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The limit distribution of the maximum increment of a random walk with regularly varying jump size distribution

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In this paper, we deal with the asymptotic distribution of the maximum increment of a random walk with a regularly varying jump size distribution. This problem is motivated by a long-standing problem on change point detection for epidemic alternatives. It turns out that the limit distribution of the maximum increment of the random walk is one of the classical extreme value distributions, the Fréchet distribution. We prove the results in the general framework of point processes and for jump sizes taking values in a separable Banach space.

Keywords: Banach space valued random element; epidemic change point; extreme value theory; Fréchet distribution; maximum increment of a random walk; point process convergence; regular variation

1. Introduction

We commence by considering a sequence (X_i) of independent random variables and denote the partial sums by

$$S_0 = 0, \qquad S_n = X_1 + \cdots + X_n, \qquad n \ge 1.$$

Our original goal is to investigate the asymptotic behavior of the quantities

$$T_n = \max_{1\le \ell < n} \max_{0\le k\le n-\ell} \bigl(\ell(1-\ell/n)\bigr)^{-1/2}(S_{k+\ell} - S_k - \ell\overline{X}_n), \qquad n \ge 1, \tag{1.1}$$

where $\overline{X}_n$ denotes the sample mean of $X_1,\dots,X_n$. The normalization in T_n is motivated by the fact that, under the assumption of i.i.d. finite variance X_i, $\operatorname{var}(S_{k+\ell} - S_k - \ell\overline{X}_n)$ is proportional to $\ell(1-\ell/n)$. In their book on change point analysis, Csörgő and Horvath [5] mention that nothing seems to be known about the distributional properties of T_n. There exist several approaches to replace the original problem by a more tractable one. One way is to restrict the range over which the maximum is taken to $\ell_n \le \ell \le n - \ell_n$ for some $\ell_n \to \infty$ satisfying $\ell_n = o(n)$; see, for example, Yao [24]. Alternatively, one can change the normalizing constants $\sqrt{\ell(1-\ell/n)}$ in a suitable way; see, for example, Račkauskas and Suquet [20].

Statistics of type T_n appear in the context of tests for change points in the mean under epidemic alternatives. This problem can be formulated as follows: given that $X_1, \dots, X_n$ are independent random variables, test the null hypothesis of constant mean

- $H_0 : EX_1 = EX_2 = \cdots = EX_n = \mu$

against the *epidemic alternative*

- H_A : There exist integers $1 \le k^* < m^* < n$ such that

$$EX_1 = \cdots = EX_{k^*} = EX_{m^*+1} = \cdots = EX_n = \mu,$$
$$EX_{k^*+1} = \cdots = EX_{m^*} = \nu \quad \text{and} \quad \mu \ne \nu.$$

One-sided alternatives such as $\mu > \nu$ or $\mu < \nu$ can also be considered. Under the alternative H_A, the mean value ν in the period $[k^*, m^*]$ is interpreted as an epidemic deviation from the usual mean μ and $\ell^* = m^* - k^*$ is called the *duration of the epidemic state.* To the best of our knowledge, this kind of change point problem was formulated for the first time by Levin and Kline [19] in the context of abortion epidemiology. In the one-sided case, they proposed the test statistic $\max_{1\le\ell\le n} \max_{0\le k\le n-\ell}(S_{k+\ell} - S_k - \ell\overline{X}_n - \ell\delta/2)$, where δ represents the smallest increment in the mean which is sufficiently important to be detected. Simultaneously, epidemic-type models were introduced by Commenges, Seal and Pinatel [4] in connection with experimental neurophysiology. They suggested a circular representation of the model, allowing both ℓ^* and $n - \ell^*$ to be interpreted as durations of the epidemic state. Models with an epidemic-type change in the mean were also used for detecting changed segments in non-coding DNA sequences [1] and for studying structural breaks in econometric contexts [3].

The form of the test statistics T_n is motivated by a log-likelihood argument. Indeed, assuming (X_i) to be i.i.d. normal under the hypothesis H_0 against the epidemic alternative $\mu < \nu$, the test statistics T_n is asymptotically equivalent to the square root of a slightly generalized log-likelihood ratio statistics. In the case of a two-sided epidemic alternative $\mu \ne \nu$, the log-likelihood ratio statistics under H_0 is asymptotically equivalent to the quantity

$$\widetilde{T}_n = \max_{1\le\ell<n} \big(\ell(1-\ell/n)\big)^{-1/2} \max_{0\le k\le n-\ell} |S_{k+\ell} - S_k - \ell\overline{X}_n|. \tag{1.2}$$

Two-sided epidemic alternatives, and hence test statistics such as $\widetilde{T}_n$, are also meaningful in the case of multivariate observations X_i. In this paper, we will even deal with sequences (X_i) of i.i.d. random elements with values in a separable Banach space.

Under the null hypothesis, when $\mu = EX_1$ is assumed to be known, it is reasonable to replace the sample mean $\overline{X}_n$ in the quantities T_n and $\widetilde{T}_n$ by μ. One then obtains the following ramifications of T_n and $\widetilde{T}_n$:

$$\begin{aligned} M_n &= \max_{1\le\ell\le n} \ell^{-1/2} \max_{0\le k\le n-\ell} (S_{k+\ell} - S_k - \ell\mu), \\ \widetilde{M}_n &= \max_{1\le\ell\le n} \ell^{-1/2} \max_{0\le k\le n-\ell} |S_{k+\ell} - S_k - \ell\mu|. \end{aligned} \tag{1.3}$$

Here, the choice of normalizing constants is again motivated by the fact that the variance $\operatorname{var}(S_{k+\ell} - S_k)$ is proportional to ℓ. An inspection of the quantities T_n, $\widetilde{T}_n$, M_n and $\widetilde{M}_n$ shows that under H_0, we may assume, without loss of generality, that the random variables X_i, $i \geq 1$, have mean zero.

Various maximal elements of the random field $(\ell^{-1/2}(S_{k+\ell} - S_k))_{\ell=1,\dots,n,k=0,\dots,n-\ell}$ have been widely discussed in the literature. Darling and Erdős [7] proved for a sequence (X_i) of i.i.d. standard normal random variables and suitable constants $a_n > 0$ and $b_n \in \mathbb{R}$ that

$$\lim_{n\to\infty} P\Big(a_n^{-1}\Big(\max_{\ell=1,\dots,n} \ell^{-1/2} S_\ell - b_n\Big) \leq x\Big) = \Lambda(x) = \mathrm{e}^{-\mathrm{e}^{-x}}, \qquad x \in \mathbb{R}. \tag{1.4}$$

Einmahl [11] showed that the Darling–Erdős result (1.4) holds for suitable $a_n > 0$ and $b_n \in \mathbb{R}$ if and only if $E(X^2 I_{\{|X|\geq x\}}) = \mathrm{o}((\log\log x)^{-1})$ as $x \to \infty$. The limit distribution Λ is the *Gumbel* or *double exponential* extreme value distribution. Note that for a sequence (X_i) of i.i.d. standard normal random variables, there exist constants $c_n > 0$ and $d_n \in \mathbb{R}$ such that

$$\lim_{n\to\infty} P\Big(c_n^{-1}\Big(\max_{i=1,\dots,n} X_i - d_n\Big) \leq x\Big) = \Lambda(x), \qquad x \in \mathbb{R};$$

see [14] and, for example, [12], Example 3.3.29. A result in the same spirit was obtained by Siegmund and Venkatraman [23], who showed that for i.i.d. standard normal random variables $X_i, i = 1, 2, \dots,$ there exist constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that $(a_n^{-1}(M_n - b_n))$ has a Gumbel limit distribution. Another proof of this result is given in [16]. Finally, the famous Erdős–Rényi laws are also closely related to the maximum increments of a random walk. These laws study the maxima of the random sequence $(S_{k+\ell_n} - S_k)_{k=1,\dots,n}$ for sequences $\ell_n \to \infty$ with $\ell_n = \mathrm{o}(n)$; see, for example, [10] for distributional convergence of the Erdős–Rényi statistic.

In this paper, we are concerned with limit results for the quantities $\widetilde{T}_n, T_n$ and $\widetilde{M}_n, M_n$, in the case where (X_i) is an i.i.d. sequence of heavy-tailed random variables. We will obtain results which parallel those in [23] in the light-tailed case. A useful definition of a heavy-tailed random variable X with distribution F is given via regular variation. The random variable X is *regularly varying with index* $\alpha > 0$ if there exists a slowly varying function L such that F satisfies the tail balance condition

$$F(-x) \sim q\frac{L(x)}{x^\alpha} \quad \text{and} \quad 1 - F(x) \sim p\frac{L(x)}{x^\alpha}, \qquad x \to \infty, \tag{1.5}$$

where $p \in (0, 1)$, $p + q = 1$; see [2] for an encyclopedic treatment of regular variation.

Under the assumption of regular variation with index α on a generic element X of the i.i.d. sequence (X_i), the theory developed in Section 2 shows that the class of scaling factors $\ell^{0.5}$ and $(\ell(1 - \ell/n))^{0.5}$ which appear in the quantities $\widetilde{T}_n, T_n$ and $\widetilde{M}_n, M_n$ is too narrow. Indeed, the square root character of the normalizations suggests a relationship with the central limit theorem, at least when $\mathrm{var}(X) < \infty$. However, this argument is potentially misleading. As a matter of fact, the scaling factors of the maximum increments of such a random walk have to be chosen depending on the index α. They can range over a large class of scaling functions. For $\gamma \geq 0$, we define the class of functions

$$\begin{aligned}\mathcal{F}_\gamma = \{f : f &\text{ is a positive non-decreasing function on } [0, \infty),\ f(1) = 1,\ f(\ell) \geq \ell^\gamma,\\ &\ell \geq 1 \text{ and for any increasing sequence } (d_n) \text{ of positive numbers}\\ &\text{such that } d_n^2/n \to 0,\ \text{ it holds that } \lim\nolimits_{n\to\infty} \inf\nolimits_{1\leq\ell\leq d_n} f(\ell(1 - \ell/n))/f(\ell) = 1\}.\end{aligned}$$

Examples of functions in the class $\mathcal{F}_\gamma$ are $f(x) = x^{\gamma'}$, where $\gamma' \ge \gamma$, and $f(x) = x^\gamma \log^\beta(1+x)$, where $\beta > 0$.

For any $f \in \mathcal{F}_\gamma$, we introduce the following quantities:

$$\widetilde{M}_n^{(\gamma)} = \max_{1\le\ell\le n} (f(\ell))^{-1} \max_{0\le k\le n-\ell} |S_{k+\ell} - S_k|, \qquad n \ge 1,$$

$$\widetilde{T}_n^{(\gamma)} = \max_{1\le\ell<n} \big(f\big(\ell(1-\ell/n)\big)\big)^{-1} \max_{0\le k\le n-\ell} |S_{k+\ell} - S_k - \ell\overline{X}_n|, \qquad n \ge 1.$$

We suppress the dependence of the quantities $\widetilde{M}_n^{(\gamma)}$ and $\widetilde{T}_n^{(\gamma)}$ on the function f. It will also turn out that the asymptotic results of this section do not depend on the concrete form of the function f; they only depend on the choice of γ. We observe that $\widetilde{M}_n = \widetilde{M}_n^{(0.5)}$ for $f(\ell) = \ell^{0.5}$ and $\widetilde{T}_n = \widetilde{T}_n^{(0.5)}$ for $f(\ell) = \ell^{0.5}$ (cf. (1.3) and (1.2)).

The following result is a consequence of the general theory given in Section 2; see Theorem 2.2. In particular, the result describes the asymptotic behavior of the quantities $\widetilde{M}_n$ and $\widetilde{T}_n$.

Theorem 1.1. *Consider an i.i.d. sequence (X_i) of random variables which are regularly varying with index $\alpha > 0$ and have mean zero if it exists. Then, for any function $f \in \mathcal{F}_\gamma$, $\gamma > \max(0, 0.5 - \alpha^{-1})$,*

$$\lim_{n\to\infty} P\big(a_n^{-1}\widetilde{M}_n^{(\gamma)} \le x\big) = \Phi_\alpha(x) = \mathrm{e}^{-x^{-\alpha}}, \qquad x > 0, \tag{1.6}$$

where the normalizing sequence is given by

$$a_n = \inf\{x \in \mathbb{R} : P(|X| \le x) \ge 1 - 1/n\}. \tag{1.7}$$

Moreover,

$$\lim_{n\to\infty} P\big(a_n^{-1}\widetilde{T}_n^{(\gamma)} \le x\big) = \Phi_\alpha(x), \qquad x > 0. \tag{1.8}$$

Note that Φ_α is the *Fréchet* extreme value distribution. In particular, for any i.i.d. sequence of regularly varying random variables X_i with index $\alpha > 0$ and (a_n) defined in (1.7),

$$\lim_{n\to\infty} P\Big(a_n^{-1} \max_{i=1,\dots,n} |X_i| \le x\Big) = \Phi_\alpha(x), \qquad x \in \mathbb{R}.$$

This relation follows from classical results by Gnedenko [14]; see, for example, [12], Theorem 3.3.7, for a more recent reference. Relation (1.6) can thus be interpreted in the sense that the maximum of the normalized increments $|S_{k+\ell} - S_k|$, $\ell = 1, \dots, n$, $k = 0, \dots, n-\ell$, of the random walk $(S_k)_{k=1,\dots,n}$ is essentially determined by the maximum of the i.i.d. random variables $|X_1|, \dots, |X_n|$. The proof of Theorem 2.2, in particular Lemma 2.4, explains the asymptotic extreme value behavior. We mentioned above that Siegmund and Venkatraman [23] proved an analogous limit relation for $(a_n^{-1}(M_n - b_n))$, assuming that (X_i) is a sequence of i.i.d. standard normal random variables. In this case, the Gumbel distribution appears in the limit. The scaling factor $\ell^{1-1/2}$ in M_n is critical for their result to hold. It can be interpreted as a boundary case for the distributional limits of $(\widetilde{M}_n^{(\gamma)})$ when $\alpha \to \infty$. We also mention that the results in [23] go well beyond proving convergence in distribution; they also give bounds for the probabilities $P(a_n^{-1}(M_n - b_n) > x)$ as $n \to \infty$. Such bounds cannot be achieved with the methods used in this paper.

It is worth mentioning that the statistics $\widetilde{T}_n^{(\gamma)}$, with $f(x) = x^\gamma$ and γ close to $\max\{0, 1/2 - 1/\alpha\}$, allow one to detect epidemic changes in the mean, provided that the duration of the epidemic state is of the order $\ell^* = \mathrm{O}(n^\theta)$, where $\theta > \max\{1/\alpha, 2/(2+\alpha)\}$. Thus, for large α, it is possible to detect short epidemics. We refer to Csörgő and Horvath [5] and Račkauskas and Suquet [20] for details on applications of statistics of the type T_n to epidemic change problems.

The paper is organized as follows. In Section 2.1, we introduce the notion of a regularly varying random element with values in a Banach space and give several examples of such elements. The main result of this paper (Theorem 2.2) is given in Section 2.2. It proves that the normalized maximum increment of a driftless random walk with values in a separable Banach space and with regularly varying jump sizes converges in distribution to a Fréchet distribution. We complement this result with one-sided versions for real-valued random variables. Section 3 contains the proofs of the results of Section 2.

2. General results

In this section, we work in a framework more general than that of Section 1. Our generalizations are twofold: (1) we consider i.i.d. sequences (X_i) of Banach space valued, regularly varying random elements; (2) we allow for more general normalizations of the increments $S_{k+\ell} - S_k$, $\ell = 1, \ldots, n$, $k = 0, \ldots, n-\ell$. In the following subsection, we introduce the notion of a regularly varying random element and in the subsequent subsection, we develop the asymptotic theory for $\widetilde{T}_n$, $\widetilde{M}_n$ and related maximum increment quantities.

2.1. Regular variation in a Banach space

Consider a separable Banach space $(\mathcal{B}, \|\cdot\|)$. We say that a $\mathcal{B}$-valued random element X is *regularly varying with index* $\alpha > 0$ if there exists a boundedly finite non-null measure μ on $\mathcal{B}_0 = \mathcal{B}\backslash\{\mathbf{0}\}$ such that

$$\mu_n(\cdot) = nP(a_n^{-1}X \in \cdot) \stackrel{\widehat{w}}{\to} \mu(\cdot), \qquad n \to \infty,$$

where $\stackrel{\widehat{w}}{\to}$ is convergence in the sense that $\int_{\mathcal{B}_0} f \,\mathrm{d}\mu_n \to \int_{\mathcal{B}_0} f \,\mathrm{d}\mu$ for any bounded and continuous function f on $\mathcal{B}_0$ with bounded support and where

$$a_n = \inf\{x \geq 0 : P(\|X\| \leq x) \geq 1 - n^{-1}\} \tag{2.1}$$

denotes the $(1-n^{-1})$-quantile of the distribution function of $\|X\|$. For locally compact $\mathcal{B}$, in particular for $\mathcal{B} = \mathbb{R}^d$ for some $d \geq 1$, $\widehat{w}$-convergence coincides with vague convergence and the boundedly finite measures are the Radon measures; see [6], Appendix A2.6. The measure μ necessarily satisfies the relation $\mu(t\cdot) = t^{-\alpha}\mu(\cdot)$, $t > 0$. Moreover,

$$P(\|X\| > x) = x^{-\alpha}L(x) \qquad \text{for a slowly varying function } L.$$

In the case $\mathcal{B} = \mathbb{R}$, the notion of regular variation of X coincides with the definition given in (1.5), provided that $P(X > x) \sim pP(|X| > x)$ for some positive p. We refer to Hult and Lindskog [15] for an insightful survey on regular variation in complete separable metric spaces. There, one also finds a useful relation in terms of spherical coordinates which is equivalent to regular variation of X with index $\alpha > 0$: for every $t > 0$,

$$nP(\|X\| > ta_n, X/\|X\| \in \cdot) \xrightarrow{w} t^{-\alpha}\widetilde{P}(\cdot), \qquad n \to \infty, \tag{2.2}$$

where (a_n) is given by (2.1) and $\widetilde{P}(\cdot)$ is a probability measure on the unit sphere $\mathbb{S} = \{x \in \mathcal{B} : \|x\| = 1\}$, called the *spectral measure of* X, and $\xrightarrow{w}$ denotes weak convergence on the Borel σ-field $\mathcal{B}_{\mathbb{S}}$ of $\mathbb{S}$.

Examples of regularly varying random elements with values in a separable Banach space can be found in, for example, [8] or [18]. Those examples include max-stable random fields on $[0, 1]^d$ with a.s. continuous sample paths and regularly varying finite-dimensional distributions [8]. In this case, the index α can be any positive number. Infinite variance stable processes with values in a separable Banach space constitute another class of regularly varying random elements; see [18], Chapter 5, in particular Corollary 5.5. In this case, α is necessarily smaller than 2.

Another important example which is of interest in the context of epidemic change point detection is a regularly varying sample covariance operator, which we define next. Denote the dual of $\mathcal{B}$ by $\mathcal{B}^*$ and let $L(\mathcal{B}^*, \mathcal{B})$ be the Banach space of bounded linear operators $u : \mathcal{B}^* \to \mathcal{B}$ with norm

$$\|u\| = \sup_{x^* \in \mathcal{B}^* : \|x^*\| \le 1} \|u(x^*)\|.$$

For $x, y \in \mathcal{B}$, the operator $x \otimes y : \mathcal{B}^* \to \mathcal{B}$ is defined by $(x \otimes y)(x^*) = x^*(x)y$, $x^* \in \mathcal{B}^*$. It is immediate that $x \otimes y \in L(\mathcal{B}^*, \mathcal{B})$ and $\|x \otimes y\| = \|x\|\|y\|$.

Let X be a $\mathcal{B}$-valued random element with mean zero and finite second moment. The covariance operator $\operatorname{cov}(X) = Q$ of X then maps $\mathcal{B}^*$ into $\mathcal{B}$ and is defined by

$$Qx^* = E(x^*(X)X), \qquad x^* \in \mathcal{B}^*,$$

where the expectation is defined in the Bochner sense.

Assume that X is defined on the probability space $(\Omega, \mathcal{F}, P)$. Then, for each $\omega \in \Omega$, $(X \otimes X)(\omega) = X(\omega) \otimes X(\omega) \in L(\mathcal{B}^*, \mathcal{B})$. Evidently, $X \otimes X : \Omega \to L(\mathcal{B}^*, \mathcal{B})$ is measurable, that is, $X \otimes X$ is a random element with values in the separable Banach space $L(\mathcal{B}^*, \mathcal{B})$. To see this, let (X_n) be a sequence of simple functions that converge to X a.s. One then checks that $\|X_n \otimes X_n - X \otimes X\| \le (\|X_n\| + \|X\|)\|X_n - X\|$ and, hence, $X_n \otimes X_n$ converge to $X \otimes X$ a.s. Moreover, $X_n \otimes X_n$ are simple functions in $L(\mathcal{B}^*, \mathcal{B})$. For this random linear operator, one can define regular variation with index $\alpha > 0$ in the usual way. We give the following result without a proof. It follows by means of a standard continuous mapping argument.

Lemma 2.1. *If a random element X with values in $\mathcal{B}$ is regularly varying with index $\alpha > 0$, then $X \otimes X$ is regularly varying with index $\alpha/2$.*

2.2. Results on the maximum increment of random walks

2.2.1. *Formulation of the main result*

Throughout this section, we consider an i.i.d. sequence of random elements X_i, $i = 1, 2, \dots,$ with values in a separable Banach space $\mathcal{B}$. We assume that a generic element X of this sequence is regularly varying with index $\alpha > 0$. If $\alpha > 1$, $E\|X\| < \infty$ and then its expectation $\mu = EX$ exists in the Bochner sense. Since we are interested in quantities of the type $\widetilde{T}_n$ and $\widetilde{M}_n$ defined in (1.2) and (1.3), respectively, we assume, without loss of generality, that $\mu = 0$ whenever μ exists. Recall the definition of the class of functions $\mathcal{F}_\gamma$, $\gamma \ge 0$, from Section 1 and the definitions of the quantities $\widetilde{M}_n^{(\gamma)}$ and $\widetilde{T}_n^{(\gamma)}$ which we adjust to the case of Banach space valued random elements. Of course, $f(\ell) = \ell^\gamma$, $\ell \ge 0$, is a possible choice for $f \in \mathcal{F}_\gamma$.

The following theorem is the main result of this paper.

Theorem 2.2. *Let (X_i) be a sequence of i.i.d. random elements with values in a separable Banach space $\mathcal{B}$ and assume that X is regularly varying with index $\alpha > 0$. In addition, assume that $EX = 0$ if $E\|X\| < \infty$ and*

$$\sup_{n\ge1} E\|n^{-1/\beta} S_n\| < \infty \qquad \text{for } \begin{cases} \beta = 2, & \text{if } \alpha > 2, \\ \text{every } \beta < \alpha, & \text{if } 1 < \alpha \le 2. \end{cases} \tag{2.3}$$

Then, for $f \in \mathcal{F}_\gamma$, $\gamma > \max(0, 0.5 - \alpha^{-1})$, with the normalizing sequence (a_n) defined as in (2.1),

$$\lim_{n\to\infty} P\big(a_n^{-1} \widetilde{M}_n^{(\gamma)} \le x\big) = \Phi_\alpha(x), \qquad x > 0, \tag{2.4}$$

$$\lim_{n\to\infty} P\big(a_n^{-1} \widetilde{T}_n^{(\gamma)} \le x\big) = \Phi_\alpha(x), \qquad x > 0. \tag{2.5}$$

Remark 2.3. It follows from (2.4) and classical extreme value theory for i.i.d. sequences that the limit distributions of $(a_n^{-1} M_n^{(\gamma)})$ and $(a_n^{-1} \max_{i=1,\dots,n} \|X_i\|)$ coincide; see, for example, [12], Theorem 3.3.7. A theoretical explanation of this phenomenon is provided by Lemma 2.4.

The following result is the key to the proof of Theorem 2.2. In particular, it explains why the quantities

$$a_n^{-1} (f(\ell))^{-1} \max_{0\le k\le n-\ell} \|S_{k+\ell} - S_k\|, \qquad \ell \ge 2,$$

do not have any influence on the limit behavior of $a_n^{-1} \widetilde{M}_n^{(\gamma)}$.

Lemma 2.4. *Assume that (X_n) is an i.i.d. sequence of regularly varying random elements with values in a separable Banach space $\mathcal{B}$ and index $\alpha > 0$. The following statements then hold:*

(1) *for any* $f \in \mathcal{F}_\gamma$, $\gamma \geq 0$ *and* $h \geq 1$,

$$\lim_{n\to\infty} P\Big(a_n^{-1} \max_{1\leq \ell \leq h} (f(\ell))^{-1} \max_{0\leq k\leq n-\ell} \|S_{k+\ell} - S_k\| \leq x\Big) = \Phi_\alpha(x), \qquad x > 0;$$

(2) *if we assume, in addition, that* $EX = 0$ *if* $E\|X\| < \infty$ *and that* (X_n) *satisfies the condition* (2.3), *then for any* $\delta > 0$ *and* $f \in \mathcal{F}_\gamma$, $\gamma > \max(0, 0.5 - \alpha^{-1})$, *we have*

$$\lim_{h\to\infty} \limsup_{n\to\infty} P\Big(\max_{h\leq \ell \leq n} (f(\ell))^{-1} \max_{0\leq k\leq n-\ell} \|S_{k+\ell} - S_k\| > \delta a_n\Big) = 0.$$

2.2.2. *A discussion of Theorem* 2.2 *and its assumptions*

In the following remarks, we provide a detailed discussion of the statements and assumptions of Theorem 2.2.

Remark 2.5. For $\gamma \geq 1$, both relations (2.4) and (2.5) are trivially satisfied. Indeed,

$$\max_{0\leq k\leq n-1} \|X_k\| \leq \widetilde{M}_n^{(\gamma)} \leq \max_{1\leq \ell\leq n} (f(\ell))^{-1} \max_{0\leq k\leq n-\ell} \sum_{i=k+1}^{k+\ell} \|X_k\| \leq \max_{1\leq k\leq n} \|X_k\|$$

for each $f \in \mathcal{F}_\gamma$ with $\gamma \geq 1$. (2.4) then follows in view of Remark 2.3.

Remark 2.6. Under the assumptions of Theorem 2.2, the sequence $(a_n^{-1} \widetilde{M}_n^{(\gamma)})$ has the same limit distribution as the sequence

$$a_n^{-1} \zeta_n^{(\gamma)} = a_n^{-1} \max_{1\leq \ell\leq n} (f(\ell))^{-1} \max_{k=0,\dots,n-\ell} \|S_{k+\ell} - S_k - \ell \overline{X}_n\|, \qquad n \geq 1.$$

To prove this statement, first assume that $\gamma \geq 1$. The argument of Remark 2.5 then shows that it suffices to consider the asymptotic behavior of the sequence $(a_n^{-1} \max_{1\leq k\leq n} \|X_k - \overline{X}_n\|)$. If $E\|X\| < \infty$, then the strong law of large numbers ensures that $\overline{X}_n \xrightarrow{\text{a.s.}} EX$ as $n \to \infty$ and therefore $a_n^{-1}\overline{X}_n \xrightarrow{P} 0$ as $n \to \infty$. If $\alpha \in (0, 1)$, then $a_n^{-1}\|\overline{X}_n\| \leq a_n^{-1} n^{-1} \sum_{i=1}^n \|X_i\| \xrightarrow{P} 0$ as $n \to \infty$. In this case, $a_n^{-1} \sum_{i=1}^n \|X_i\| \xrightarrow{d} Y_\alpha$ for some α-stable random variable Y_α as $n \to \infty$ since $\|X\|$ is regularly varying with index α; see [13], Section XVII, 5. The remaining case $\alpha = 1$ with $E\|X\| = \infty$ is similar. In this case, again applying [13], Section XVII, 5, $a_n^{-1} \sum_{i=1}^n (\|X_i\| - E(\|X\| I_{\{\|X\|\leq a_n\}})) \xrightarrow{d} Y_\alpha$ as $n \to \infty$. Then, as $n \to \infty$,

$$a_n^{-1}\|\overline{X}_n\| \leq n^{-1} a_n^{-1} \sum_{i=1}^n \big(\|X_i\| - E\big(\|X\| I_{\{\|X\|\leq a_n\}}\big)\big) + a_n^{-1} E\big(\|X\| I_{\{\|X\|\leq a_n\}}\big) = o_P(1).$$

In the last step, we also applied Karamata's theorem; see [2], Section 1.6.

We now consider the case $\gamma \in (0, 1)$. Since $f \in \mathcal{F}_\gamma$, it suffices to show that as $n \to \infty$,

$$a_n^{-1} \max_{1\leq \ell\leq n} \ell^{-\gamma} \|\ell \overline{X}_n\| = a_n^{-1} n^{1-\gamma} \|\overline{X}_n\| \xrightarrow{P} 0. \tag{2.6}$$

Assume that $\alpha > 2$ and $\gamma > 0.5 - 1/\alpha$. Then, by virtue of (2.3), as $n \to \infty$,

$$a_n^{-1} n^{1-\gamma} \|\overline{X}_n\| = a_n^{-1} n^{-\gamma+0.5} \|n^{-0.5} S_n\| \xrightarrow{P} 0.$$

If $\alpha \in (1, 2]$, choose β in (2.3) such that $\gamma > 1/\beta - 1/\alpha$. Then, as $n \to \infty$,

$$a_n^{-1} n^{1-\gamma} \|\overline{X}_n\| = a_n^{-1} n^{-\gamma+1/\beta} \|n^{-1/\beta} S_n\| \xrightarrow{P} 0.$$

For $\alpha \in (0, 1)$, we again use the fact that $(a_n^{-1} \sum_{k=1}^n \|X_k\|)$ has an α-stable limit as $n \to \infty$:

$$a_n^{-1} n^{1-\gamma} \|\overline{X}_n\| \le n^{-\gamma} a_n^{-1} \sum_{k=1}^n \|X_k\| \xrightarrow{P} 0.$$

Similarly, for $\alpha = 1$, as $n \to \infty$,

$$a_n^{-1} n^{1-\gamma} \|\overline{X}_n\| \le n^{-\gamma} \left[a_n^{-1} \left(\sum_{k=1}^n \|X_k\| - n E\|X\| I_{\{\|X\| \le a_n\}} \right) + n^{1-\gamma} a_n^{-1} E\|X\| I_{\{\|X\| \le a_n\}} \right] \xrightarrow{P} 0.$$

Remark 2.7. If $\alpha > 2$, then condition (2.3) is fulfilled if the sequence (X_i) satisfies the central limit theorem in $\mathcal{B}$, that is, $n^{-1/2} S_n \xrightarrow{d} Y$ as $n \to \infty$ for some Gaussian element in $\mathcal{B}$ (see Corollary 10.2 in [18]). If the space $\mathcal{B}$ is of type 2 (e.g., any finite-dimensional space, Hilbert space or Lebesgue space L_p with $p \ge 2$), then (2.3) follows from regular variation for $\alpha > 2$. Similarly, if $\alpha \in (1, 2)$ and $a_n^{-1} S_n \xrightarrow{d} Y_\alpha$ as $n \to \infty$ for some α-stable limit in $\mathcal{B}$, then (2.3) is satisfied. This limit always exists in a finite-dimensional space as a consequence of regular variation; see [22].

Remark 2.8. The condition $\gamma > \max(0, 0.5 - \alpha^{-1})$ divides the α values into two sets. For $\alpha \le 2$, this condition is satisfied for all $\gamma > 0$, whereas it restricts γ to $(0.5 - \alpha^{-1}, \infty)$ for $\alpha > 2$. Under the assumption (2.3), this condition is a natural one. Indeed, assume for the moment that the X_i's are real-valued. By the definition of (a_n), $a_n = n^{1/\alpha}/\ell(n)$ for some slowly varying function ℓ and hence

$$a_n^{-1} \widetilde{M}_n^{(\gamma)} \ge n^{-\alpha^{-1}-\gamma+0.5} \ell(n) |n^{-0.5} S_n|.$$

If $\gamma < 0.5 - \alpha^{-1}$, then the left-hand side converges in probability to infinity since $(n^{-0.5} S_n)$ converge in distribution to a Gaussian random variable. Hence, the normalization (a_n) does not ensure the stochastic boundedness of $(a_n^{-1} \widetilde{M}_n^{(\gamma)})$.

With a stronger normalization, a non-degenerate limit distribution of the sequence $(\widetilde{M}_n^{(\gamma)})$ can be achieved by an application of the invariance principle in Hölder space. Following [21], choose $f(\ell) = \ell^\gamma$ for $\gamma < 0.5 - \alpha^{-1}$ and some $\alpha > 2$ and assume that the central limit theorem for (X_n) holds. Then, as $n \to \infty$,

$$n^{-0.5+\gamma} \widetilde{M}_n^{(\gamma)} \xrightarrow{d} R_{W,Q} = \sup_{s,t \in [0,1], s \ne t} \frac{\|W_Q(t) - W_Q(s)\|}{|t-s|^\gamma}, \tag{2.7}$$

where $(W_Q(t))_{0\le t\le 1}$ is the $\mathcal{B}$-valued Wiener process corresponding to the covariance operator $Q=\operatorname{cov}(X)$.

Remark 2.9. Remark 2.8 shows that $\gamma = 0.5-\alpha^{-1}$ for $\alpha>2$ is the borderline which divides the possible limit distributions of the normalized sequence $(\widetilde{M}_n^{(\gamma)})$ into two classes: the Fréchet extreme value distribution, as described in Theorem 2.2, and the distribution of the functional $R_{W,Q}$ of a Wiener process given in (2.7). In the former case, only the extremes in the sample $\|X_1\|,\dots,\|X_n\|$ are responsible for the limit distribution, whereas in the latter case, the limit distribution is obtained by an application of the functional central limit theorem acting on the increments of the random walk (S_n).

The limit distribution of the normalized sequence $(\widetilde{M}_n^{(0.5-\alpha^{-1})})$ is, in general, unknown; it very much depends on the asymptotic behavior of the slowly varying function L in the tail $P(\|X\|>x)=x^{-\alpha}L(x)$. To illustrate the complexity of the situation, we briefly consider two different cases. If $L(x)\to 0$ as $x\to\infty$, then the limiting relation (2.7) proved in [21] still applies with $f(\ell)=\ell^{\gamma}$. If $L(x)\sim c\in(0,\infty)$ as $x\to\infty$, then one can show that $(a_n^{-1}\widetilde{M}_n^{(0.5-\alpha^{-1})})$ is stochastically bounded. However, none of the sequences $(a_n^{-1}M_{n1})$ and $(a_n^{-1}M_{n2})$ is asymptotically negligible in this case, where, for any $h\ge 1$ and $\varepsilon_n\to 0$ as $n\to\infty$,

$$M_{n1} = \max_{1\le \ell\le h} \ell^{-0.5+\alpha^{-1}} \max_{0\le k<n-\ell} \|S_{k+\ell}-S_k\|,$$

$$M_{n2} = \max_{n\varepsilon_n\le \ell\le n} \ell^{-0.5+\alpha^{-1}} \max_{0\le k<n-\ell} \|S_{k+\ell}-S_k\|.$$

This is in stark contrast to the situation described in Lemma 2.4. By the latter result, $(a_n^{-1}M_{n1})$ converges in distribution to a Fréchet-distributed random variable. On the other hand, by adapting the proof of Theorem 8 in [20], one can deduce that $a_n^{-1}M_{n2}\xrightarrow{d} R_{W,Q}$ as $n\to\infty$ is possible, at least for $\mathcal{B}=\mathbb{R}$.

2.2.3. *One-sided results for real-valued random variables*

In the remainder of this section, we restrict our attention to a real-valued i.i.d. sequence (X_i). We note that the two-sided relations (1.6) and (1.8) in Theorem 1.1 immediately follow from Theorem 2.2 by choosing $\mathcal{B}=\mathbb{R}$. However, in the real-valued case, one can also study one-sided versions of Theorem 2.2, for example, the asymptotic behavior of the quantities, for $f\in\mathcal{F}_\gamma$, $\gamma\ge 0$ and $n\ge 1$,

$$M_n^{(\gamma)} = \max_{1\le \ell\le n} (f(\ell))^{-1} \max_{0\le k\le n-\ell} (S_{k+\ell}-S_k),$$

$$m_n^{(\gamma)} = \min_{1\le \ell\le n} (f(\ell))^{-1} \min_{0\le k\le n-\ell} (S_{k+\ell}-S_k).$$

Theorem 2.10. *Assume that (X_i) is an i.i.d. sequence of real-valued random variables with distribution F which is regularly varying with index $\alpha>0$, in the sense of* (1.5). *In addition,* *assume that* $EX = 0$ *if the mean of* X *exists. Then, for* $f \in \mathcal{F}_\gamma$, $\gamma > \max(0, 0.5 - \alpha^{-1})$,

$$\begin{aligned} &\lim_{n\to\infty} P\big((p^{1/\alpha} a_n)^{-1} m_n^{(\gamma)} \le -x, (p^{1/\alpha} a_n)^{-1} M_n^{(\gamma)} \le y\big) \\ &\quad = \Phi_\alpha(y)\big(1 - \Phi_\alpha^{q/p}(x)\big), \qquad x, y > 0, \end{aligned} \tag{2.8}$$

where the normalizing sequence (a_n) *is defined as in* (1.7) *and* $p \in (0,1)$ *appears in the tail balance condition* (1.5).

Remark 2.11. We note that as $n \to \infty$,

$$(p^{1/\alpha} a_n)^{-1}\big(m_n^{(\gamma)}, M_n^{(\gamma)}\big) \xrightarrow{d} \big(y^{(\gamma)}, Y^{(\gamma)}\big),$$

where the limit distribution is given in (2.8). In particular, $y^{(\gamma)}$ is independent of $Y^{(\gamma)}$ and the range statistic $M_n^{(\gamma)} - m_n^{(\gamma)}$ has the limit, as $n \to \infty$,

$$(p^{1/\alpha} a_n)^{-1}\big(M_n^{(\gamma)} - m_n^{(\gamma)}\big) \xrightarrow{d} Y^{(\gamma)} - y^{(\gamma)}.$$

The limit distribution is the convolution $\Phi_\alpha * \Phi_\alpha^{q/p}$, corresponding to the sum of two independent Fréchet-distributed random variables.

Consider the following one-sided version of the statistics $\widetilde{T}_n^{(\gamma)}$:

$$T_n^{(\gamma)} = \max_{1\le \ell < n} \big(\ell(1-\ell/n)\big)^{-\gamma} \max_{0\le k\le n-\ell} (S_{k+\ell} - S_k - \ell \overline{X}_n), \qquad n \ge 1.$$

Theorem 2.12. *Assume that* (X_i) *is an i.i.d. sequence of real-valued random variables with distribution* F *which is regularly varying with index* $\alpha > 0$, *in the sense of* (1.5). *In addition, assume that* $EX = 0$ *if the mean of* X *exists. Then, for any* $\gamma > \max(0, 0.5 - \alpha^{-1})$,

$$\lim_{n\to\infty} P\big((p^{1/\alpha} a_n)^{-1} T_n^{(\gamma)} \le x\big) = \Phi_\alpha(x), \qquad x > 0. \tag{2.9}$$

The following quantity has a structure similar to $M_n^{(\gamma)}$ for $f \in \mathcal{F}_\gamma$:

$$\widehat{M}_n^{(\gamma)} = \max_{\ell=1,\dots,n} (f(\ell))^{-1} \max_{k=\ell+1,\dots,n-\ell} (S_{k+\ell} + S_{k-\ell} - 2S_k).$$

In contrast to the quantities $M_n^{(\gamma)}$, where we need to assume that $EX = 0$ for $\alpha > 1$ in order to guarantee the asymptotic results of Theorem 2.10, centering of the X_i's in $\widehat{M}_n^{(\gamma)}$ is automatic. Indeed, the random variables $S_{k+\ell} + S_{k-\ell} - 2S_k$ are symmetric.

The following result is analogous to Theorem 2.10.

Theorem 2.13. *Assume that* (X_i) *is an i.i.d. sequence of real-valued random variables with distribution* F *which is regularly varying with index* $\alpha > 0$, *in the sense of* (1.5). *Then, with* (a_n)

defined in (2.1), *for* $f \in \mathcal{F}_\gamma$, $\gamma > \max(0, 0.5 - \alpha^{-1})$,

$$\lim_{n\to\infty} P\big(a_n^{-1} \widehat{M}_n^{(\gamma)} \le x\big) = \Phi_\alpha^2(x), \qquad x > 0.$$

3. Proofs

3.1. Proof of Lemma 2.4(1)

The following analog of Davis and Resnick [9], Theorem 2.2, in the case $\mathcal{B} = \mathbb{R}$ is the key to this result.

Lemma 3.1. *Let* (X_i) *be an i.i.d. sequence of random elements with values in* $\mathcal{B}$. *Assume that* X *is regularly varying with index* $\alpha > 0$ *and limit measure* μ. *Then, for any* $h \ge 1$,

$$\widehat{N}_n = \sum_{t=1}^{n} \varepsilon_{a_n^{-1}(X_t,\dots,X_{t+h-1})}$$
$$\xrightarrow{d} \quad \widehat{N} = \sum_{i=1}^{\infty} \varepsilon_{(J_i,0,\dots,0)} + \sum_{i=1}^{\infty} \varepsilon_{(0,J_i,0,\dots,0)} + \cdots + \sum_{i=1}^{\infty} \varepsilon_{(0,\dots,0,J_i)}, \qquad n \to \infty,$$

where ε_x *is Dirac measure at* x *and* $J_1, J_2, \dots$ *are the points of a Poisson random measure with mean measure* μ *on* $\mathcal{B}_0$ *equipped with the Borel* σ*-field. Moreover, on the right-hand side, in the subscripts of the* ε*'s, there are vectors of length* h. *Here, convergence in distribution is in the space of point measures* M_p *on* $\mathcal{B}_0^h$, *equipped with the vague topology*; *see* [6], *Section* 9.1.

We postpone the proof until the end of this subsection.

Remark 3.2. According to Daley and Vere-Jones [6], Theorem 9.1.VI, $\widehat{N}_n \xrightarrow{d} \widehat{N}$ is equivalent to the convergence of the finite-dimensional distributions

$$(\widehat{N}_n(B_1), \dots, \widehat{N}_n(B_m)) \xrightarrow{d} (\widehat{N}(B_1), \dots, \widehat{N}(B_m)), \qquad n \to \infty,$$

for any choice of bounded continuity sets B_i of $\mathcal{B}_0$. Moreover, according to [6], Corollary 9.1.VIII, it suffices that the sets B_i run through any covering semiring of bounded continuity sets for the limiting measure $P_{\widehat{N}}(\cdot) = P(\widehat{N} \in \cdot)$. This means that every open set in $\mathcal{B}_0$ can be represented as a finite or countable union of sets from this semiring. Since $\mathcal{B}_0$ is separable, an important example of such a semiring is obtained by first taking the open spheres $S(d_k, r_j)$ with centers at the points d_k of a countable dense set and radii r_j forming a countable dense set in $(0, 1)$, then forming intersections and finally taking proper differences; see [6], page 617. We will make use of such a semiring in the proof of Lemma 3.1.

A combination of this lemma and the continuous mapping argument analogous to the one in the proof of Davis and Resnick [9], Theorem 2.4, yields

$$\begin{aligned} N_n &= \sum_{t=1}^{n} \varepsilon_{a_n^{-1}(X_t, X_t+X_{t+1},\dots,X_t+\dots+X_{t+h-1})} \\ &\xrightarrow{d} \sum_{i=1}^{\infty} \varepsilon_{(J_i,\dots,J_i)} + \sum_{i=1}^{\infty} \varepsilon_{(0,J_i,\dots,J_i)} + \cdots + \sum_{i=1}^{\infty} \varepsilon_{(0,\dots,0,J_i)} = N, \qquad n \to \infty. \end{aligned} \tag{3.1}$$

Here, the vectors in the subscripts of the ε's have length h. Write

$$B(y) = \{(x_1,\dots,x_h) \in \mathcal{B}^h : \|x_i\| \le y, i = 1,\dots,h\} \tag{3.2}$$

and, for $f \in \mathcal{F}_\gamma$, $\gamma \ge 0$,

$$\widetilde{M}_{n\ell}^{(\gamma)} = (f(\ell))^{-1} \max_{0\le k\le n} \|S_{k+\ell} - S_k\|, \qquad \ell = 1,2,\dots.$$

Then, for $y > 0$, by (3.1), as $n \to \infty$,

$$\begin{aligned} P\big(N_n(B(y)^c) = 0\big) &= P\big(a_n^{-1}\widetilde{M}_{n1}^{(0)} \le y,\dots,a_n^{-1}\widetilde{M}_{nh}^{(0)} \le y\big) \\ &\to P\big(N(B(y)^c) = 0\big) \\ &= P\Big(\sup_{i\ge1}\|J_i\| \le y, \sup_{i\ge1}\|J_i\| \le y,\dots,\sup_{i\ge1}\|J_i\| \le y\Big). \end{aligned} \tag{3.3}$$

Since (J_i) constitute a Poisson random measure on $\mathcal{B}_0$ with mean measure μ, the transformed points $(\|J_i\|)$ constitute a Poisson random measure on $(0,\infty)$ with mean measure ν given by

$$\nu(y,\infty) = \mu(\{x \in \mathcal{B}_0 : \|x\| > y\}) = y^{-\alpha}\mu(\{x \in \mathcal{B}_0 : \|x\| > 1\}) = y^{-\alpha}, \qquad y > 0.$$

In the last step, we used the definition of (a_n). Moreover, we assumed that $P(N(\partial B(y)^c) = 0) = 1$. However,

$$\begin{aligned} N(\partial B(y)^c) &= N(\{x \in \mathcal{B}^h : \|x_i\| = y, \|x_j\| \le y, j \ne i, \text{ for any } i = 1,\dots,h\}) \\ &\le \sum_{i=1}^{h} N(\{x \in \mathcal{B}^h : \|x_i\| = y\}) = 0 \qquad \text{a.s.} \end{aligned}$$

since the expectation of the right-hand expression is zero. Hence, $\nu(y,\infty) = y^{-\alpha}$, $y > 0$. Writing the points $\|J_i\|$ in descending order, they have the representation

$$\Gamma_1^{-1/\alpha} > \Gamma_2^{-1/\alpha} > \cdots,$$

where (Γ_i) are the arrivals of a unit-rate homogeneous Poisson process on $(0,\infty)$. Therefore, and by (3.3), we conclude that for $h \geq 1$, as $n \to \infty$,

$$\begin{aligned}
&P\Big(a_n^{-1} \max_{\ell \leq h} \widetilde{M}_{n\ell}^{(\gamma)} \leq x\Big) \\
&= P\Big(a_n^{-1} \max_{\ell \leq h} (f(\ell))^{-1} \widetilde{M}_{n\ell}^{(0)} \leq x\Big) \\
&\to P\Big(\sup_{i \geq 1} \Gamma_i^{-1/\alpha} \leq x, (f(2))^{-1} \sup_{i \geq 1} \Gamma_i^{-1/\alpha} \leq x, \dots, (f(h))^{-1} \sup_{i \geq 1} \Gamma_i^{-1/\alpha} \leq x\Big) \\
&= P(\Gamma_1^{-1/\alpha} \leq x) \\
&= \mathrm{e}^{-x^{-\alpha}} = \Phi_\alpha(x), \qquad x > 0.
\end{aligned}$$

This concludes the proof of Lemma 2.4(1).

Proof of Lemma 3.1. We follow the proofs of Davis and Resnick [9], Proposition 2.1 and Theorem 2.2, for the case $\mathcal{B} = \mathbb{R}$. For $h = 1$, the points of $\widehat{N}_n$ are independent and therefore the convergence of the finite-dimensional distributions of $\widehat{N}_n$ to those of $\widehat{N}$ follows from $\mu_n \stackrel{\widehat{w}}{\to} \mu$ as $n \to \infty$. Therefore, we consider the case $h > 1$. We write

$$\widetilde{I}_n = \sum_{t=1}^{n} \varepsilon_{a_n^{-1}(X_t,0,\dots,0)} + \cdots + \sum_{t=1}^{n} \varepsilon_{a_n^{-1}(0,\dots,0,X_t)},$$

where the points of this process are in $\mathcal{B}_0^h$. Our first aim is to show that $\widehat{N}_n(B) - \widetilde{I}_n(B) \stackrel{P}{\longrightarrow} 0$ as $n \to \infty$ for bounded Borel sets $B \subset \mathcal{B}_0^h$ which are bounded away from zero. For this reason (see Remark 3.2), it suffices to show that $\widehat{N}_n(B) - \widetilde{I}_n(B) \stackrel{P}{\longrightarrow} 0$ as $n \to \infty$ for sets B from a covering semiring of $\mathcal{B}_0^h$. Therefore, it suffices to consider sets $B = B_1 \times \cdots \times B_h$, where each of the sets $B_i \in \mathcal{B}$ is an element of the semiring generated by the open spheres $S(d_k, r_j)$, as explained in Remark 3.2. We also assume that $\mu(\partial B_i) = 0$, which is possible because (d_k) is dense in $\mathcal{B}$ and (r_j) in $(0,1)$, and there exist only countably many atoms of μ because it is finite on bounded sets.

Since B is bounded away from zero, exactly one of the following two distinct situations may occur: (C_1): B has no intersection with the sets $M_i = \{(0,\dots,0,y,0,\dots,0) \in \mathcal{B}^h : y \in \mathcal{B}\}$, that is, this set consists of the vectors $(0,\dots,0,y,0,\dots,0)$ with y at the ith position; (C_2): $B \cap M_i = B_{i'}$ for $i = i'$ and $B \cap M_i = \varnothing$ for $i \neq i'$. This means that B is either bounded away from the 'axes' M_i or exactly one set $B_i \subset \mathcal{B}$ contains zero. We can now essentially follow the lines of the proof of Davis and Resnick [9], Proposition 2.1, in order to prove $\widehat{N}_n(B) - \widetilde{I}_n(B) \stackrel{P}{\longrightarrow} 0$ as $n \to \infty$. (They prove the result for $\mathcal{B} = \mathbb{R}$ for the more complicated point processes involving the points $(t/n, a_n^{-1}(X_t,\dots,X_{t+h-1}), t = 1,2,\dots.)$

The proof of [9], Theorem 2.2, can be adapted by replacing the semiring S in [9] by the semiring of the sets described above. Moreover, Davis and Resnick [9] apply Kallenberg [17], Theorem 4.2 (which applies to the convergence of point processes on locally compact spaces).

In our situation, this result can be replaced by the results in [6] on the convergence of point processes in a separable complete metric space which were mentioned in Remark 3.2 above. An adaptation of [9], Theorem 2.2, yields that $\widehat{N}_n \xrightarrow{d} \widehat{N}$ as $n \to \infty$. □

3.2. Proof of Lemma 2.4(2)

By the definition of the class $\mathcal{F}_\gamma$, it suffices to prove the result for the functions $f(\ell) = \ell^\gamma$. We will show that the quantities $a_n^{-1} \widetilde{M}_{n\ell}^{(\gamma)}$ for large values of ℓ do not contribute to the limit distribution of $a_n^{-1} \widetilde{M}_n^{(\gamma)}$. The key to the proof is the following inequality.

Lemma 3.3. *Let (X_i) be an i.i.d. sequence with values in $\mathcal{B}$. Then, for any $\delta, \gamma > 0$, $h \geq 1$ and $H \leq n$,*

$$P\Big(\max_{h\le \ell\le H} \ell^{-\gamma} \sup_{k\le n} \|S_{k+\ell} - S_k\| > \delta a_n\Big) \le 2 \sum_{j=J_1}^{J_0} 2^j P\Big(\max_{1\le k\le 2n2^{-j}} \|S_k\| > \delta (n2^{-j})^\gamma a_n\Big),$$

where $J_0 = \log_2(n/h)$, $J_1 = \log_2(n/H) + 1$ and $\log_2 x$ denotes the dyadic logarithm.

Here, and in what follows, we abuse notation when we write $\sum_{j=a}^b x_j$ instead of $\sum_{j:a\le j\le b} x_j$ for real values $a < b$.

Proof of Lemma 3.3. We use a dyadic splitting of the ℓ- and k-index ranges. Recall the definitions of J_0, J_1, where we assume, for simplicity, that these numbers are integers. Setting

$$I_j = (n2^{-j}, n2^{-j+1}], \qquad j = J_1, \dots, J_0,$$

we obtain

$$\bigcup_{j=J_1}^{J_0} I_j = \{h, h+1, \dots, H\}$$

and, therefore,

$$\begin{aligned}
\max_{h\le\ell\le H} \ell^{-\gamma} \max_{k\le n} \|S_{k+\ell} - S_k\| &= \max_{J_1\le j\le J_0} \max_{\ell\in I_j} \ell^{-\gamma} \max_{1\le k\le n} \|S_{k+\ell} - S_k\| \\
&\le \max_{J_1\le j\le J_0} (n^{-1}2^j)^\gamma \max_{\ell\in I_j} \max_{1\le k\le n} \|S_{k+\ell} - S_k\| \qquad (3.4)\\
&\le \max_{J_1\le j\le J_0} (n^{-1}2^j)^\gamma \max_{\ell\in I_j} \max_{1\le i<2^j} \max_{(i-1)n2^{-j}\le k<in2^{-j}} \|S_{k+\ell} - S_k\|.
\end{aligned}$$

We observe that for $n2^{-j} < \ell \le n2^{-j+1}$ and $(i-1)n2^{-j} \le k < in2^{-j}$,

$$\begin{aligned}
\|S_{k+\ell} - S_k\| &\le \big\|S_{k+\ell} - S_{[in2^{-j}]}\big\| + \big\|S_{[in2^{-j}]} - S_k\big\| \\
&\le \max_{in2^{-j}<u<(i+2)n2^{-j}} \big\|S_u - S_{[in2^{-j}]}\big\| + \max_{(i-1)n2^{-j}\le k<in2^{-j}} \big\|S_{[in2^{-j}]} - S_k\big\|.
\end{aligned}$$

Hence, by virtue of (3.4), we obtain the bound

$$P\Big(\max_{h\le \ell\le H} \ell^{-\gamma} \max_{k\le n} \|S_{k+\ell}-S_k\| > \delta a_n\Big) \le P_1 + P_2,$$

where

$$P_1 = P\Big(\max_{J_1\le j\le J_0} (n^{-1}2^j)^\gamma \max_{1\le i<2^j} \max_{in2^{-j}<u<(i+2)n2^{-j}} \big\|S_u - S_{[in2^{-j}]}\big\| > \delta a_n\Big),$$

$$P_2 = P\Big(\max_{J_1\le j\le J_0} (n^{-1}2^j)^\gamma \max_{1\le i<2^j} \max_{(i-1)n2^{-j}\le k<in2^{-j}} \big\|S_{[in2^{-j}]} - S_k\big\| > \delta a_n\Big).$$

Finally, we obtain

$$\begin{aligned} P_1 &\le \sum_{j=J_1}^{J_0} \sum_{1\le i<2^j} P\Big(\max_{in2^{-j}<u<(i+2)n2^{-j}} \big\|S_u - S_{[in2^{-j}]}\big\| > \delta (n2^{-j+2})^\gamma a_n\Big) \\ &= \sum_{j=J_1}^{J_0} 2^j P\Big(\max_{1\le k\le 2n2^{-j}} \|S_k\| > \delta (n2^{-j})^\gamma a_n\Big). \end{aligned}$$

In the last step, we used the i.i.d. property of (X_i). The corresponding bound for P_2 is similar. □

We are now ready for the second part of Lemma 2.4. We consider the truncated elements

$$X_i' = X_i I_{\{\|X_i\|\le h^\gamma a_n\}}, \qquad \widetilde{X}_i = X_i' - EX_i', \qquad i = 1,\dots,n,$$

and the corresponding partial sums $S_k' = \sum_{i=1}^k X_i'$ and $\widetilde{S}_k = \sum_{i=1}^k \widetilde{X}_i$, $k=1,\dots,n$, with $S_0' = \widetilde{S}_0 = 0$. By virtue of Lemma 3.3, we conclude that for any $\delta > 0$,

$$\begin{aligned} &P\Big(\max_{h\le \ell\le n} \ell^{-\gamma} \max_{0\le k\le n-\ell} \|S_{k+\ell}-S_k\| > \delta a_n\Big) \\ &\quad\le P\Big(\max_{1\le k\le n} \|X_k\| \ge h^\gamma a_n\Big) + P\Big(\max_{h\le \ell\le n} \ell^{-\gamma} \max_{0\le k\le n-\ell} \|S_{k+\ell}'-S_k'\| > \delta a_n\Big) \qquad (3.5) \\ &\quad\le P\Big(\max_{1\le k\le n} \|X_k\| \ge h^\gamma a_n\Big) + 2\sum_{j=1}^{\log_2(n/h)} 2^j Q_j, \end{aligned}$$

where

$$Q_j = P\Big(\max_{1\le k\le 2n2^{-j}} \|S_k'\| > \delta (n2^{-j})^\gamma a_n\Big).$$

Since ([14]; see, for example, [12], Theorem 3.3.7, for a more recent reference)

$$\lim_{h\to\infty}\lim_{n\to\infty} P\Big(\max_{1\le k\le n} \|X_k\| \ge h^\gamma a_n\Big) = 1 - \lim_{h\to\infty} \mathrm{e}^{-h^{-\gamma\alpha}} = 0,$$

it suffices to show that

$$\lim_{h\to\infty}\limsup_{n\to\infty}\sum_{j=1}^{\log_2(n/h)} 2^j Q_j = 0. \tag{3.6}$$

Write

$$\Delta_{nj} = a_n (n2^{-j})^\gamma \quad \text{and} \quad N = [2n2^{-j}].$$

First, we consider the case $\alpha > 1$. By assumption, $EX = 0$ and, therefore, we have

$$\max_{1\le k\le N} \|ES_k'\| = N\big\|EXI_{\{\|X\|\ge h^\gamma a_n\}}\big\| \le NE\big(\|X\| I_{\{\|X\|\ge h^\gamma a_n\}}\big).$$

Since $\|X\|$ is regularly varying with index α, an application of Karamata's theorem yields that as $n\to\infty$,

$$E\big(\|X\| I_{\{\|X\|\ge h^\gamma a_n\}}\big) \sim c_\alpha n^{-1} a_n h^{\gamma(1-\alpha)}. \tag{3.7}$$

Hence, since $N \le n$ and, therefore, $cn^{-1}N^{1-\gamma}h^{\gamma(1-\alpha)} \le \delta/2$ for large n and some constant $c > 0$, we have

$$Q_j \le P\Big(\max_{1\le k\le N}\|S_k' - ES_k'\| > \delta N^\gamma a_n - cNh^{\gamma(1-\alpha)} a_n n^{-1}\Big) \le \widetilde{Q}_j,$$

where

$$\widetilde{Q}_j = P\Big(\max_{1\le k\le N}\|\widetilde{S}_k\| > (\delta/2)N^\gamma a_n\Big).$$

Since the sequence $(\|\widetilde{S}_k\|)_{k=0,1,\dots}$ constitutes a submartingale, an application of the Chebyshev and Doob inequalities for $p > 1$ yields, for some constant $c > 0$, that

$$\widetilde{Q}_j \le (\delta/2)^{-p}\Delta_{nj}^{-p} E\Big(\max_{1\le k\le N}\|\widetilde{S}_k\|^p\Big) \le c\Delta_{nj}^{-p} E\|\widetilde{S}_N\|^p. \tag{3.8}$$

We proceed by applying an L_p-inequality for sums of independent mean zero random elements (see [18], Theorem 6.20). We obtain, for $p > 2$,

$$E\|\widetilde{S}_N\|^p \le c[(E\|\widetilde{S}_N\|)^p + NE\|\widetilde{X}_1\|^p] \tag{3.9}$$

with a constant c depending on p only. For fixed $\gamma > 0$ and $\alpha > 1$, let us choose $\beta > 0$ such that $\beta < \alpha$ and $\gamma > \beta^{-1} - \alpha^{-1}$. We then have

$$\begin{aligned} E\|\widetilde{S}_N\| &= E\Big\|\sum_{i=1}^N X_i' - NEX_1'\Big\| = E\Big\|S_N - \sum_{i=1}^N X_i I_{\{\|X_i\|>h^\gamma a_n\}} - NE\big(XI_{\{\|X\|>h^\gamma a_n\}}\big)\Big\| \\ &\le E\|S_N\| + 2NE\big(\|X\| I_{\{\|X\|>h^\gamma a_n\}}\big). \end{aligned}$$

By (3.7) and assumption (2.3), we conclude that

$$E\|\widetilde{S}_N\| \le c\big[N^{1/\beta} + Nn^{-1}a_n h^{\gamma(1-\alpha)}\big]. \tag{3.10}$$

Again, by regular variation of $\|X\|$ and Karamata's theorem, for $p > \max(2, \alpha)$, as $n \to \infty$,

$$E\|\widetilde{X}_1\|^p \sim c_\alpha a_n^p n^{-1} h^{\gamma(p-\alpha)}. \tag{3.11}$$

Combining (3.8)–(3.11), we obtain

$$\begin{aligned}\sum_{j=1}^{\log_2(n/h)} 2^j \widetilde{Q}_j &\le c \sum_{j=1}^{\log_2(n/h)} 2^j a_n^{-p} N^{-p\gamma}\left[N^{p/\beta} + N^p n^{-p} a_n^p h^{p\gamma(1-\alpha)} + N a_n^p n^{-1} h^{\gamma(p-\alpha)}\right] \\ &\le c[I_1 + I_2 + I_3],\end{aligned}$$

where

$$I_1 = a_n^{-p} n^{-p\gamma+p/\beta} \sum_{j=1}^{\log_2(n/h)} 2^{j(1+p\gamma-p/\beta)},$$

$$I_2 = n^{-p\gamma} h^{p(1-\alpha)} \sum_{j=1}^{\log_2(n/h)} 2^{j-pj+p\gamma j},$$

$$I_3 = n^{-p\gamma} h^{\gamma(p-\alpha)} \sum_{j=1}^{\log_2(n/h)} 2^{p\gamma j}.$$

If $\gamma \le 1/\beta$, then using the fact that $p > \max(2, \alpha)$, for some constants $c > 0$ and a slowly varying function ℓ,

$$I_1 \sim c a_n^{-p} n^{-p\gamma+p/\beta} (n/h)^{1+p\gamma-p/\beta} = c h^{-1-p\gamma+p/\beta} (\ell(n))^{-p} n^{-p/\alpha+1} = o(1), \qquad n \to \infty.$$

If $\gamma > 1/\beta - 1/p$, then $I_1 = o(1)$ as $n \to \infty$ by choosing $p > 1/(\gamma - 1/\beta)$. Next, we see that $I_2 = O(n^{-p+1})$ as $n \to \infty$ if $\gamma \ge 1$ and $I_2 = O(n^{-p\gamma})$ as $n \to \infty$ if $\gamma < 1$ and $p > 1/(1-\gamma)$. Finally, $I_3 \le c h^{-\gamma\alpha}$ for some $c > 0$ and the right-hand side converges to zero as $h \to \infty$. This proves (3.6) for $\alpha > 1$.

The case $\alpha = 1$, $EX = 0$, can be handled following the lines of the proof above. Then, (3.7) does not remain valid. However, Karamata's theorem yields that $f_1(x) = E\|X\|_{\{\|X\|>x\}}$ is a slowly varying function. This fact suffices to derive the corresponding relations after (3.7).

We now consider the cases $0 < \alpha < 1$ and $\alpha = 1$, $E\|X\| = \infty$. We have

$$\sum_{j=1}^{\log_2(n/h)} 2^j Q_j \le \sum_{j=1}^{\log_2(n/h)} 2^j P(T_N - N E\|X'\| > \delta N^\gamma a_n - N E\|X'\|),$$

where $T_k = \sum_{i=1}^k \|X_i'\|$. Another application of Karamata's theorem yields, as $n \to \infty$,

$$E\left(\|X\| I_{\{\|X\| \le h^\gamma a_n\}}\right) \sim \begin{cases} c_\alpha a_n n^{-1} h^{\gamma(1-\alpha)}, & \text{when } \alpha \ne 1, \\ \text{slowly varying}, & \text{when } \alpha = 1. \end{cases}$$

We now easily deduce that $NE\|X'\| = \mathrm{o}(N^\gamma a_n)$ as $n\to\infty$ for $\gamma>0$. Thus, we have, for large n, by Kolmogorov's inequality,

$$P(T_N - NE\|X'\| > \delta N^\gamma a_n - NE\|X'\|) \le P\big(T_N - NE\|X'\| > (\delta/2)N^\gamma a_n\big) \le c_\delta N^{-2\gamma} a_n^{-2} N \operatorname{var}(\|X'\|).$$

By Karamata's theorem, $\operatorname{var}(\|X'\|) \sim c_\alpha a_n^2 n^{-1} h^\gamma$ as $n\to\infty$. So, for large n,

$$\sum_{j=1}^{\log_2(n/h)} 2^j Q_j \sim c_{\alpha,\delta} \sum_{j=1}^{\log_2(n/h)} 2^j N^{1-2\gamma} n^{-1} h^\gamma \le c_{\alpha,\delta} \sum_{j=1}^{\log_2(n/h)} 2^{2\gamma j} n^{-2\gamma} h^\gamma \le c_{\alpha,\delta} h^{-\gamma}$$

and we conclude that (3.6) indeed holds. This completes the proof of the lemma.

3.3. Proof of Theorem 2.2

The proof of (2.4) is immediate from Lemma 2.4. It thus suffices to prove (2.5). We achieve this by showing that the sequences $(a_n^{-1}\widetilde{M}_n^{(\gamma)})$ and $(a_n^{-1}\widetilde{T}_n^{(\gamma)})$ have the same asymptotic behavior.

Throughout the proof, we set

$$V_\ell(i,j) = \max_{i\le k\le j} \|S_{k+\ell} - S_k - \ell \overline{X}_n\|, \qquad 0\le i<j\le n.$$

The argument of Remark 2.5 allows us to assume that $\gamma\in(0,1)$ and we start by observing that, in view of Remark 2.6, the sequences $(a_n^{-1}\widetilde{M}_n^{(\gamma)})$ and $(a_n^{-1}\zeta_n^{(\gamma)})$ have the same limit distribution. Next, we observe that $(a_n^{-1}\zeta_n^{(\gamma)})$ has the same limit distribution as

$$a_n^{-1}\zeta_n'^{(\gamma)} = a_n^{-1} \max_{1\le\ell\le d_n} \big(f\big(\ell(1-\ell/n)\big)\big)^{-1} V_\ell(0,n-\ell), \qquad n\ge 1,$$

for any sequence $d_n^2\to\infty$ such that $d_n/n\to 0$ as $n\to\infty$. Indeed, we have

$$\inf_{1\le\ell\le d_n} \big(f\big(\ell(1-d_n/n)\big)/f(\ell)\big)\zeta_n'^{(\gamma)} \le \zeta_n^{(\gamma)} \le \max\big(\zeta_n'^{(\gamma)}, \Delta_n\big),$$

where $\Delta_n = \max_{\ell\ge d_n}(f(\ell))^{-1}V_\ell(0,n-\ell) = \mathrm{o}_P(a_n)$ due to Lemma 2.4 and Remark 2.6. By the definition of the class $\mathcal{F}_\gamma$, we have that $\inf_{1\le\ell\le d_n}(f(\ell(1-d_n/n))/f(\ell)) \to 1$ as $n\to\infty$. Again by Lemma 2.4 and Remark 2.6, we conclude that the sequence $(a_n^{-1}\zeta_n'^{(\gamma)})$ has the same asymptotic distribution as $(a_n^{-1}\zeta_n''^{(\gamma)})$, where

$$\zeta_n''^{(\gamma)} = \max_{1\le\ell\le n/2} \big(f\big(\ell(1-\ell/n)\big)\big)^{-1} V_\ell(0,n-\ell), \qquad n\ge 1.$$

Finally, we show that

$$\Delta_n = a_n^{-1} \max_{n/2<\ell<n} \big(\ell(1-\ell/n)\big)^{-\gamma} V_\ell(0,n-\ell) \xrightarrow{P} 0 \qquad \text{as } n\to\infty. \tag{3.12}$$

We use the following identity:

$$S_{k+\ell} - S_k - \ell\overline{X}_n = \sum_{i=k+1}^{k+\ell} (X_i - \overline{X}_n)$$
$$= -\Bigg[\sum_{i=k+\ell+1}^{n} (X_i - \overline{X}_n) + \sum_{i=1}^{k}(X_i - \overline{X}_n)\Bigg].$$

In view of the identical distributions of the X_i's, the proof of (3.12) reduces to showing that, as $n \to \infty$,

$$\Delta_n' = a_n^{-1} \max_{n/2<\ell<n} \big(\ell(1-\ell/n)\big)^{-\gamma} \max_{0\le k\le n-\ell} \Bigg\| \sum_{i=k+1}^{k+n-\ell} (X_i - \overline{X}_n) \Bigg\| \xrightarrow{P} 0. \tag{3.13}$$

By virtue of (2.6), we have

$$\begin{aligned}
\Delta_n' &= a_n^{-1} \max_{1\le\ell<n/2} \big(\ell(1-\ell/n)\big)^{-\gamma} \max_{0\le k\le\ell} \Bigg\| \sum_{i=k+1}^{k+\ell} (X_i - \overline{X}_n) \Bigg\| \\
&\le 2a_n^{-1} \max_{1\le\ell<n/2} \big(\ell(1-\ell/n)\big)^{-\gamma} \max_{0\le k\le 2\ell} [\|S_k\| + \ell\|\overline{X}_n\|] \\
&\le 2^{\gamma+1} a_n^{-1} \max_{1\le\ell<n/2} \ell^{-\gamma} \max_{0\le k\le 2\ell} \|S_k\| + \mathrm{o}_P(1) \\
&\le 2^{2\gamma+1} a_n^{-1} \max_{1\le k\le n} \|k^{-\gamma} S_k\| + \mathrm{o}_P(1), \qquad n \to \infty.
\end{aligned}$$

By assumption (2.3), choosing $\beta = 2$ if $\gamma > 0.5 - 1/\alpha$ or $\beta < \alpha$ such that $\gamma > 1/\beta - 1/\alpha$, we have

$$\begin{aligned}
a_n^{-1} \max_{1\le k\le n} k^{-\gamma} \|S_k\| &= a_n^{-1} \max_{1\le k\le n} k^{-\gamma+1/\beta} \|k^{-1/\beta} S_k\| \\
&\le a_n^{-1} \max(1, n^{-\gamma+1/\beta}) \max_{1\le k\le n} \|k^{-1/\beta} S_k\| \xrightarrow{P} 0, \qquad n \to \infty.
\end{aligned}$$

This concludes the proof of the theorem.

3.4. Proof of Theorem 2.10

The proof is similar to that of Theorem 2.2. Lemma 3.1 remains valid with a_n replaced by $b_n = p^{1/\alpha} a_n$, but the limiting Poisson random measure with state space $\mathbb{R}\backslash\{0\}$ has mean measure μ given by $\mu(x, \infty) = x^{-\alpha}$ and $\mu(-\infty, -x) = (q/p)x^{-\alpha}$ for $x > 0$. Consider the set

$$B(x, y)^c = (-\infty, -x) \cup (y, \infty), \qquad x, y > 0.$$

Recall the definition of N_n (with a_n replaced by b_n) from (3.1). Then, using Lemma 3.1 and the same ideas as in the proof of Lemma 2.4(1), for $h \geq 1$ and $x, y > 0$,

$$\begin{aligned}
&P\big(N_n(B(x,y)^c) = 0\big) \\
&= P\Big(b_n^{-1} \max_{\ell=1,\dots,h} (f(\ell))^{-1} \max_{k=0,\dots,n-\ell} (S_{k+\ell} - S_k) \leq y, \\
&\qquad b_n^{-1} \min_{k=1,\dots,h} (f(\ell))^{-1} \min_{k=0,\dots,n-\ell} (S_{k+\ell} - S_k) \geq -x\Big) \\
&\to \exp\big\{-\mu\big((-\infty,-x) \cup (y,\infty)\big)\big\} \\
&= \exp\{-(q/p)x^{-\alpha} - y^{-\alpha}\} \\
&= \Phi_\alpha^{q/p}(x)\Phi_\alpha(y).
\end{aligned}$$

Furthermore, for fixed $h \geq 1$,

$$\begin{aligned}
&P\Big(b_n^{-1} \max_{\ell=1,\dots,h} (f(\ell))^{-1} \max_{k=0,\dots,n-\ell} (S_{k+\ell} - S_k) \leq y, \\
&\quad b_n^{-1} \min_{k=1,\dots,h} (f(\ell))^{-1} \min_{k=0,\dots,n-\ell} (S_{k+\ell} - S_k) \leq -x\Big) \\
&= P\Big(b_n^{-1} \max_{\ell=1,\dots,h} (f(\ell))^{-1} \max_{k=0,\dots,n-\ell} (S_{k+\ell} - S_k) \leq y\Big) \\
&\quad - P\Big(b_n^{-1} \max_{\ell=1,\dots,h} (f(\ell))^{-1} \max_{k=0,\dots,n-\ell} (S_{k+\ell} - S_k) \leq y, \\
&\qquad b_n^{-1} \min_{k=1,\dots,h} (f(\ell))^{-1} \min_{k=0,\dots,n-\ell} (S_{k+\ell} - S_k) > -x\Big) \\
&\to \Phi_\alpha(y)\big(1 - \Phi_\alpha^{q/p}(x)\big), \qquad x, y > 0.
\end{aligned}$$

The right-hand side can be extended to a bivariate distribution in a natural way. An application of Lemma 2.4(2) shows that this distribution is the joint limit distribution of $b_n^{-1}(m_n^{(\gamma)}, M_n^{(\gamma)})$.

3.5. Proof of Theorem 2.12

One can follow the lines of the proof of (2.5) to show that the sequences $(b_n^{-1} M_n^{(\gamma)})$ and $(b_n^{-1} T_n^{(\gamma)})$ have the same limiting distribution.

3.6. Proof of Theorem 2.13

The proof is similar to that of Theorem 2.10. We sketch the main ideas. We first observe that the symmetric random variable $\widehat{X} = X_1 - X_2$ is regularly varying:

$$P(X_1 - X_2 > x) \sim P(X > x) + P(X < -x) = P(|X| > x), \qquad x \to \infty;$$

see [12], Lemma A.3.26. Hence, $nP(X_1 - X_2 > a_n) \to 1$ as $n \to \infty$.

Following the lines of the proof of Theorem 2.2, one can show that

$$\lim_{h\to\infty} \limsup_{n\to\infty} P\Big(a_n^{-1} \max_{\ell=h,\dots,n} (f(\ell))^{-1} \max_{k=\ell+1,\dots,n-\ell} |S_{k+\ell} + S_{k-\ell} - 2S_k| > \delta\Big) = 0, \qquad \delta > 0.$$

Hence, it suffices to show that for any fixed $h \ge 1$,

$$\lim_{n\to\infty} P\Big(a_n^{-1} \max_{\ell=1,\dots,h} (f(\ell))^{-1} \max_{k=\ell+1,\dots,n-\ell} (S_{k+\ell} + S_{k-\ell} - 2S_k) \le x\Big) = \Phi_\alpha^2(x), \qquad x > 0.$$

This is again achieved by a point process argument in the spirit of Davis and Resnick [9]. The same argument as in Section 3.1 yields, for any fixed $\ell \ge 1$,

$$\begin{aligned}
&\sum_{k=\ell+1}^{n} \varepsilon_{a_n^{-1}(X_{k+1}-X_{k-1},(X_{k+1}-X_{k-1})+(X_{k+2}-X_{k-2}),\dots,(X_{k+1}-X_{k-1})+\cdots+(X_{k+\ell}-X_{k-\ell}))} \\
&\quad \xrightarrow{d} \sum_{i=1}^{\infty}\big[\varepsilon_{(J_i,,\dots,J_i)} + \varepsilon_{(-J_i,\dots,-J_i)}\big] + \sum_{i=1}^{\infty}\big[\varepsilon_{(0,J_i,\dots,J_i)} + \varepsilon_{(0,-J_i,\dots,-J_i)}\big] + \cdots \\
&\qquad + \sum_{i=1}^{\infty}\big[\varepsilon_{(0,\dots,0,J_i)} + \varepsilon_{(0,\dots,0,-J_i)}\big], \qquad n \to \infty,
\end{aligned}$$

where (J_i) are the points of a Poisson random measure on $\mathcal{B}_0$ with mean measure μ satisfying $\mu(x,\infty) = \mu(-\infty,-x] = x^{-\alpha}$, $x > 0$. This limit result implies that for $h \ge 1$ and $x > 0$,

$$\begin{aligned}
&P\Big(a_n^{-1} \max_{l=1,\dots,h} (f(\ell))^{-1} \max_{k=l+1,\dots,n-\ell} (S_{k+l} + S_{k-l} - 2S_k) \le x\Big) \\
&\quad \to P\Big(\sup_{i\ge1} |J_i| \le x, (f(2))^{-1} \sup_{i\ge1} |J_i| \le x, \dots, (f(h))^{-1} \sup_{i\ge1} |J_i| \le x\Big) \\
&\quad = \Phi_\alpha^2(x), \qquad n \to \infty.
\end{aligned}$$

This concludes the proof.

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