Numerical bounds for the distribution of the maximum of a one- or two-parameter process.

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Goal and motivations

$X$ stochastic process, $M_T = \sup_{t \in [0, T]} X_t$.

**Goal:**
Numerical estimation of $\mathbb{P}(M_T \geq u)$ for every $u$.

**Motivations:**
- LRT in non identifiable mixture:
  - Definition of a test,
  - Computation of its power.
- Estimation of the persistence exponent:
  \[
  q(u) = \lim_{T \to +\infty} -\frac{1}{T} \ln \mathbb{P}(M_T < u) .
  \]
- Computation of the significative height of wave.
Known Results

- Comparison Lemma, *Plackett (1954).*
- Some Processes based on the Brownian motion.
- Some stationary Gaussian processes (periodic covariance, OU).
- Rice’s upper bound:

\[
P(M_T \geq u) \leq P(X_0 \geq u) + E(N_u),
\]

where \( N_u \) denotes the number of \( u \)-upcrossings.

- Extreme values:

\[
P(M_T \leq x) \sim_{T \to +\infty} \exp(-\exp(x/a_T + b_T)).
\]

- Rice series, *Azaïs and Wschebor (2002):*

\[
P(M_T \geq u) = P(X_0 \geq 0) + \sum_{m=1}^{+\infty} (-1)^{m+1} \frac{E(U_u^{[m]} \mathbb{1}_{X_0 < u})}{m!}.
\]
Statement of the integral formula

Framework:
$X$ Gaussian with $C^1$ sample paths and $\sigma_T^2 = \inf_{t \in [0,T]} \text{Var}(X_t) > 0$.

Time of first passage:

$$\tau_u = \inf \{ t \in (0, T), X_s < X_t, \forall s < t, X_t = u \}.$$  

Formula:

$$\mathbb{P}(M_T \geq u) = \mathbb{P}(X_0 \geq u) + \mathbb{P}(\tau_u \in (0, T))$$

$$= \mathbb{P}(X_0 \geq u) + \int_0^T \mathbb{E}(X_t' \mathbb{1}_{\{X_s < u, \forall s < t\}} / X_t = u) p_X(u) dt.$$  

Remark:

Rychlik (1987) gives density of $\tau_u$.  

Numerical bounds for the distribution of the maximum
Numerical procedure

MAGP tool-box gives bounds for $\mathbb{P}(M_T \geq u)$

\textit{rind.m}: WAFO tool-box, Brodtkorb et al. (2000).

Arguments: $(T, u, r)$.

Lower bound: For \( \{t_1, \ldots, t_m\} \) a subdivision of \([0, T]\),

$$1 - \mathbb{P}(X_{t_1} < u, \ldots, X_{t_m} < u).$$

Upper bound: From integral formula

$$\mathbb{P}(X_0 \geq u) + \int_0^T \mathbb{E}(X_t^+ \mathbb{1}_{\{X_{s_k} < u, (s_k)_{k=1,\ldots,m} \in (0,t)\}}) / X_t = u) p_{X_t}(u) dt.$$
Numerical Results

Example: Gaussian process with the arguments $(T, 1, \exp(-t^2/2))$

Application of the Rice’s upper bound

Estimation de $P(M_T > 1)$
Numerical Results

Example: Gaussian process with the arguments \((T, 1, \exp(-t^2/2))\)

Application of the Rice series
Numerical Results

Example: Gaussian process with the arguments \((T, 1, \exp(-t^2/2))\)

Application of the MAGP tool-box

![Graph showing estimated probability of \(M_T > 1\) as a function of \(T\).]
Comments

Error of the procedure:
For $T \leq 25$ and $u \geq 1$ upper bounded by $10^{-3}$.

Stationary and non stationary frameworks.

Non differentiable sample paths.

Estimation of the persistence exponent:

$$q(u) = \lim_{T \to +\infty} -\frac{1}{T} \ln \mathbb{P} (M_T < u).$$

Example, $r(t) = (\cosh(t))^{-1}$
- $q(0) \geq 0.25$ Li-Shao (2002).
- $q(0) \geq 0.365$ Dembo, Poonen, Shao and Zeitouni (2002).
- $q(0) \geq 0.371$
Equivalents when \( u \) is large

**Framework:** \( M_S = \sup_{t \in S} X(t) \),
\( X \) Gaussian process defined on \( S \subset \mathbb{R}^2 \) compact.

**References:** Adler (1981); Adler and Taylor (2005).

**Notations:** \( \lambda(S) \) Lebesgue measure of \( S \); \( \Delta = \text{Var}(X'(t)) \).

- **One term:**
  \[
  \frac{\lambda(S)|\det(\Delta)|^{\frac{1}{2}}}{(2\pi)^{\frac{3}{2}}}ue^{-\frac{u^2}{2}} + o\left(u e^{-\frac{u^2}{2}}\right)
  \]

- **Two terms:**
  \[
  \forall \delta > 0 : \quad 1 - \Phi(u) + \left[\frac{T^2|\det(\Delta)|^{\frac{1}{2}}}{(2\pi)^{\frac{3}{2}}} + \frac{T(\Delta_{11}^{\frac{1}{2}} + \Delta_{22}^{\frac{1}{2}})}{2\pi}\right] e^{-\frac{u^2}{2}} + o\left(e^{-(1+\delta)\frac{u^2}{2}}\right)
  \]
Extension of the integral formula

Integral formula:

↩️ Study on the border of $S$,

↩️ Study of the interior of $S$. 
Extension of the integral formula

Integral formula:
↬ Study on the border of $S$,
↬ Study of the interior of $S$.

Past of $t$:
$$\Gamma_t = \{ s, s_2 \leq t_2 \}$$

Record points:
$$\{ t, X(s) < X(t), \forall s \in \Gamma_t \}.$$
Extension of the integral formula

Integral formula:
✓ Study on the border of $S$,
✓ Study of the interior of $S$.

Crossings:
On the border: $X(t) = u$,
In the interior: $\left(X(t), \frac{\partial X}{\partial t_1}(t)\right) = (u, 0)$. 

Numerical bounds for the distribution of the maximum
Extension of the integral formula

Integral formula:

⇔ Study on the border of $S$,

⇔ Study of the interior of $S$.

First explicit upper bound in the "unit-speed" case:

$$\mathbb{P} \left( M_{[0,T]} \geq u \right) \leq$$

$$1 - \Phi(u) + \frac{T}{\pi} \exp \left( -\frac{u^2}{2} \right) + \frac{T^2}{(2\pi)^{\frac{3}{2}}} \left[ c \varphi\left( \frac{u}{c} \right) + u \Phi\left( \frac{u}{c} \right) \right] \exp \left( -\frac{u^2}{2} \right)$$

with $c = \left( \text{Var} \left( X_{20} \right) - 1 \right)^{\frac{1}{2}}$
Numerical Procedure and Results

Upper bound: Integral formula and discretization
Lower bound: discretization

rind.m: WAFO tool-box, Brodtkorb et al. (2000)
Numerical Procedure and Results

Example: $S = [0, T]^2$ and arguments $(10, u, \exp(-\|s-t\|^2/2))$

Application of equivalents
Numerical Procedure and Results

Example: \( S = [0, T]^2 \) and arguments \((10, u, \exp(-\|s - t\|^2/2))\)

Application of the MAGP tool-box

Numerical bounds for the distribution of the maximum
Conclusion and perspectives

Conclusion

⇝ Effective tool of computation.

⇝ Geometry of the problem.

Perspectives

⇝ rind.m and simultaneous statistics.

⇝ Numerical extension $n = 3$.

⇝ Explicit upper bound for all $n$. 