Solvency Capital Requirement calculation for different hedge funds strategies.

in collaboration with T. Mikosch and S. Darolles

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1 Motivation: realistic SCR calculation
2 Peaks Over Threshold approach
3 Precise large deviations in the iid case
4 Precise large deviations in the dependent case
Outline

1. Motivation: realistic SCR calculation
2. Peaks Over Threshold approach
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4. Precise large deviations in the dependent case
**Definition (Solvency Capital Requirement)**

"The SCR is the capital required to ensure that the (re)insurance company will be able to meet its obligations over the next 12 months with a probability of at least 99.5%." (Wikipedia)

**Solvency II: a challenge for the mathematician**

Use a standard formula or find a more realistic calculation of the SCR (quantiles, VaR) using an internal model.

Extrapolation of the magnitude of an event that occurs once per 200 years, i.e. that is not observed!
Motivation: insurance companies as Hedge Fund investors

Standard formula for ”other equities”

The capital requirement is 48% of the investment whatever is the HF strategy.

Annualized returns Lyxor indices
Limitation of the standard formula

Log ratios \( r_t = \log(P_{t+1}/P_t) \) where \( (P_t) \) are weakly prices of HF indices.

3 different strategies, same SCRs under standard formula
Outline

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Gaussian modeling works on average

Convertible.Bonds.Arbitrage

N = 560   Bandwidth = 0.085
Density

Dark: marginal density of the log-ratio of an index
Red: Gaussian model
Blue: Gaussian model excluding the 40 worst days
Gaussian modeling underestimate tails

\[ \mathbb{P}\left( \frac{P_{T+1} - P_T}{P_T} \leq SCR \right) = 0.005 \quad \sim \quad \mathbb{P}\left( \log \left( \frac{P_{T+1}}{P_T} \right) \leq SCR \right) = 0.005 \]

\[ \approx \quad \mathbb{P}\left( \sum_{t=T}^{T+52} r_t \leq SCR \right) = 0.005. \]

Modern portfolio theory, Markowitz (1952)
Peaks Over Threshold approach

To extrapolate, "Let the tails speak by themselves"! (EKM, 1997)

**Theorem (Pickands-Balkema-de Haan, 1975-1974)**

Let \((X_t)\) iid with distribution \(F\). Denote \(F_u\) the distribution of the exceedances

\[
F_u(x) = \mathbb{P}(X - u \leq x \mid X > u), \quad x \geq 0, \quad X \sim F.
\]

Then for a large class of underlying distribution functions \(F\) with endpoint \(x_F\)

\[
\lim_{u \to x_F} \sup_{0 < x < x_F - u} |F_u(x) - G_{\xi,\sigma}(x)| = 0
\]

where \(G_{\xi,\sigma}\) is the Generalized Pareto Distribution

\[
G_{\xi,\sigma}(x) = \begin{cases} 
1 - (1 - \xi x / \sigma)^{-1/\xi}, & \text{if } \xi \neq 0, \\
1 - e^{-x / \sigma}, & \text{if } \xi = 0.
\end{cases}
\]
Regularly varying distribution

Extrapolation to high quantiles possible iff $\xi > 0$ iff $X \sim F$ is an $\alpha = \xi^{-1}$ regularly varying r.v.:

**Definition (Feller, 1971)**

$\exists p, q \geq 0$ with $p + q = 1$ and a slowly varying function $L$ such that

$$
\mathbb{P}(X > x) \sim p \frac{L(x)}{x^\alpha} \quad \text{and} \quad \mathbb{P}(X \leq -x) \sim q \frac{L(x)}{x^\alpha}, \quad x \to \infty.
$$

POT approach: fit by ML a GPD on the exceedances under low thresholds. $\hat{\xi}_u$:
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The iid case

**Theorem (A.V. Nagaev, 1969)**

If \((X_i)\) iid random variables with \(\alpha > 0\) regularly varying (centered if \(\alpha > 1\)) distribution then \(S_n = \sum_{i=1}^{n} X_i\) satisfies the **precise large deviations** relation

\[
\lim_{n \to \infty} \sup_{x \geq b_n} \left| \frac{\mathbb{P}(S_n > x)}{n \mathbb{P}(|X| > x)} - p \right| = 0 \quad \text{and} \quad \lim_{n \to \infty} \sup_{x \geq b_n} \left| \frac{\mathbb{P}(S_n \leq -x)}{n \mathbb{P}(|X| > x)} - q \right| = 0
\]

with \(b_n = n^{\delta+1/\left(\alpha^2\right)}\) for any \(\delta > 0\).

Assume \((r_t)\) iid \(\Rightarrow \mathbb{P}\left( \sum_{j=T}^{T+52} r_t \leq \text{SCR} \right) = 0.005 \sim \mathbb{P}\left( r_t \leq \text{SCR} \right) = 0.0001\).
SCR extrapolation, iid case

SCR with POT approach based on iid \((r_1, \ldots, r_n)\)

\[
\hat{SCR} = \max_{1 \leq m \leq 40} u_m + \hat{\sigma} \left[ \left( \frac{n \ast 0.0001}{m} \right)^{-\hat{\xi}} - 1 \right]
\]

where \(m\) is the number of exceedances below \(u_m < 0\).
SCR extrapolation when extremes cluster

What is happening for dependent sequences for whom extremes cluster?
1 Motivation: realistic SCR calculation
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Regularly varying processes

Definition (Basrak & Segers, 2009)
A stationary sequence \((X_t)\) is regularly varying of order \(\alpha > 0\) if \(\exists\) spectral tail process \((\Theta_t)\) defined for any \(k \geq 0,\) any \(u > 0\) by the relation

\[
P(|X_0|^{-1}(X_0, \ldots, X_k) \in \cdot, |X_0| > ux \mid |X_0| > x) \xrightarrow{w} u^{-\alpha} P((\Theta_0, \ldots, \Theta_k) \in \cdot).
\]
Large deviations in the $m$-dependent case

Assume $(X_t, t \leq 0)$ is independent of $\sigma(X_t, t \geq m)$ then $\Theta_t = 0$ for $|t| \geq m$.

**Theorem (Mikosch & W., 2012)**

Assume $(X_t)$ is $\alpha > 0$ regularly varying (centered if $\alpha > 1$) distribution then

$$
\lim_{n \to \infty} \sup_{x \geq b_n} \left| \frac{P(S_n > x)}{n \frac{1}{P(|X| > x)}} - b_+ \right| = 0 \text{ and } \lim_{n \to \infty} \sup_{x \geq b_n} \left| \frac{P(S_n \leq -x)}{n \frac{1}{P(|X| > x)}} - b_- \right| = 0,
$$

with $b_n = n^{\delta + 1/\alpha^2}$ for any $\delta > 0$ and *cluster indices*

$$
b_\pm = \mathbb{E} \left[ \left( \sum_{t=0}^{m-1} \Theta_t \right)_\pm^\alpha - \left( \sum_{t=1}^{m-1} \Theta_t \right)_\pm^\alpha \right] = \mathbb{E} \left[ \left( \sum_{t=0}^{m-1} \Theta_t 1_{\Theta_\pm = 0, 0 < j < m} \right)_\pm^\alpha \right].
$$
**Examples**

**Definition (Conditional spectral tail process)**

Define for $m$-dependent RV$(\alpha)$ processes $(\Theta'_0, \ldots, \Theta'_m) = (\Theta_0, \ldots, \Theta_k)$ conditionally to $\Theta_{-j} = 0$, $0 < j < m$.

1. $(X_t)$ iid then $\Theta'_0 = 1$,
2. $X_t = \max(Z_{t-1}, Z_t)$ then
   
   $$(\Theta_0, \Theta_1)_{1_{\Theta_{-1} = 0}} = \begin{cases} 
   (\Theta'_0, \Theta'_1) = (1, 1), & \text{if first exc. at 0, i.e. w.p. } 1/2, \\
   (0, 0). & \text{else,}
   \end{cases}$$
3. $X_t = Z_t + \frac{1}{2}Z_{t-1}$ then
   
   $$(\Theta_0, \Theta_1)_{1_{\Theta_{-1} = 0}} = \begin{cases} 
   (\Theta'_0, \Theta'_1) = (1, \frac{1}{2}), & \text{w.p. } = \text{extr. index } \theta_+, \\
   (0, 0). & \text{else.}
   \end{cases}$$
Application to risk management

Definition (Empirical conditional spectral tail process)

Define \((\hat{\Theta}_j', \ldots, \hat{\Theta}_{j+k}') = (r_j/|r_j|, \ldots, r_{j+k}/|r_j|)\) if \(|r_t| > u\), \(t \in I = \{j, \ldots, j+k\}\), \(|r_{j-1}| < \varepsilon u\) and \(|r_{j+k+1}| < u\).

Approximation of \((\Theta_0, \ldots, \Theta_k)\) conditionally on \(\Theta_{-j} = 0, j \geq 1\).
Approximation of the cluster index

\[ P \left( \sum_{j=T}^{T+52} r_t \leq SCR \right) = 0.005 \approx P \left( |r_t| > SCR \right) = 0.0001/b_- , \]

with \( b_- = \mathbb{E} \left[ \left( \sum_{t=0}^{m-1} \Theta_t 1_{j=0, 0 < j < m} \right)^{\alpha} \right] = \theta + \mathbb{E} \left[ \left( \sum_{t=0}^{m-1} \Theta'_t \right)^{\alpha} \right] . \)

Definition (Empirical cluster index)

\[ \hat{b}_-^\ell = \frac{1}{\ell} \sum_{i=1}^{\ell} \frac{1}{k_i} \left( \sum_{t=0}^{k_i} \hat{\Theta}'_{ji+t} \right)^{\hat{\alpha}} \text{ with } \max(X_t, t \in I_i) \geq \max(X_t, t \in I_{i+1}) . \]
Calculation of the SCR when extremes cluster

**Definition**

\[
\hat{SCR} = \max_{15 \leq m \leq 40} \max_{\ell} u_m + \frac{\hat{\sigma}}{\xi} \left[ \left( \frac{n \times 0.0001}{m \times \hat{b}_\ell} \right)^{-\xi} - 1 \right].
\]
New calculation of the SCR

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</table>
Conclusion

- Cluster indices $b_{\pm}$ determine the large deviations of the sums of $m$-dependent regularly varying sequences,
- A new approach of risk management based on clusters of extremes to take into account the dependence.
Thank you for your attention!