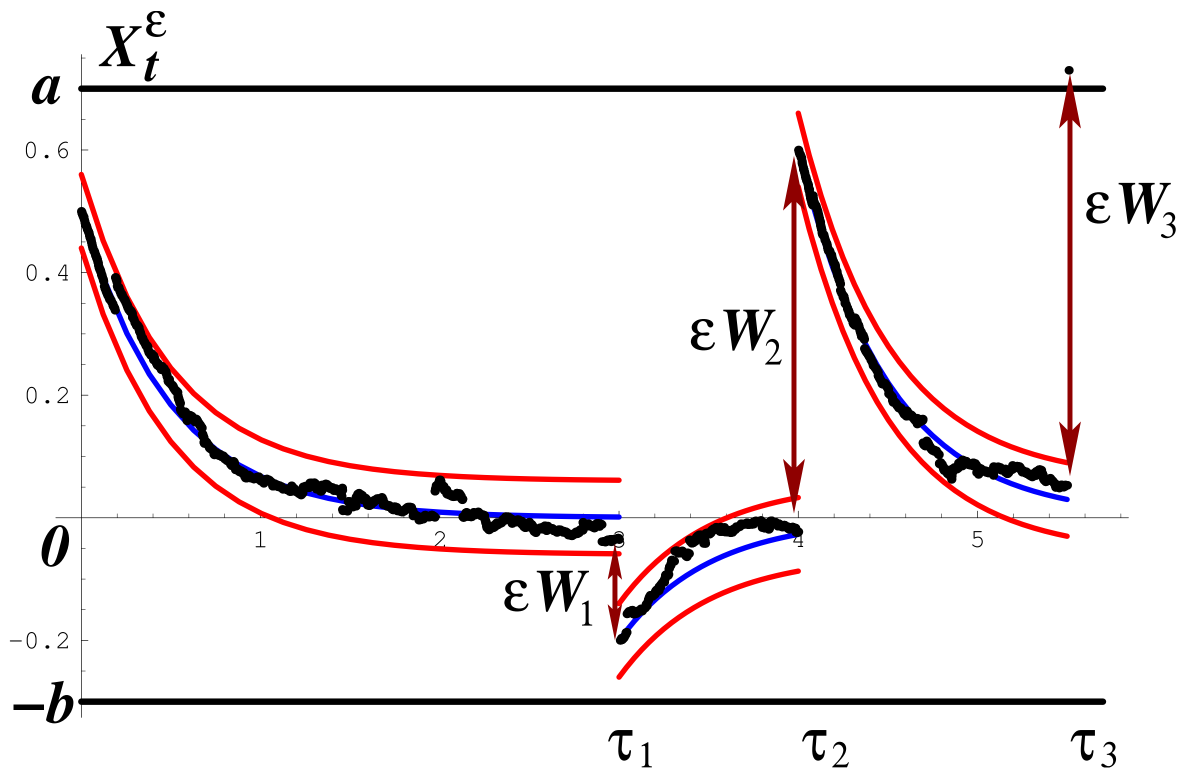
$$\mathbf{P}\left(\sup_{[0,T_1)} |X_t^\varepsilon - Y_t| \geq \tfrac{\varepsilon^\gamma}{2}\right) \leq \mathbf{P}\left(\sup_{[0,T_1)} |\varepsilon\xi_t^\varepsilon| \geq \tfrac{\varepsilon^\gamma}{C}\right)$$

$$\leq \int_0^\infty \mathbf{P}\left(\sup_{[0,u)} |\varepsilon\xi_t^\varepsilon| \geq \tfrac{\varepsilon^\gamma}{C}\right) \beta_\varepsilon e^{-\beta_\varepsilon u}\, du \leq e^{-1/\varepsilon^\delta}$$

$$T(x,\varepsilon) = \int_{\varepsilon^\gamma/2}^x \frac{dy}{|U'(y)|} \approx \int_\delta^x \frac{dy}{|U'(y)|} + \int_{\varepsilon^\gamma/2}^\delta \frac{dy}{My}$$

$$\approx \text{Const} + \frac{\gamma}{M}|\ln\varepsilon| \leq R(\varepsilon) = \mathcal{O}(|\ln\varepsilon|)$$

11. Predominant Behaviour



Expected inter-jump time	$\mathbf{E}T_k = \frac{1}{\beta_\varepsilon} = \frac{\alpha}{2}\varepsilon^{-\alpha/2}$	**polynomial in** ε
Relaxation time	$R(\varepsilon) = \mathcal{O}(\lvert \ln \varepsilon \rvert)$	**logarithmic in** ε
Between big jumps	X^ε is driven by	**"small jumps"** $\varepsilon\xi^\varepsilon$
Deviation probability		**small**

$\Rightarrow$ Typically, X^ε **jumps from a neighbourhood of** 0 by εW_k at τ_k.
Typically, X^ε exits $I = [-b, a]$ by jumping at times τ_k.

12. Exit Time Law. Heuristic Proof of Theorem 2

$$\tau_k = T_1 + T_2 + \cdots + T_k, \qquad \mathbf{E}T_1 = 1/\beta_\varepsilon$$

$$\mathbf{P}(\varepsilon W_1 \notin [-b,a]) = \frac{1}{\beta_\varepsilon}\left(\int_{\frac{a}{\varepsilon}}^{\infty} + \int_{\frac{b}{\varepsilon}}^{\infty}\right)\frac{dy}{y^{1+\alpha}} = \frac{\varepsilon^\alpha}{\alpha\beta_\varepsilon}\left[\frac{1}{a^\alpha} + \frac{1}{b^\alpha}\right]$$

$$\begin{aligned}
\mathbf{E}_x\sigma(\varepsilon) &\approx \sum_{k=1}^{\infty} \mathbf{E}\tau_k \cdot \mathbf{P}(\sigma(\varepsilon) = \tau_k) \\
&\approx \sum_{k=1}^{\infty} k \cdot \mathbf{E}T_1 \cdot \mathbf{P}(\varepsilon W_1 \in I, \ldots, \varepsilon W_{k-1} \in I, \varepsilon W_k \notin I) \\
&= \sum_{k=1}^{\infty} k \cdot \mathbf{E}T_1 \cdot [1 - \mathbf{P}(\varepsilon W_1 \notin I)]^{k-1} \cdot \mathbf{P}(\varepsilon W_1 \notin I) \\
&= \frac{\mathbf{P}(\varepsilon W_1 \notin I)}{\beta_\varepsilon}\frac{1}{\mathbf{P}(\varepsilon W_1 \notin I)^2} = \frac{\alpha}{\varepsilon^\alpha}\left[\frac{1}{a^\alpha} + \frac{1}{b^\alpha}\right]^{-1}
\end{aligned}$$

13. Exponential Exit. Heuristic Proof of Theorem 1

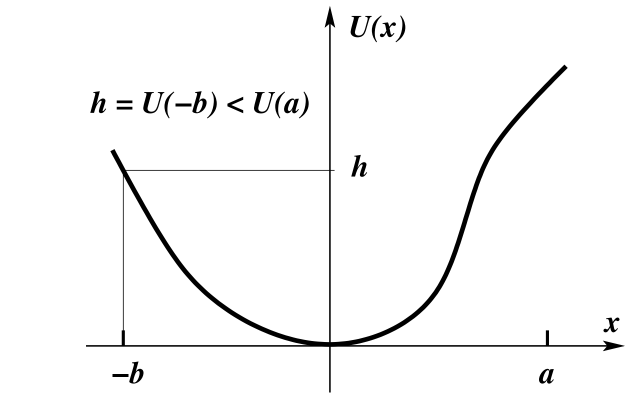
$$\tau_k = T_1 + T_2 + \cdots + T_k \sim \text{Gamma}(\beta_\varepsilon, k)$$

$$\mathbf{P}(\tau_k \in [t, t+dt]) = \beta_\varepsilon e^{-\beta_\varepsilon t}\frac{(\beta_\varepsilon t)^{k-1}}{(k-1)!}dt$$

$$\mathbf{P}(\varepsilon W_1 \notin [-b, a]) = \frac{1}{\beta_\varepsilon}\left(\int_{\frac{a}{\varepsilon}}^{\infty} + \int_{\frac{b}{\varepsilon}}^{\infty}\right)\frac{dy}{y^{1+\alpha}} = \frac{1}{2}\varepsilon^{\alpha/2}\left[\frac{1}{a^\alpha} + \frac{1}{b^\alpha}\right]$$

$$\begin{aligned}
&\mathbf{P}_x\left(\tfrac{\varepsilon^\alpha}{\alpha}\left[\tfrac{1}{a^\alpha} + \tfrac{1}{b^\alpha}\right]\sigma(\varepsilon) > u\right) \\
&\approx \sum_{k=1}^{\infty}\mathbf{P}\left(\tfrac{\varepsilon^\alpha}{\alpha}\left[\tfrac{1}{a^\alpha} + \tfrac{1}{b^\alpha}\right]\tau_k > u\right)\cdot \mathbf{P}_x(\sigma(\varepsilon) = \tau_k) \\
&\approx \sum_{k=1}^{\infty}\mathbf{P}\left(\tfrac{\varepsilon^\alpha}{\alpha}\left[\tfrac{1}{a^\alpha} + \tfrac{1}{b^\alpha}\right]\tau_k > u\right)\cdot \mathbf{P}(\varepsilon W_1 \in I, \ldots, \varepsilon W_{k-1} \in I, \varepsilon W_k \notin I) \\
&= \exp(-u)
\end{aligned}$$

14. Comparison with Gaussian Case



$$\hat{X}_t^\varepsilon = x - \int_0^t U'(\hat{X}_s^\varepsilon)\,ds + \varepsilon W_t$$

Freidlin–Wentzell (large deviations):

$$\mathbf{P}_x\big(e^{(2h-\delta)/\varepsilon^2} < \hat{\sigma} < e^{(2h+\delta)/\varepsilon^2}\big) \to 1$$

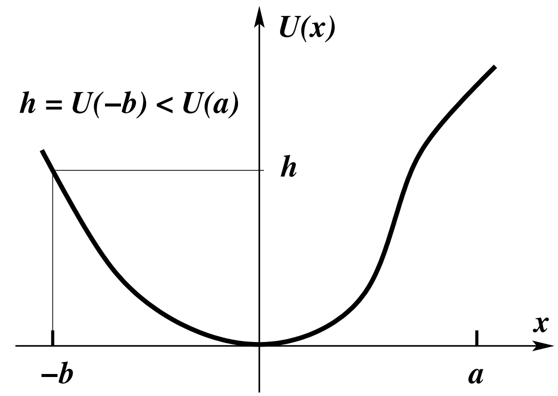
Kramers' law ('40, Williams, Bovier):

$$\mathbf{E}_x\hat{\sigma} \approx \frac{\varepsilon\sqrt{\pi}}{|U'(-b)|\sqrt{U''(0)}} e^{2h/\varepsilon^2}$$

Exponential exit (Day, Bovier)

$$\mathbf{P}_x\big(\tfrac{\hat{\sigma}}{\mathbf{E}_x\hat{\sigma}} > u\big) \sim \exp(-u)$$

Diffusion 'climbs up and out'



$$X_t^\varepsilon = x - \int_0^t U'(X_{s-}^\varepsilon)\,ds + \varepsilon L_t$$

$$\mathbf{P}_x\big(\tfrac{1}{\varepsilon^{\alpha-\delta}} < \sigma < \tfrac{1}{\varepsilon^{\alpha+\delta}}\big) \to 1$$

$$\mathbf{E}_x\sigma \approx \frac{\alpha}{\varepsilon^\alpha}\Big[\frac{1}{a^\alpha} + \frac{1}{b^\alpha}\Big]^{-1}$$

$$\mathbf{P}_x\big(\tfrac{\sigma}{\mathbf{E}_x\sigma} > u\big) \sim \exp(-u)$$

Lévy motion driven SDE 'jumps out'

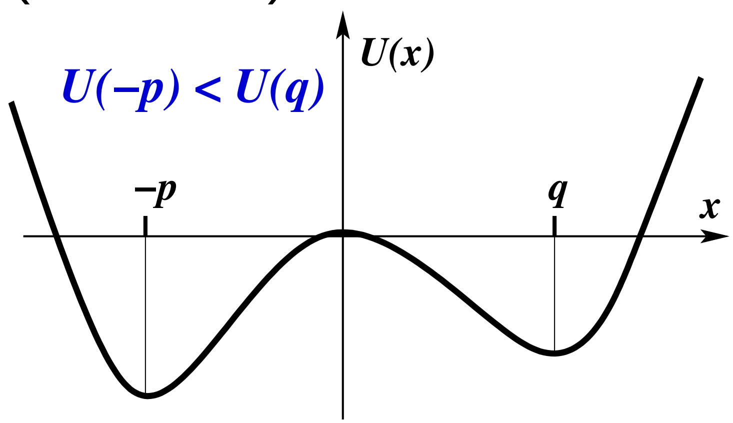
15. Double-well Potential

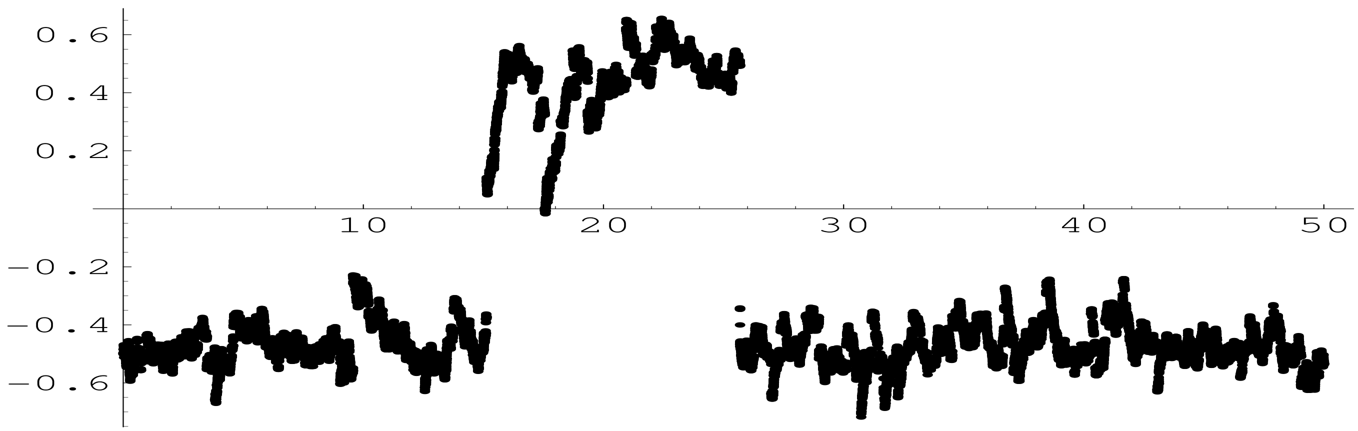
$$X_t^\varepsilon = x - \int_0^t U'(X_{s-}^\varepsilon)\, ds + \varepsilon L_t, \quad \varepsilon \downarrow 0.$$

- L — α-stable symmetric Lévy motion ($0 < \alpha < 2$) $+$ Brownian Motion

Potential $U \in \mathcal{C}^{(3)}(\mathbb{R})$:

- $U'(-p) = U'(0) = U'(q) = 0$
- $U''(-p), U''(q) > 0$, $U''(0) < 0$
- $|U'(x)| > c_1|x|^{1+c_2}$, $x \to \pm\infty$





Main inconvenience: **Saddle Point 0**, characteristic boundary

Gaussian Case: $\mathbf{E}_x\hat{\sigma} \approx \frac{2\pi}{\sqrt{U''(0)|U''(q)|}} e^{2|U(q)|/\varepsilon^2}$

16. Metastable Behaviour

Theorem 3. *For any* $0 < t_1 < t_2 < \cdots < t_n$*,*

$$(X^\varepsilon_{t_1/\varepsilon^\alpha}, X^\varepsilon_{t_2/\varepsilon^\alpha}, \cdots, X^\varepsilon_{t_n/\varepsilon^\alpha}) \xrightarrow{\mathcal{D}} (Y_{t_1}, Y_{t_2}, \cdots, Y_{t_n}), \quad \varepsilon \downarrow 0,$$

where Y *is a* $\{-p, q\}$*-valued Markov process with generator*

$$\begin{pmatrix} -\frac{1}{\alpha p^\alpha} & \frac{1}{\alpha p^\alpha} \\ \frac{1}{\alpha q^\alpha} & -\frac{1}{\alpha q^\alpha} \end{pmatrix}, \text{ and } Y_0 = \begin{cases} -p, & \text{if } x < 0, \\ q, & \text{if } x > 0. \end{cases}$$

Gaussian case (Kipnis and Newman '85, Mathieu '95).
Assume the left well is deeper: $U(-p) < U(q) < 0$
There is a scaling $\lambda(\varepsilon)$: $\varepsilon^2 \ln \lambda(\varepsilon) \to -2U(q)$ such that

$$(\hat{X}^\varepsilon_{t_1\lambda(\varepsilon)}, \hat{X}^\varepsilon_{t_2\lambda(\varepsilon)}, \cdots, \hat{X}^\varepsilon_{t_n\lambda(\varepsilon)}) \xrightarrow{\mathcal{D}} (\hat{Y}_{t_1}, \hat{Y}_{t_2}, \cdots, \hat{Y}_{t_n}), \quad \varepsilon \downarrow 0,$$

where $\hat{Y}$ is a $\{-p, q\}$-valued Markov process with generator

$$\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}, \text{ and } \hat{Y}_0 = \begin{cases} -p, & \text{if } x < 0, \\ q, & \text{if } x > 0. \end{cases}$$

Sources and References

Page 1 Pictures:

http://www.ngdc.noaa.gov/paleo/globalwarming/gallery/icecore_4.jpg

http://ess.geology.ufl.edu/ess/Notes/Paleoclimatology/Paleoclimate Slides/greenland.gif

Page 2 Pictures:
Ganopolski, A. and Rahmstorf, S., 2002: Abrupt glacial climate changes due to stochastic resonance. Phys. Rev. Let. 88(3), 038501.

Page 3 Picture:
P. D. Ditlevsen, 1999: Observation of alpha-stable noise and a bistable climate potential in an ice-core record. Geophys. Res. Lett. 26, 1441-1444.

[1] P. IMKELLER AND I. PAVLYUKEVICH, First exit times of solutions of non-linear stochastic differential equations driven by symmetric Lévy process with α-stable component, arXiv:math.PR/0409246, 2004.

[2] M. FREIDLIN AND A. WENTZELL, Random perturbations of dynamical systems, 1998.

[3] H.A. KRAMERS, Brownian motion in a field of force and the diffusion model of chemical reactions, Physica 7, 284–304, 1940.

[4] M. WILLIAMS, Asymptotic exit time distributions, SIAM J. Appl. Math. 42, 149–154, 1982.

[5] A. BOVIER, M. ECKHOFF, V. GAYRARD AND M. KLEIN, Metastability in reversible diffusion processes I. Sharp asymptotics for capacities and exit times, WIAS Berlin, Preprint No. 767, 2002, to appear in J. Eur. Math. Soc.

[6] M. DAY, On the exponential exit law in the small parameter exit problem, Stochastics 8, 297–323, 1983.

[7] I. PAVLYUKEVICH, Metastable Behaviour of Lévy-Driven Diffusion (Lévy Flights in a Double-Well Potential), in preparation, 2005

[8] C. KIPNIS AND C. NEWMAN, The metastable behavior of infrequently observed, weakly random, one-dimensional diffusion processes, SIAM J. Appl. Math. 45 (6), 972–982, 1985

[9] P. MATHIEU, Spectra, exit times and long time asymptotics in the zero white noise limit, Stoch. Stoch. Rep. 55, 1–20 1995

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