On a storage process for fluid networks with multiple Lévy input

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Presentation at EVA 2005

Gothenburg, Sweden

August, 2005
Outline of the talk

- Two-node tandem network (Mandjes, van Uitert, K.D.)
  - New representation for $Q_2$
  - Lévy case: distribution of $Q_2$
  - Examples

- n-node tandem network (Dieker, Rolski, K.D.)
  - Skorokhod problem
  - Stationary representation
  - Laplace transform
Two-node tandem network

\[ J(t) \rightarrow r_1 \rightarrow r_2 \]

- \( r_1 > r_2 > 0 \)

- \( J(t) - \text{with stationary increments}, \mathbb{E}J(1) < r_2 \)

We are interested in \( \mathbb{P}(Q_2 > u) \)

- \( Q_2 - \text{stationary buffer content at the second node} \)

- Kella, Whitt, Rubin, Shalmon, Mandjes, van Uitert,...
Following Reich's representation we have

$$Q_1 = \sup_{t \geq 0} \{ J(t) - r_1 t \}$$

and

$$Q_{\text{total}} = \sup_{t \geq 0} \{ J(t) - r_2 t \}.$$ 

Hence

$$Q_2 = \sup_{t \geq 0} \{ J(t) - r_2 t \} - \sup_{t \geq 0} \{ J(t) - r_1 t \}.$$
New representation for $Q_2$

Let

$$t_u = \frac{u}{r_1 - r_2}.$$

**Theorem 1.** For each $u \geq 0$,

$$\mathbb{P}(Q_2 > u) = \mathbb{P}\left( \sup_{t \in [t_u, \infty)} \{ J(t) - r_2 t \} - \sup_{t \in [0, t_u]} \{ J(t) - r_1 t \} > u \right).$$

This representation enables us to analyze the distribution of $Q_2$ for following classes of input processes:

- processes with independent increments
- Gaussian processes
Theorem 2. Let \( \{J(t), t \in \mathbb{R}\} \) be a stochastic process with stationary independent increments and let \( \mu = \mathbb{E}J(1) < r_2 \). Then for each \( u \geq 0 \), and \( J_1(\cdot) \) and \( J_2(\cdot) \) independent copies of the process \( J(\cdot) \),

\[
P(Q_2 > u) = \mathbb{P}\left( \sup_{t \in [0, \infty)} \{J_1(t) - r_2t\} > \sup_{t \in [0, tu]} \{-J_2(t) + r_1t\} \right).
\]
Let $J(t)$ be a spectrally positive Lévy process. Introduce

$$\theta(s) := \log(\mathbb{E}e^{-s(J(1)-r_1)}).$$

**Theorem 3.** Let $\{J(t), t \in \mathbb{R}\}$ be a spectrally positive Lévy process with $\mu := \mathbb{E}J(1) < r_2$. Then, for each $x > 0$,

$$\mathbb{E}e^{-xQ_2} = \frac{r_2 - \mu}{r_1 - r_2} \cdot \frac{\theta^{-1}(x(r_1 - r_2))}{x - \theta^{-1}(x(r_1 - r_2))}.$$

**Remark 1.** Theorem 3 can be considered as an analogue of the result of Zolotarev who obtained the Laplace transform of $\mathbb{P}(Q_1 < u)$ for $J(\cdot)$ being a spectrally positive Lévy process.
Pollaczek-Khintchine representation

**Theorem 4.** Let \( \{J(t), t \in \mathbb{R}\} \) be a spectrally positive Lévy process with \( \mu := \mathbb{E}J(1) < r_2 \). Then

\[
\mathbb{P}(Q_2 \leq u) = (1 - \varrho) \sum_{i=1}^{\infty} \varrho^{i-1} H^*(u),
\]

where

- \( \varrho := (r_1 - r_2)/(r_1 - \mu) \)
- \( H(\cdot) \) is a distribution function such that \( H(x) = 0 \) for \( x < 0 \)

and

\[
\int_0^{\infty} e^{-xv} dH(v) = \frac{\theta^{-1}(x)}{\varrho x}
\]

for \( x \geq 0 \).
Examples: exact distributions

- \( J(t) \) is a standard Brownian motion.

\[
\mathbb{P}(Q_2 > u) = \frac{r_1 - 2r_2}{r_1 - r_2} e^{-2r_2u} \left( 1 - \Psi \left( \frac{r_1 - 2r_2}{\sqrt{r_1 - r_2}} \frac{1}{\sqrt{u}} \right) \right) \\
+ \frac{r_1}{r_1 - r_2} \Psi \left( \frac{r_1}{\sqrt{r_1 - r_2}} \frac{1}{\sqrt{u}} \right),
\]

where \( \Psi(x) = \mathbb{P}(N > x) \).

\( \checkmark \) If \( c_1 > 2c_2 \), then

\[
\mathbb{P}(Q_2 > u) \sim \frac{r_1 - 2r_2}{r_1 - r_2} e^{-2r_2u}.
\]

\( \checkmark \) If \( c_1 \leq 2c_2 \), then

\[
\mathbb{P}(Q_2 > u) \sim \frac{1}{\sqrt{2\pi(r_1 - r_2)}} \frac{1}{\sqrt{u}} \exp \left( -\frac{r_1^2}{2(r_1 - r_2)u} \right).
\]

Also we can get the exact distribution if
- \( J(t) \) is a Poisson process.
Examples: asymptotic results

- $J(t) = X_{\alpha,1,\beta}(t)$ with $\alpha \in (1,2)$ and $\beta \in (-1,1]$.

  Then

  \[
  C_1 u^{1-\alpha} \leq \mathbb{P}(Q_2 > u) \leq C_2 u^{1-\alpha} \quad \text{as} \quad u \to \infty.
  \]

- $J(t) = X_{\alpha,1,1}(\cdot)$ with $\alpha \in (1,2)$.

  Then

  \[
  \mathbb{P}(Q_2 > u) \sim \frac{1}{\Gamma(2-\alpha) \cos(\pi(\alpha-2)/2)} \frac{1}{r_2} \left( \frac{r_1}{r_1 - r_2} \right)^{1-\alpha} u^{1-\alpha}.
  \]

  Also we can get the asymptotic for

  - $J(t)$ compound Poisson input, with regularly varying jumps.
n-node tandem network

\[ J(t) = (J_1(t), ..., J_n(t))' - n\text{-dimensional Lévy process with mutually independent components and } J_1(t) \text{ is a spectrally positive Lévy process, } J_2(t), \ldots, J_n(t) \text{ are subordinators} \]

- \( r = (r_1, ..., r_n)' - \text{output rates} \)
- \( P = (p_{ij})_{i,j=1}^n - \text{routing matrix} \)
  
  \[ 0 < p_{ii+1} \leq 1 \text{ and } p_{ij} = 0 \text{ if } j \neq i + 1 \]

Moreover, we tacitly assume that

\[ \mathcal{N}1 \text{ (Work-conserving)} \quad p_{ii+1} > \frac{r_{i+1}}{r_i}, \]

\[ \mathcal{N}2 \text{ (Stability)} \quad (I - P')^{-1} \mathbb{E}J(1) < r. \]
We are interested in the transient joint distribution of

- $Q(t) = (Q_1(t), ..., Q_n(t))'$ - storage process
- $B(t) = (B_1(t), ..., B_n(t))'$ - running busy period process,

where

$$B_i(t) = t - \sup\{0 \leq s \leq t : Q_i(s) = 0\}.$$ 

$Q(t)$ is the unique solution of the Skorokhod problem of $J(t) - (I - P')rt$ with reflection matrix $I - P'$, that is

- S1 $Q(t) = w + J(t) - (I - P')rt + (I - P')L(t), t \geq 0$,
- S2 $Q(t) \geq 0, t \geq 0$ and $Q(0) = w$,
- S3 $L(0) = 0$ and $L$ is nondecreasing, and
- S4 $\sum_{i=1}^{n} \int_{0}^{\infty} Q_i(t) \, dL_i(t) = 0$. 

n-node tandem network, ctd.
n-node tandem network, ctd.

**Proposition 1.** Suppose that $Q(t)$ is the storage process associated to the stochastic network $(J, r, P)$. Then

$$(I - P')^{-1}Q(t)$$

is the solution to the Skorokhod problem of

$$X(t) = (I - P')^{-1}J(t) - rt$$

with reflection matrix $I$.

**Theorem 5.** The stationary distribution $(W(\infty), B(\infty))$ is the same as the distribution of $((I - P')\bar{X}, G)$, where

$$\bar{X} = (\bar{X}_1, ..., \bar{X}_n)'$$

and $G = (G_1, ..., G_n)'$ with

$$\bar{X}_i = \sup_{t \geq 0} \left( \sum_{k=1}^{i} \left( \prod_{j=1}^{k-1} p_{jj+1} \right) J_k(t) - r_i t \right)$$

$$G_k = \sup \{ t \geq 0 : X_k(t) = \bar{X}_k(t) \}.$$
n-node tandem network, ctd.

Theorem 6. Consider a tandem stochastic network \((J, r, P)\) that N1-N2 hold. Then for \(\alpha = (\alpha_1, \ldots, \alpha_n)' > 0\), \(\beta = (\beta_1, \ldots, \beta_n)' > 0\)

\[
\mathbb{E}e^{-<\alpha, Q(\infty)> - <\beta, B(\infty)> } = \\
\mathbb{E}e^{-\alpha_nX_n - \beta_nG_n} \times \\
\prod_{j=1}^{n-1} \frac{\mathbb{E}e^{-\alpha_jX_j - \sum_{\ell=j+1}^{n} \Psi^I(\alpha_{\ell}) + \sum_{\ell=j+1}^{n} (p_{\ell-1}r_{\ell-1} - r_{\ell})\alpha_{\ell} + \sum_{p=j+1}^{n} \beta_p}G_j}{\mathbb{E}e^{-p_{j+1}\alpha_j+1X_j - \sum_{\ell=j+1}^{n} \Psi^I(\alpha_{\ell}) + \sum_{\ell=j+1}^{n} (p_{\ell-1}r_{\ell-1} - r_{\ell})\alpha_{\ell} + \sum_{p=j+1}^{n} \beta_p}G_j},
\]

where

\[
X(t) = (I - P')^{-1} J(t) - rt \\
\Psi^J_i(\lambda) = -\log \left( \mathbb{E}e^{-\lambda J_i(1)} \right).
\]

The formula can be made explicit by the use of fluctuation identities. But is a bit long...