On the extremal behavior of random variables observed at renewal times

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based on the joint work with

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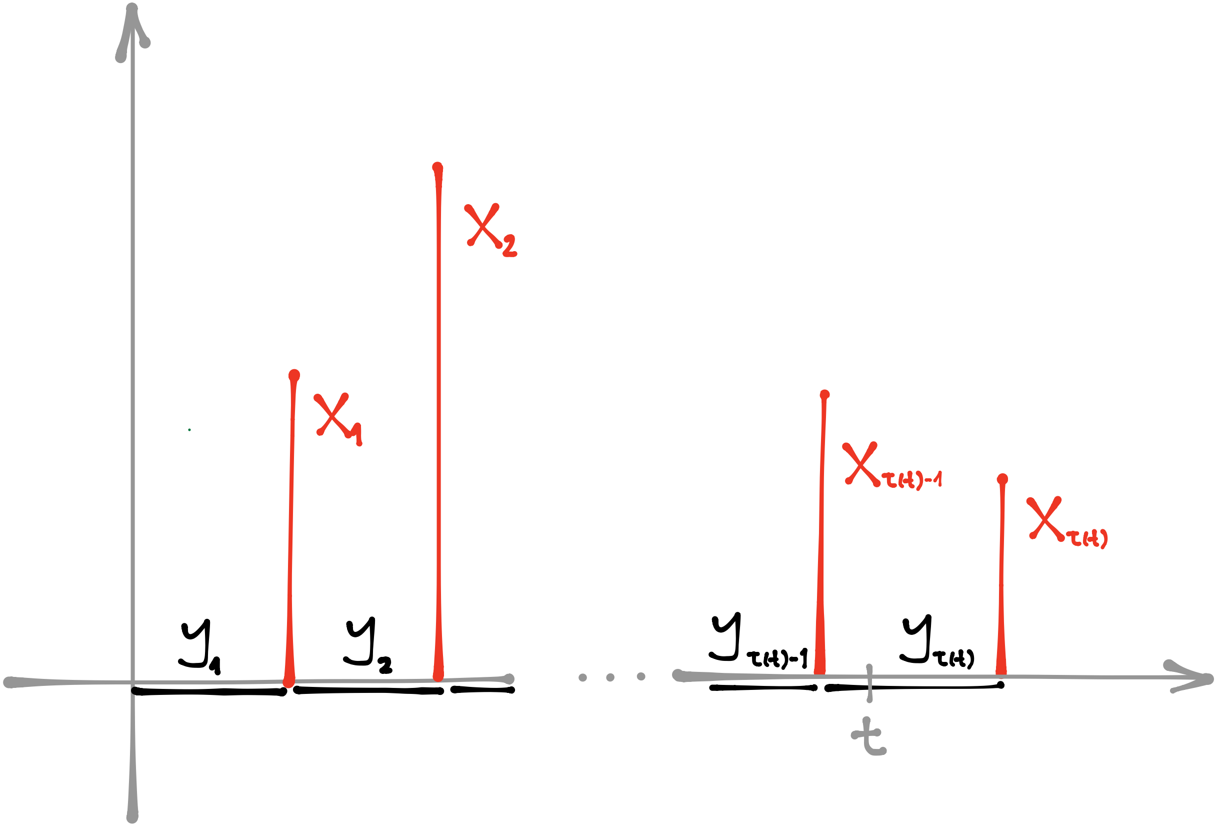
The problem



observations at renewal times

Main assumptions

- $((X_n, Y_n))_n$ are iid random vectors, with $Y_n \geq 0$
- $X_i \in \mathrm{MDA}(G)$ (enough to study $G = \Lambda$ or Φ_α)

For the sequence of interarrival times (Y_i) we study renewal process

$$\tau(t) = \inf\{k : Y_1 + \cdots + Y_k > t\}.$$

and partial maxima

$$M(t) = M^\tau(t) = \max_{i=1,\ldots,\tau(t)} X_i\,,$$

Berman and Barndorff-Nielsen in 1960's did the same for more general random variables $\tau(t)$.

Related results

an incomplete list

- Shantikumar and Sumita (1983)
- Anderson (1987)
- Silvestrov and Teugels (1998, 2004)
- Meerschaert and Scheffler (2004)
- Meerschaert and Stoev (2009)
- Pancheva, Mitov, Mitov (2009)

Our goal is to move beyond (or rather below) the maximum and understand the asymptotics for all upper order statistics of observations X_i until $\tau(t)$ and relax some assumption on the dependence between (X_n) and (Y_n)

Since $X_i \in$MDA(G) there exist functions $\tilde{a}(t)$ and $\tilde{b}(t)$ such that

$$tP\left(\frac{X_1 - \tilde{b}(t)}{\tilde{a}(t)} > x\right) \to -\log G(x),$$

but for iid (X_i) this is known to be equivalent to convergence of point processes

$$N_t = \sum_{i=1}^{\infty} \delta_{\left(\frac{i}{t}, X_{t,i}\right)}$$

towards

$$N \sim \text{PRM}(Leb \times \mu_G)$$

where $X_{t,i}$ represent appropriate affine transformations of the observations and the state space depends on MDA but can be written as

$$[0, \infty) \times \mathbb{E}$$

We actually sometimes need more general time normalization, and therefore we consider

$$N_t = \sum_{i=1}^{\infty} \delta_{\left(\frac{i}{g(t)}, X_{t,i}\right)}$$

where

$$X_{t,i} = \frac{X_i - \tilde{b}(g(t))}{\tilde{a}(g(t))} .$$

for a function $g \nearrow \infty$.

Although, often

$$g(t) = t$$

For simplicity we write $a(t) = \tilde{a}(g(t))$ and $b(t) = \tilde{b}(g(t))$.

Clearly

$$\left\{\frac{M(t)-b(t)}{a(t)} \leq x\right\} = \left\{\max_{i \leq \tau(t)} \frac{X_i - b(t)}{a(t)} \leq x\right\} = \left\{\max_{\frac{i}{g(t)} \leq \frac{\tau(t)}{g(t)}} \frac{X_i - b(t)}{a(t)} \leq x\right\}$$

Therefore

$$P\left(\frac{M(t)-b(t)}{a(t)} \leq x\right) = P\left(N_t\Big|_{[0,\frac{\tau(t)}{g(t)}]\times(x,\infty]} = 0\right)$$

On the rhs above we have an object of the form

$$N_t\big|_{[0,Z_t]\times\mathbb{E}}$$

Since $N_t \stackrel{d}{\to} N$, if this happens jointly with

$$Z_t \stackrel{d}{\to} Z$$

one could expect

$$N_t\big|_{[0,Z_t]\times\mathbb{E}} \stackrel{d}{\to} N\big|_{[0,Z]\times\mathbb{E}}$$

Lemma

Assume

$$(N_t, Z_t) \xrightarrow{d} (N, Z)$$

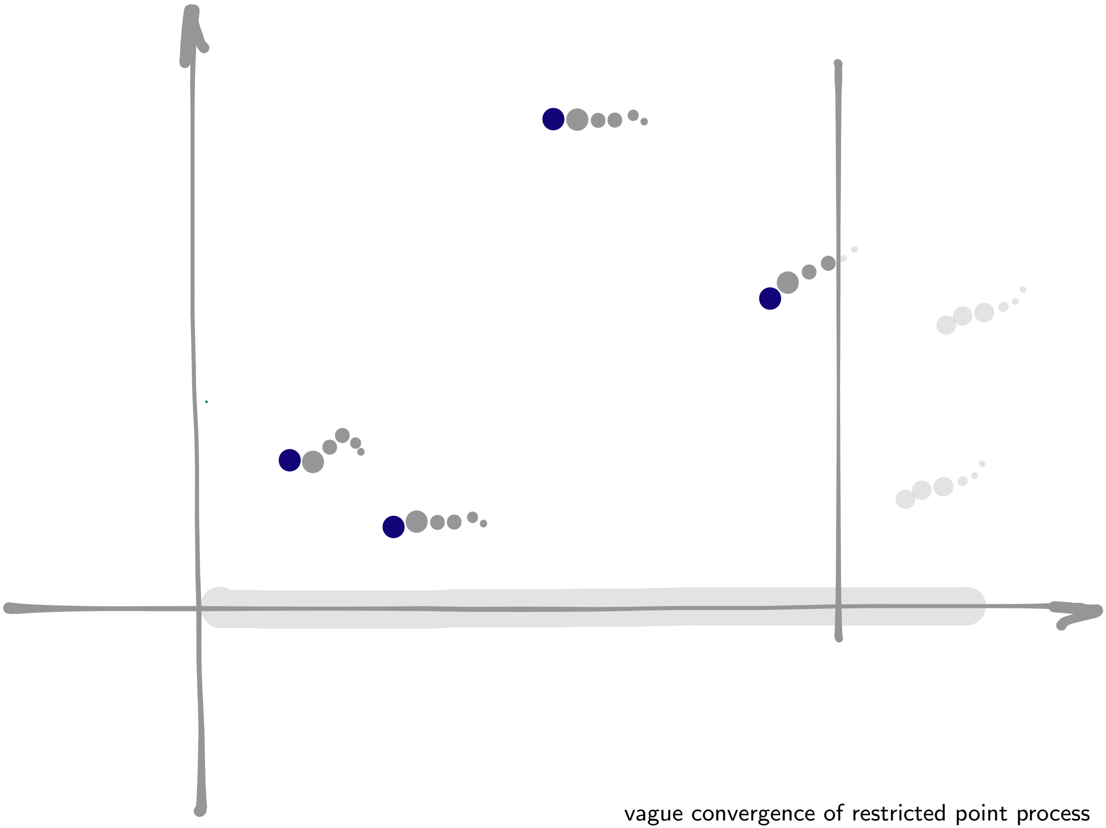
as $t \to \infty$ and

$$P(N(\{Z\} \times \mathbb{E}) > 0) = 0$$

then

$$N_t\big|_{[0,Z_t]\times\mathbb{E}} \xrightarrow{d} N\big|_{[0,Z]\times\mathbb{E}}$$

as $t \to \infty$.



vague convergence of restricted point process

The finite mean case

Assume $\mu = EY_1 < \infty$, the by SLLN

$$\frac{\tau(t)}{t} \stackrel{\text{a.s.}}{\rightarrow} \frac{1}{\mu},$$

and therefore for $g(t) = t$

$$(N_t, \frac{\tau(t)}{t}) \stackrel{d}{\rightarrow} (N, \frac{1}{\mu})$$

Remark i) dependence between X_i's and Y_i's is irrelevant ii) all upper order statistics are covered iii) Y_i's do not have to be iid actually

Example

Erdös and Rényi 1970

Note

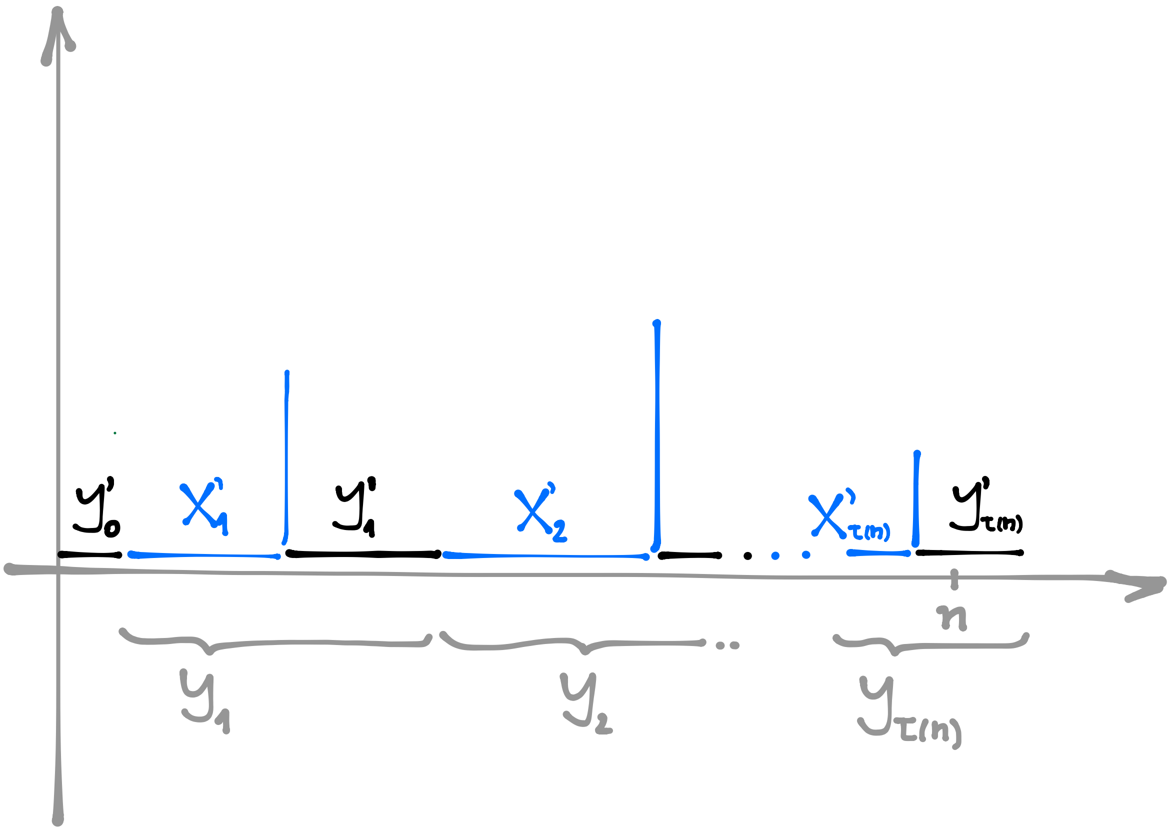
- ▷ runs of heads $X_i' \sim \text{Geom}(p)$
- ▷ runs of tails $Y_i' \sim \text{Geom}(q)$
- ▷ initial $Y_0' \sim \text{Geom}(q)$ but on $0, 1, \ldots$

We study

$$L_n = M_{\tau(n)} = \max\{X_i' : 1 \leq i \leq \tau(n)\}$$

where

$$\tau(n) = \inf\{k : Y_0' + \sum_{i=1}^{k}(X_i' + Y_i') > n\}$$



Y'_0
X'_1
Y'_1
X'_2
$X'_{\tau(n)}$
$Y'_{\tau(n)}$
n
Y_1
Y_2
$Y_{\tau(n)}$

Although geometric rv's do not belong to any MDA, still they are not far since we can always write

$$X_i' = \lfloor X_i \rfloor + 1$$

for an iid sequence

$$X_i \sim \mathsf{Exp}(-\ln p)$$

which satisfies $X_i \in \mathsf{MDA}(\Lambda)$

Set $Y_0 = Y_0'$ and

$$Y_i = X_i' + Y_i'$$

clearly corresponding renewal process satisfies

$$\frac{\tau(t)}{t} \stackrel{\text{a.s.}}{\to} \frac{1}{\mu}$$

with

$$\mu = EY_1 = \frac{1}{pq}$$

Finally

$$M^{\tau}(t) - \log_{1/p}(npq) \leq L_n - \log_{1/p}(npq) \leq M^{\tau}(t) + 1 - \log_{1/p}(npq)$$

and since the lhs and the rhs converge to G and $G + 1$ by our results, it follows that

$$L_n - \log_{1/p}(npq)$$

is tight in distribution and not far from G as well cf. EKM, but much more can be said.

The infinite mean case

As before assume

- $((X_n, Y_n))_n$ are iid random vectors, with $Y_n \geq 0$
- $X_i \in \text{MDA}(G)$ (enough to study $G = \Lambda$ or Φ_α)
- $Y_i \sim \text{RV}(\alpha)$, $\alpha \in (0, 1)$

Therefore for (d_n) such that

$$nP(Y_1 > d_n) \to 1$$

we have

$$d_n^{-1}(Y_1 + \cdots + Y_n) \xrightarrow{d} S_\alpha$$

Moreover

$$S_n(s) = \frac{1}{d_n} \sum_{i=1}^{\lfloor ns \rfloor} Y_i \xrightarrow{d} S_\alpha(s), \quad s \geq 0$$

in $D[0,\infty)$ with J_1 metric, for a positive α-stable process $S_\alpha(\cdot)$

It is known that

$$S_\alpha(s) = \sum_{T_i \le s} P_i$$

where

$$\sum_i \delta_{T_i, P_i} \sim \mathsf{PRM}(Leb \times d(-u^{-\alpha}))$$

It will be useful to study S_t indexed over all $t \in [0, \infty)$, with normalization

$$d_t = d_{\lfloor t \rfloor} \,.$$

One can also find an asymptotic inverse $\tilde{d}$ of d such that

$$d(\widetilde{d}(t)) \sim \widetilde{d}(d(t)) \sim t$$

see Seneta (1976) for instance.

Recall

$$\tau(t) = \inf\{k : Y_1 + \cdots + Y_k > t\}.$$

thus

$$\frac{\tau(t)}{\tilde{d}(t)} = \frac{1}{\tilde{d}(t)} \inf\{k : Y_1 + \cdots + Y_k > t\}$$

$$= \inf\{s : \frac{1}{t} \sum_{i=1}^{\lfloor \tilde{d}(t)s \rfloor} Y_i > 1\} \approx S^{\leftarrow}_{\tilde{d}(t)}(1)$$

since $d(\widetilde{d}(t)) \sim t$.

It is known that

$$S_t^{\leftarrow}(1) \xrightarrow{d} W_\alpha := S_\alpha^{\leftarrow}(1)$$

where W_α has Mittag-Leffler distribution. This is even true on the level of stoch. processes. Since $\tilde{d}(t) \to \infty$ also

$$\frac{\tau(t)}{\tilde{d}(t)} \xrightarrow{d} W_\alpha \, .$$

Suppose X_i and Y_i are independent. Then jointly

$$N_t \stackrel{d}{\to} N \sim \mathsf{PRM}(Leb \times \mu)$$

and

$$\frac{\tau(t)}{\tilde{d}(t)} \stackrel{d}{\to} W_\alpha \,,$$

with N independent of W_α.

In particular

$$P\left(\frac{M(t)-b(t)}{a(t)} \leq x\right) = P\left(N_t\big|_{[0,\frac{\tau(t)}{\tilde{d}(t)}]\times(x,\infty]} = 0\right)$$

as $t\to\infty$ converges to

$$P\left(N\big|_{[0,W_\alpha]\times(x,\infty]} = 0\right) = \cdots = E\left(G(x)^{W_\alpha}\right)$$

cf. Berman (1962).

Dependence between X_i and Y_i

Define

$$U_X = \frac{1}{1 - F_X} \text{ and } U_Y = \frac{1}{1 - F_Y}$$

Note that $\widetilde{d}(t) \sim U_Y(t)$. One can describe the limit quite precisely even under some sorts of dependence between observations and interarrival times.

▷ **Asymptotic tail independence**

$$\lim_{x \to \infty} P(X_1 > U_X^{\leftarrow}(x) | Y_1 > U_Y^{\leftarrow}(x)) = 0 \ .$$

▷ **Asymptotic full tail dependence**

$$\lim_{x \to \infty} P(X_1 > U_X^{\leftarrow}(x) | Y_1 > U_Y^{\leftarrow}(x)) = 1 \ .$$

Now we need to include interarrival times in the point processes and define

$$N_t = \sum_{i=1}^{\infty} \delta_{\left(\frac{i}{\widetilde{d}(t)}, X_{t,i}, Y_{t,i}\right)}$$

with

$$Y_{t,i} = \frac{Y_i}{t}$$

Tail independence

In this case (see Thm 6.2.3 in de Haan & Ferreira)

$$\widetilde{d}(t)P\left((X_{t,i},Y_{t,i})\in\cdot\right)\xrightarrow{v}\mu_0(\cdot)$$

for a measure μ_0 **concentrated on the axes** and given as

$$\mu_0(([-\infty,x]\times[0,y])^c)=e^{-x}+y^{-\alpha}\ .$$

if $X_1\in$ MDA(Λ) or as

$$\mu_0(([0,x]\times[0,y])^c)=x^{-\beta}+y^{-\alpha}\ .$$

if $X_1\in \mathrm{MDA}(\Phi_\beta)$

Recall that

$$\widetilde{d}(t)P\left((X_{t,i}, Y_{t,i}) \in \cdot\right) \xrightarrow{v} \mu_0(\cdot)$$

is necessary and sufficient for

$$N_t \xrightarrow{d} N\,,$$

where N is $\mathrm{PRM}(\lambda \times \mu_0)$

Similarly as before we obtain the joint convergence

$$(N_t, \frac{\tau(t)}{\tilde{d}(t)}) \xrightarrow{d} (N, W_\alpha)$$

with $W_\alpha =$ the first hitting time of the level 1 by a positive α-stable process S_α (note $\tau(t)/\tilde{d}(t)$ is just a transformation of N_t)

Moreover, if

$$N = \sum_i \delta_{T_i, P_i^X, P_i^Y}$$

then

$$N' = \sum_i \delta_{T_i, P_i^X} \text{ and } W_\alpha$$

are independent.

Therefore

$$P\left(\frac{M(t)-b(t)}{a(t)} \leq x\right) = P\left(N'_t\Big|_{[0,\frac{\tau(t)}{\tilde{d}(t)}]\times(x,\infty]} = 0\right)$$

as $t \to \infty$ converges to

$$E\left(G(x)^{W_\alpha}\right)$$

as before.

Full dependence

de Haan & Resnick 1977

Again

$$N_t \stackrel{d}{\to} N \,,$$

where N is $\mathrm{PRM}(\lambda \times \mu_0)$ with μ_0 concentrated on the set

$$C = \{(u,v) \in (-\infty,\infty) \times (0,\infty) : e^{-u} = v^{-\alpha}\}, \quad \text{if } G_1 = \Lambda$$

$$C = \{(u,v) \in (0,\infty) \times (0,\infty) : u^{-\beta} = v^{-\alpha}\}, \quad \text{if } G_1 = \Phi_\beta \,.$$

and

$$\mu_0(\{(u,v) : (u,v) \in C,\ v > y\}) = y^{-\alpha} \,,$$

for $y > 0$.

Again

$$(N_t, \frac{\tau(t)}{\tilde{d}(t)}) \xrightarrow{d} (N, W_\alpha)$$

However, N' and W_α are not independent, just the opposite since if $X_i, Y_i \in$ MDA(α) for instance, then

$$N = \delta_{T_i, P_i, P_i}$$

Conclusion

- ▷ point processes approach turns out to be simple and very efficient
- ▷ all upper order statistics are covered
- ▷ extremal process can be understood as well