The Extremogram in Space (and Time):

Richard A. Davis*
Columbia University

Collaborators:
Yongbum Cho, Columbia University
Souvik Ghosh, LinkedIn
Claudia Klüppelberg, Technical University of Munich
Christina Steinkohl, Technical University of Munich

* Support of the Villum Kann Rasmussen Foundation is gratefully acknowledged.

Plan

- Extremogram in space
  - lattice vs continuous space
- Estimating extremogram—random pattern
- Limit theory for empirical extremogram
- Simulation examples
Extremal Dependence in Space and Time

Setup: Let $X(s)$ be a stationary (isotropic?) spatial process defined on $s \in \mathbb{R}^2$ (or on a regular lattice $s \in \mathbb{Z}^2$).

Space-time domain: $[(s, t) \in \mathbb{R}^d \times [0, \infty)]$
Lattice vs cont space

Setup: Let $X(s)$ be a RV stationary (isotropic?) spatial process defined on $s \in \mathbb{R}^2$ (or on a regular lattice $s \in \mathbb{Z}^2$). Consider the former—latter is more straightforward.

\[
\rho_{ABA} (h) = \lim_{x \to \infty} P(X(s + h) \in XB \mid X(s) \in xA), \quad h \in \mathbb{R}^2
\]

Regular grid

- $h = 1$; # of pairs = 4
- $h = \sqrt{2}$; # of pairs = 4
- $h = \sqrt{10}$; # of pairs = 8
- $h = \sqrt{18}$; # of pairs = 4

Random pattern
Random pattern

Estimate of extremogram at lag h = 1 for red point: weight
"indicators of points" in the buffer.
Bandwidth: half the width of the buffer.
Random pattern

Estimating extremogram--random pattern

Setup: Suppose we have observations, $X(s_1), \ldots, X(s_N)$ at locations $s_1, \ldots, s_{N_n}$ of some Poisson process $N$ with rate $\nu$ in a domain $S_n \uparrow \mathbb{R}^2$.

Here, $N_n = N(S_n) =$ number of Poisson points in $S_n$. $N_n \sim \text{Pois}(\nu|S_n|)$.

Weight function $w_n(x)$: Let $w(\cdot)$ be a bounded pdf and set

$$w_n(x) = \frac{1}{\lambda_n^2} w\left(\frac{x}{\lambda_n}\right),$$

where the bandwidth $\lambda_n \to 0$ and $\lambda_n^2 |S_n| \to \infty$. 

Note:
- Expanding domain asymptotics: domain is getting bigger.
- Not infill asymptotics: insert more points in fixed domain.


**Estimating extremogram--random pattern**

\[ \rho_{A,B}(h) = \lim_{x \to \infty} P(X(s + h) \in xB, X(s) \in xA) / PX(s) \in xA), \quad h \in \mathbb{R}^2 \]

Kernel estimate of \( \rho \):

\[
\hat{\rho}_{A,B}(h) = \frac{m_n}{v^2 |S_n|} \sum_{i,j=1}^{N_n} w_n(h - s_i + s_j) I(X(s_i) \in a_mB) I(X(s_j) \in a_mA),
\]

\[
\hat{\rho}_{A,B}(h) = \frac{m_n}{v|S_n|} \int_{S_n} w_n(h + s_1 - s_2) I(X(s_1) \in a_mB) I(X(s_2) \in a_mA) N^2(ds_1, ds_2)
\]

\[
\frac{m_n}{v|S_n|} \int_{S_n} I(X(s) \in a_mA) N(ds)
\]

**Note:** \( N^2(ds_1, ds_2) = N(ds_1)N(ds_2)I(s_1 \neq s_2) \) is product measure off the diagonal.

---

**Limit Theory**

**Theorem:** Under suitable conditions on \( X(s) \), (i.e., regularly varying, mixing, local uniform negligibility, etc.), then

\[
\left( \frac{|S_n|^2}{m_n} \right)^{\frac{1}{2}} \left( \hat{\rho}_{A,B}(h) - \rho_{A,B,m}(h) \right) \to N(0, \Sigma),
\]

where \( \rho_{A,B,m}(h) \) is the pre-asymptotic extremogram,

\[
\rho_{A,B,m}(h) = P(X(s + h) \in a_mB, X(s) \in a_mA) / PX(s) \in a_mA), \quad h \in \mathbb{R}^2,
\]

\((a_m \text{ is the } 1 - 1/m \text{ quantile of } |X(s)|).)

**Remark:** The formulation of this estimate and its proof follow the ideas of Karr (1986) and Li, Genton, and Sherman (2008).
Limit Theory

Asymptotic “unbiasedness”: \( \hat{\rho}_{AB}(h) \) is a ratio of two terms;

\[
\hat{\rho}_{AB}(h) = \frac{\hat{t}_{AB,m}(h)}{\hat{t}_{A,m}}
\]

will show that both are asymptotically unbiased.

Denominator: By RV, stationarity, and independence of \( N \) and \( \{X(s)\} \),

\[
E \hat{t}_{A,m} = E \left( \frac{m_n}{v[S_n]} \int_{S_n} I(X(s) \in a_m A) N(ds) \right)
\]

\[
= \frac{m_n}{v[S_n]} P(X(0) \in a_m A) E(N(S_n))
\]

\[
= m_n P(X(0) \in a_m A)
\]

\[
\rightarrow \mu(A)
\]

Limit Theory

Numerator:

\[
E \left( \frac{m_n}{v[S_n]} \int_{S_n} \int_{S_n} w_n(h + s_1 - s_2) I(X(s_1) \in a_m B) I(X(s_2) \in a_m A) N^2(ds_1, ds_2) \right)
\]

\[
= \frac{m_n}{v[S_n]} \int_{S_n} \int_{S_n} w_n(h + s_1 - s_2) P(X(0) \in a_m B, X(s_2 - s_1) \in a_m A) v^2 ds_1 ds_2
\]

\[
= \frac{1}{|S_n|} \int_{S_n} \int_{S_n} \frac{1}{\lambda_n} \left( \frac{h + s_1 - s_2}{\lambda_n} \right) \tau_m (s_2 - s_1) ds_1 ds_2
\]

where \( \tau_m (h) = m P(X(0) \in a_m B, X(h) \in a_m A) \). Making the change of variables \( y = \frac{h + s_1 - s_2}{\lambda_n} \) and \( u = s_2 \), the expected value is

\[
\frac{1}{|S_n|} \int_{S_n} \int_{S_n} w(y) \tau_m (h - \lambda_n y) du dy
\]

\[
= \int_{S_n} \int_{S_n} w(y) \tau_m (h - \lambda_n y) dy |S_n \cap (S_n - \lambda_n y + h)| / |S_n|
\]
Limit Theory

\[ \int_{S_n-S_n+h} w(y) \tau_m (h, \lambda_n y) dy \frac{|S_n \cap (S_n - \lambda_n y + h)|}{|S_n|} \]

\[ \rightarrow \int_{\mathbb{R}^2} w(y) \tau_{A,B} (h) dy = \tau_{A,B} (h). \]

Remark: We used the following in this proof.

- \( \frac{|S_n \cap (S_n - \lambda_n y + h)|}{|S_n|} \rightarrow 1 \) and \( \frac{s_n - s_n + h}{\lambda_n} \rightarrow \mathbb{R}^2 \).

- \( \tau_m (h, \lambda_n y) = mP (X(0) \in a_m B, X(h - \lambda_n y) \in a_m A) \rightarrow \tau_{A,B} (h). \)

Need a condition for the latter.

Limit theory

Local uniform negligibility condition (LUNC): For any \( \epsilon, \delta > 0 \), there exists a \( \delta' \) such that

\[ \lim sup_n nP \left( \sup_{|k| < \delta'} \frac{|X_k - X_{\alpha}|}{a_n} > \delta \right) < \epsilon. \]

Proposition: If \( (X(s)) \) is a strictly stationary regularly varying random field satisfying LUNC, then for \( \lambda_m \rightarrow 0 \),

\[ mP \left( X(0) \in A, \frac{X(s + \lambda_m)}{a_m} \in B \right) \rightarrow \tau_{A,B} (s) \]

This result generalizes to space points, \( 0, s_1 + \lambda_m, \ldots, s_k + \lambda_m \).
Limit Theory

Outline of argument:

- Under LUNC already shown asymptotic unbiasedness of numerator and denominator.
  
  \[
  E \hat{r}_{A,m} \to \mu(A)
  \]
  
  \[
  E \hat{r}_{A,B,m}(h) \to \tau_{A,B}(h)
  \]

  with \( \rho_{A,B}(h) = \tau_{A,B}/\mu_A(h) \).

Strategy: Show joint asymptotic normality of \( \hat{r}_{A,m} \) and \( \hat{r}_{A,B,m}(h) \)

\[
\frac{S_n}{m_n} \text{var} \left( \hat{r}_{A,m} \right) \to \frac{\mu(A)}{\nu} + \int_{\mathbb{R}^2} \tau_{AA}(y)dy \Rightarrow \hat{r}_{A,m}(h) \to p \mu_A(h)
\]

Limit Theory

Step 1: Compute asymptotic variances and covariances.

i. \[
\frac{S_n}{m_n} \text{var} \left( \hat{r}_{A,m} \right) \to \frac{\mu(A)}{\nu} + \int_{\mathbb{R}^2} \tau_{AA}(y)dy
\]

ii. \[
\left( \frac{S_n}{m_n} \right)^2 \text{var} \left( \hat{r}_{A,B,m}(h) \right) \to \frac{1}{\nu^2} \tau_{AB}(h) \int_{\mathbb{R}^2} w^2(y)dy
\]

Proof of (i): Sum of variances + sum of covariances

\[
\frac{S_n}{m_n} E \left( \hat{r}_{A,m}^2 \right) = \frac{m_n}{\nu^2 |S_n|} \int_{S_n} I(X(s_1) \in a_mA) N(dS_1) + \frac{m_n}{\nu |S_n|} \int_{S_n} \int_{S_n} I(X(s_1) \in a_mA, X(s_2) \in a_mA) dN^2(dS_1, dS_2)
\]

\[
\to \frac{\mu(A)}{\nu} + \int_{\mathbb{R}^2} \tau_{AA}(y)dy
\]
Step 2: Show joint CLT for $\tilde{r}_{A,m}$ and $\tilde{r}_{A,B,m}(h)$ using a blocking argument.

Idea: Focus on $\tilde{r}_{A,B,m}(h)$. Set

$A_n = \left(\frac{m_n^{1/2}}{|S_n|}\right)^{1/2} \frac{1}{\nu^2} \int_{S_n} \int_{S_n} w_n(h + s_1 - s_2) I(X(s_1) \in A_m) I(X(s_2) \in A_m B) N^2(ds_1, ds_2)$

and put $\tilde{A}_n = A_n - E(A_n)$. We will show $\tilde{A}_n$ is asymptotically normal.

Subdivide $S_n = [0, n]^2$ into big blocks and small blocks.

$S_n = \bigcup_{i=1}^{K_n} D_i$ where $D_i$ has dimensions $n^\alpha \times n^\alpha$ and size $|D_i| = n^{2\alpha}$
Limit Theory

Subdivide $B_n$ into blocks and sub-blocks:

$$|B_n| = \left( \sum_{i=1}^{k_n} D_i(n^{\gamma}) \right)^2$$

where $D_i$ has dimension $n^\alpha \times n^\alpha$ and size $|D_i| = n^{2\alpha}$.

---

Limit Theory

Recall that $A_n$ is a (mean-corrected) double integral over $S_n \times S_n$, i.e.,

$$A_n = \int_{S_n \times S_n} w_n(h + s_1 - s_2)H(s_1, s_2)N^{(2)}(ds_1, ds_2)$$

$$= \sum_{i=1}^{k_n} \int_{D_i \times D_i} w_n(h + s_1 - s_2)H(s_1, s_2)N^{(2)}(ds_1, ds_2)$$

$$= \sum_{i=1}^{k_n} \int_{B_i \times B_i} w_n(h + s_1 - s_2)H(s_1, s_2)N^{(2)}(ds_1, ds_2) + R_n$$

$$= \sum_{i=1}^{k_n} \tilde{a}_{ni} + \tilde{A}_n - \sum_{i=1}^{k_n} \tilde{a}_{ni}$$
Limit Theory

Remaining steps: $\bar{a}_{ni} = \int_{B_i \times \bar{B}_i} w_n(h + s_1 - s_2) H(s_1, s_2) N^{(C)}(ds_1, ds_2)$

i. Show $\text{var} \left( \bar{A}_n - \sum_{i=1}^{k_n} \bar{a}_{ni} \right) \rightarrow 0$.

ii. Let $(\varepsilon_{ni})$ be an iid sequence with $\varepsilon_{ni} \xrightarrow{d} \bar{a}_{ni}$ whose sum has characteristic function $\phi_n^C(t)$. Show $\phi_n^C(t) \rightarrow \exp \left( -\frac{\sigma^2}{2} t^2 \right)$.

iii. $\phi_n(t) - \phi_n(t) \rightarrow 0$.

Intuition.

(i) The sets $D_i \setminus B_i$ are small by proper choice of $\alpha$ and $\eta$.

(ii) Use a Linapounov CLT (have a triangular array).

(iii) Use a Bernstein argument (see next page).

Useful identity: $\prod_{i=1}^{k_n} a_i - \prod_{j=1}^{k_n} b_j = \sum_{i=1}^{k_n} a_i \cdots a_{i-1}(a_i - b_i)b_{i+1} \cdots b_k$

$$|\phi_n(t) - \phi_n(t)| = |E \prod_{i=1}^{k_n} e^{lt \bar{a}_{ni}} - E \prod_{i=1}^{k_n} e^{lt \bar{a}_{ni}}|$$

$$= |E \sum_{i=1}^{k_n} \prod_{j=1}^{i-1} e^{lt \bar{a}_{ni}} (e^{lt \bar{a}_{ni}} - e^{lt \bar{c}_{ni}}) \prod_{j=i+1}^{k_n} e^{lt \bar{c}_{ni}}|$$

$$\leq \sum_{i=1}^{k_n} |\text{cov}(\prod_{j=1}^{i-1} e^{lt \bar{a}_{ni}}, e^{lt \bar{a}_{ni}})| \quad \text{(by indep of } \varepsilon_{ni})$$

$$\leq \sum_{i=1}^{k_n} |E(\text{cov}(\prod_{j=1}^{i-1} e^{lt \bar{a}_{ni}}, e^{lt \bar{a}_{ni}})[N])|$$

$$\leq \sum_{i=1}^{k_n} 4E \alpha_{(r,s)}(N(\cup_{j=1}^{i-1} B_j) \cap N(B_0))(n^0)$$

where $\alpha_{(r,s)}(h)$ is a strong mixing bounding function that is based on the separation $h$ between two sets $U$ and $V$ with cardinality $r$ and $s$. 
Strong mixing coefficients

Let $X(s)$ be a stationary random field on $\mathbb{R}^2$. Then the mixing coefficients are defined by

$$\alpha_{j,k}(h) = \sup_{E_1,E_2} |P(A \cap B) - P(A)P(B)|,$$

where the sup is taking over all sets $A \in \sigma(E_1), B \in \sigma(E_2)$, with $\text{card}(E_1) \leq j, \text{card}(E_2) \leq k$, and $d(E_1,E_2) \geq h$.

Proposition (Li, Genton, Sherman (2008), Ibragimov and Linnik (1971)): Let $U$ and $V$ be closed and connected sets such that $|U| \leq s, |V| \leq t$ and $d(U,V) \geq h$. If $X$ and $Y$ are rvs measurable wrt $\sigma(U)$ and $\sigma(V)$, respectively, and bded by 1, then

$$\text{cov}(X,Y) \leq 4\alpha_{s,t}(h)$$

$$(16\alpha_{s,t}(h) \text{ if } X,Y \text{ complex}).$$

Limit Theory

Mixing condition: $\sup_s \alpha_{s,s}(h)/s = O(h^{-\epsilon})$ for some $\epsilon > 2$.

Returning to calculations:

$$|\phi_n(t) - \phi_n^C(t)| = \sum_{i=1}^{k_n} |\text{cov}(\prod_{j=1}^{i-1} e^{i\alpha_{i-j} t}, e^{i\alpha_{n-i} t})|$$

$$\leq \sum_{i=1}^{k_n} 16E\alpha_{N(U^{i-1}_{j=1} B_j),N(B_j)}(n^\eta)$$

$$\leq \sum_{i=1}^{k_n} 16EN(U^{i-1}_{j=1} B_j)n^{-\eta}$$

$$\leq \sum_{i=1}^{k_n} 16i n^{2\alpha_n-\eta} \leq Ck_n^2 n^{2\alpha_n-\eta}$$

$$= Cn^{4-2\alpha-\eta} \rightarrow 0 \text{ if } (4 - 2\alpha - \eta < 0).$$
Simulations of spatial extremogram

Extremogram for one realization of B-R process
(function of level)

Note: black dots = true; blue bands are permutation bounds

Lattice

Non-Lattice

Box-plots based on 1000 (100) replications of MMA(1) (left) and BR (right)

Lattice

Non-lattice;
\[ \lambda_n = 1 / \log n \] (left)
\[ \lambda_n = 5 / \log n \] (right)