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# EXTREME VALUE THEORY FOR SPACE-TIME PROCESSES WITH HEAVY-TAILED DISTRIBUTIONS

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ABSTRACT. Many real-life time series often exhibit clusters of outlying observations that cannot be adequately modeled by a Gaussian distribution. Heavy-tailed distributions such as the Pareto distribution have proved useful in modeling a wide range of bursty phenomena that occur in areas as diverse as finance, insurance, telecommunications, meteorology, and hydrology. Regular variation provides a convenient and unified background for studying multivariate extremes when heavy tails are present. In this paper, we study the extreme value behavior of the space-time process given by

$$X_t(\mathbf{s}) = \sum_{i=0}^{\infty} \psi_i(\mathbf{s}) Z_{t-i}(\mathbf{s})\,, \quad \mathbf{s} \in [0,1]^d\,,$$

where $(Z_t)_{t\in\mathbb{Z}}$ is an iid sequence of random fields on $[0,1]^d$ with values in the Skorokhod space $\mathbb{D}([0,1]^d)$ of càdlàg functions on $[0,1]^d$ equipped with the $J_1-$topology. The coefficients $\psi_i$ are deterministic real-valued fields on $\mathbb{D}([0,1]^d)$. The indices $\mathbf{s}$ and $t$ refer to the observation of the process at location $\mathbf{s}$ at time $t$. For example, $X_t(\mathbf{s}), t = 1, 2, \ldots,$ could represent the time series of annual maxima of ozone levels at location $\mathbf{s}$. The problem of interest is determining the probability that the maximum ozone level over the entire region $[0,1]^2$ does not exceed a given standard level $f \in \mathbb{D}([0,1]^2)$ in $n$ years. By establishing a limit theory for point processes based on $(X_t(\mathbf{s}))$, $t = 1\ldots,n$, we are able to provide approximations for probabilities of extremal events. This theory builds on earlier results of de Haan and Lin [11] and Hult and Lindskog [13] for regular variation on $\mathbb{D}([0,1]^d)$ and Davis and Resnick [7] for extremes of linear processes with heavy-tailed noise.

## 1. Introduction

Building on the recent theory developed by de Haan and Lin [11] and Hult and Lindskog [13] for random functions with values in the space of càdlàg functions, we study the asymptotic theory for point processes and extremes of filtered processes of the form

$$X_t(\mathbf{s}) = \sum_{i=0}^{\infty} \psi_i(\mathbf{s})\, Z_{t-i}(\mathbf{s})\,, \quad \mathbf{s} \in [0,1]^d\,. \tag{1.1}$$

Here $(Z_t)_{t\in\mathbb{Z}}$ is an iid sequence of random fields on $[0,1]^d$ with values in the Skorokhod space $\mathbb{D} = \mathbb{D}([0,1]^d)$ of càdlàg functions equipped with the $J_1$-topology; see Bickel and Wichura [2] for definitions and properties related to this topology. The $\psi_i$'s are deterministic real-valued fields on $\mathbb{D}$.

The indices $\mathbf{s}$ and $t$ refer to a measurement taken at location $\mathbf{s}$ at time $t$. For example, $X_t(\mathbf{s})$, $t = 1, 2, \ldots,$ could represent the time series of annual maxima of ozone levels at location $\mathbf{s}$. One of the problems of interest is determining the probability that the maximum ozone level over the entire region $[0,1]^2$ does not exceed a given standard level $f \in \mathbb{D}([0,1]^2)$ in $n$ years. Another example, mentioned in de Haan and Lin [11], concerns the probability that the water level $X_t(s)$ on day $t$

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at location $s \in [0,1]$ along the Dutch coast will not breach the dykes. Here $f(s)$ is a function that represents the height of the dykes at location $s$. Then the probability of interest is

$$P\left(\max_{t=1,\dots,n} X_t(s) \le f(s) \text{ for all } s \in [0,1]\right).$$

A third example is the windspeed $X_t(\mathbf{s})$ along a building at time $t$ and location $\mathbf{s}$ on the face of a building.

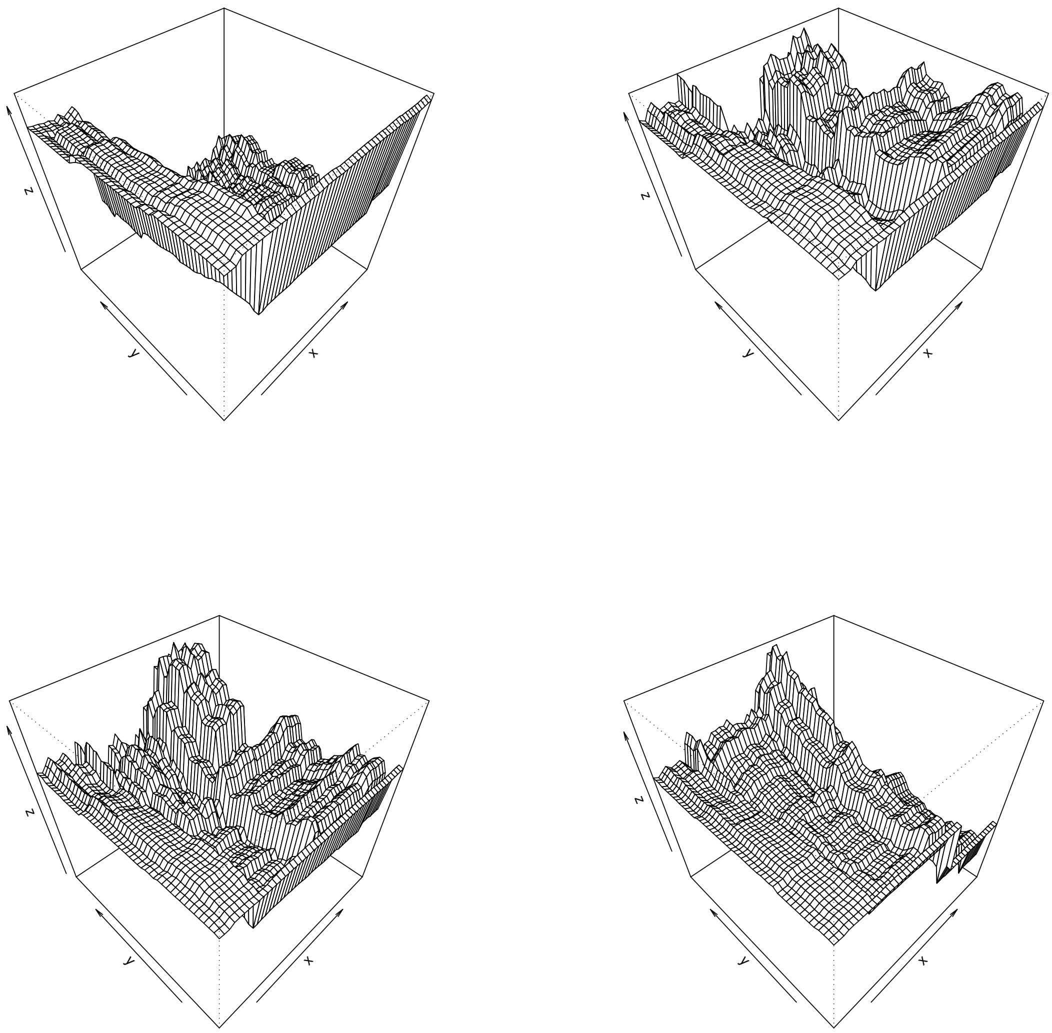


**Figure 1.1.** *An autoregressive random field* $X_t = 0.9X_{t-1} + Z_t$, $t = 0, 1, 2, 3$, *(top left to bottom right) with a regularly varying Lévy random field with index* $\alpha = 1$, *see Section 4.2.*

Serial dependence enters in the model $(X_t)$ through the linear filter with weights $\psi_j$, $j = 1, 2, \dots$. For example, at each fixed location $\mathbf{s}$ we have a linear time series model $(X_t(\mathbf{s}))_{t\in\mathbb{Z}}$. If $Z_t$ is a second order stationary random field with mean 0 and covariance function $\gamma_Z(\mathbf{s})$, then the serial

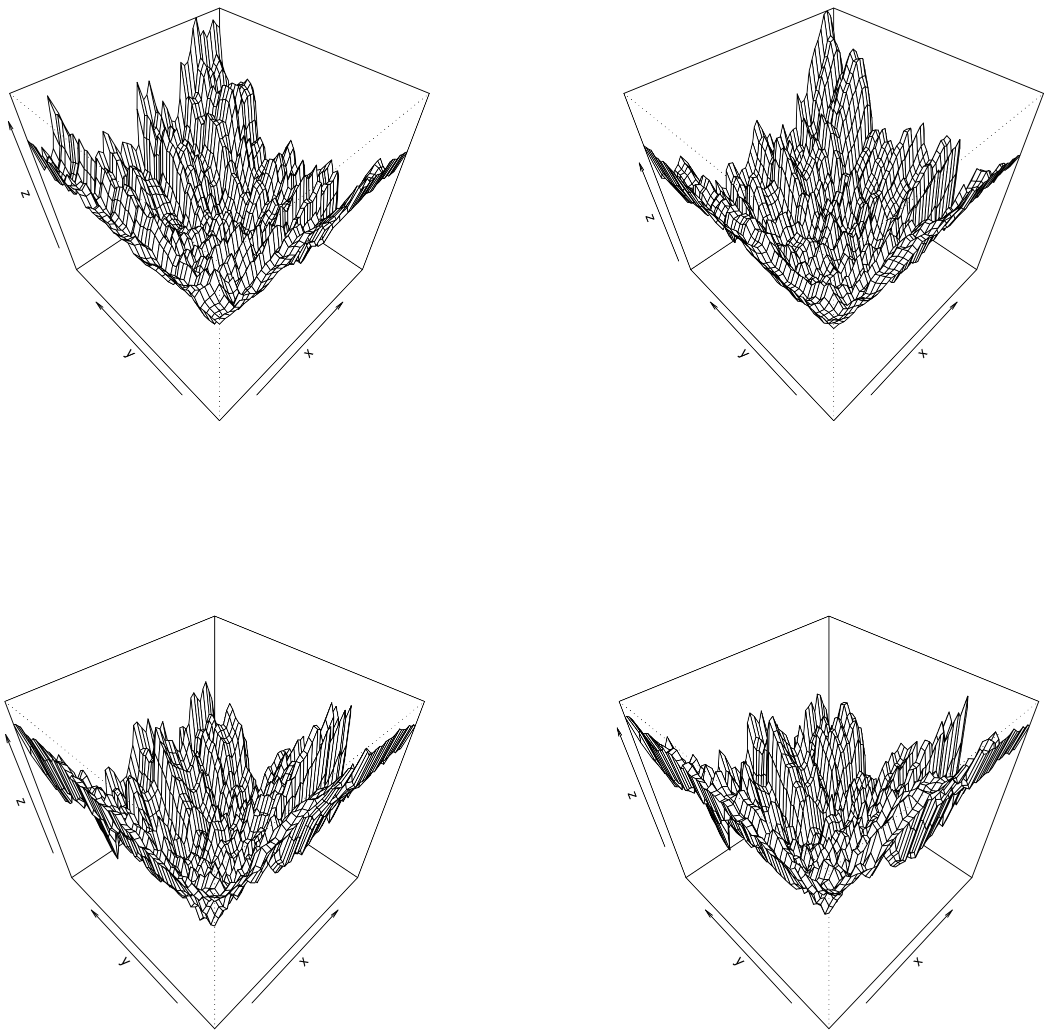


**Figure 1.2.** *An autoregressive random field* $X_t = -0.8X_{t-1} + Z_t$*,* $t = 0, 1, 2, 3$*, (top left to bottom right) with a regularly varying Lévy random field with index* $\alpha = 4$*, see Section 4.2.*

autocorrelation of $X_t$ at location $\mathbf{s}$ is given by

$$\mathrm{Cor}(X_t(\mathbf{s}), X_{t+h}(\mathbf{s})) = \sum_{i=0}^{\infty} \psi_i(\mathbf{s})\psi_{i+h}(\mathbf{s}) / \sum_{i=0}^{\infty} \psi_i^2(\mathbf{s}) \, ,$$

provided $\sum_j \psi_j^2(\mathbf{s}) < \infty$. The spatial dependence among these linear time series is governed by the spatial dependence in the noise $(Z_t)$. In particular, the spatial covariance function at fixed time $t$ is given by

$$\mathrm{Cov}(X_t(\mathbf{s}_1), X_t(\mathbf{s}_2)) = \left( \sum_{i=0}^{\infty} \psi_i(\mathbf{s}_1)\psi_i(\mathbf{s}_2) \right) \gamma_Z(\mathbf{s}_2 - \mathbf{s}_1).$$

If the linear filter weights $\psi_j(\mathbf{s})$ are space-invariant, i.e., $\psi_j(\mathbf{s}) = \psi_j$ for all $j$ and $\mathbf{s}$, then $(X_t(\mathbf{s})$ is stationary in both space and time with a multiplicative covariance function given by

$$\mathrm{Cov}(X_t(\mathbf{s}_1), X_{t+h}(\mathbf{s}_2)) = \left(\sum_{i=0}^{\infty} \psi_i \psi_{i+h}\right) \gamma_Z(\mathbf{s}_2 - \mathbf{s}_1).$$

Realizations from two autoregressive (AR) spatial processes are displayed in Figures 1.1 and 1.2. The AR(1) process is given by $X_t = \phi X_{t-1} + Z_t$ which corresponds to the linear process in (1.1) with coefficients $\psi_j(\mathbf{s}) = \phi^j$. The realizations in the figures correspond to $t = 0, 1, 2, 3$ with $\phi = .9$ (Figure 1.1), $\phi = -.8$ (Figure 1.2) and noise $(Z_t)$ which is a regularly varying Lévy random field with $\alpha = 4$ (see Section 4.2). The process defined by (1.1) allows for modeling of both the dependence in time and space in a flexible way. While one can introduce serial dependence of random fields in more complicated ways, we will restrict attention to the linear case in this paper.

Many real-life time series often exhibit clusters of outlying observations that cannot be adequately modeled by a Gaussian distribution. Heavy-tailed distributions such as the Pareto distribution have proved useful in modeling a wide range of bursty phenomena that occur in finance, insurance, telecommunications, meteorology, hydrology; see Embrechts et al. [8] and the collection of papers [9] for specific examples and references. The theory of regular variation provides a convenient and unified background for studying multivariate extremes when heavy tails are present; see Resnick [22] for the basic theory and Basrak et al. [1], de Haan and Lin [11], Hult and Lindskog [13] for some recent developments. The novelty of the papers by de Haan and Lin [11] and Hult and Lindskog [13] is the precise formulation of regular variation for random functions with values in $\mathbb{C}[0,1]$ and $\mathbb{D}[0,1]$. This serves as a starting point for what we consider in this paper.

This paper is organized as follows. In Section 2, we introduce the notion of a regularly varying random field with values in $\mathbb{D}$ and we quote some preliminary results that will be frequently used in the sequel. In Section 3, we apply the notion of regular variation on $\mathbb{D}$ to max-stable fields. The section culminates with a representation of a max-stable random field in terms of a homogeneous Poisson process and iid random fields. This representation was proposed by Schlather [25] as one possibility for simulating max-stable random fields. de Haan and Pereira [12] formulate one and two dimensional families of parametric models for spatial extremes that are based on a similar representation for stationary random fields. These parameters can be used to describe a form of dependence between the random field at any two locations. In Section 4, we continue with some examples of regularly varying random fields. These include regularly varying Lévy and s$\alpha$s random fields. In Section 5 we deal with the regular variation on $\mathbb{D}$ of the linear process $X_t$ and study some of its consequences. In Section 5.2 we establish convergence for the sequence of point processes based on the points $X_t$, properly normalized, towards a compound Poisson process. This may be viewed as an extension of the seminal result by Davis and Resnick [7] for linear processes. Applications of these results to problems in extreme value theory, including the calculation of the probability of exceedances of high thresholds by the $X_t$'s and the extremal index of the sequence $(|X_t|_\infty)$, are given in Section 5.3.

## 2. Preliminaries on regular variation on $\mathbb{D}$

2.1. **Definition and properties of regularly varying random fields.** In this section we introduce the essential ingredients about regular variation on $\mathbb{D}$ that will be required for the results in Section 5. We closely follow the discussion in de Haan and Lin [11], Hult and Lindskog [13] and Hult et al. [15]. Denote by $\mathbb{D} = \mathbb{D}([0,1]^d, \mathbb{R})$ the space of càdlàg functions $x : [0,1]^d \to \mathbb{R}$ equipped with a metric $d_0$ which is equivalent to the $J_1$-metric and such that it makes $\mathbb{D}$ a complete separable linear metric space; see Bickel and Wichura [2] and Billingsley [3]. We denote by $\mathbb{S}_{\mathbb{D}}$ the "unit sphere" $\{x \in \mathbb{D} : |x|_\infty = 1\}$ with $|x|_\infty = \sup_{\mathbf{s}\in[0,1]^d} |x(\mathbf{s})|$, equipped with the relativized topology of $\mathbb{D}$. Define $\overline{\mathbb{D}}_0 = (0,\infty] \times \mathbb{S}_{\mathbb{D}}$, where $(0,\infty]$ is equipped with the metric $\rho(x,y) = |1/x - 1/y|$ making it complete and separable. For any element $x \in \overline{\mathbb{D}}_0$, we write $x = (|x|_\infty, \widetilde{x})$, where $\widetilde{x} = x/|x|_\infty$. Then $\overline{\mathbb{D}}_0$, equipped with the metric $\max\{\rho(x,y), d_0(\widetilde{x},\widetilde{y})\}$, is a complete separable metric space. The topological spaces $\mathbb{D}\backslash\{0\}$, equipped with the relativized topology of $\mathbb{D}$, and $(0,\infty) \times \mathbb{S}_{\mathbb{D}}$, equipped with the relativized topology of $\overline{\mathbb{D}}_0$, are homeomorphic; the function $T$ given by $T(x) = (|x|_\infty, \widetilde{x})$ is a homeomorphism. Hence

$$\mathcal{B}(\overline{\mathbb{D}}_0) \cap [(0,\infty) \times \mathbb{S}_{\mathbb{D}}] = \mathcal{B}(T(\mathbb{D}\backslash\{0\})),$$

i.e., the sets of the Borel $\sigma$-field $\mathcal{B}(\overline{\mathbb{D}}_0)$ that are of interest to us can be identified with the usual Borel sets on $\mathbb{D}$ (viewed in spherical coordinates) that do not contain the zero function. For notational convenience we will throughout the paper identify $\mathbb{D}$ with the product space $[0,\infty) \times \mathbb{S}_{\mathbb{D}}$ so that expressions like $\overline{\mathbb{D}}_0\backslash\mathbb{D}$ $(= \{\infty\} \times \mathbb{S}_{\mathbb{D}})$ make sense. We denote by $\mathcal{B}(\overline{\mathbb{D}}_0) \cap \mathbb{D}$ the Borel sets $B \in \mathcal{B}(\overline{\mathbb{D}}_0)$ such that $B \cap [\{\infty\} \times \mathbb{S}_{\mathbb{D}}] = \emptyset$. Notice that a bounded set $B$ of $\overline{\mathbb{D}}_0$ is a set bounded away from zero, i.e., there exists $\delta > 0$ such that $|x|_\infty > \delta$ for all $x \in B$.

In addition, we will make use of the space $\mathbb{C} = \mathbb{C}([0,1]^d)$ of continuous functions on $[0,1]^d$ equipped with the uniform topology. Completely analogously to $\overline{\mathbb{D}}_0$, $\mathbb{S}_{\mathbb{D}}$, etc., we will use the notation $\overline{\mathbb{C}}_0$, $\mathbb{S}_{\mathbb{C}}$, etc.

Regular variation on $\mathbb{R}^d$ (for random vectors) is typically formulated in terms of vague convergence on $\mathcal{B}(\overline{\mathbb{R}}^d_0)$, where $\overline{\mathbb{R}}^d_0 = \overline{\mathbb{R}}^d\backslash\{\mathbf{0}\}$ and $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty,\infty\}$; see Resnick [21, 22]. The topology on $\overline{\mathbb{R}}^d_0$ is chosen so that $\mathcal{B}(\overline{\mathbb{R}}^d_0)$ and $\mathcal{B}(\mathbb{R}^d)$ coincide on $\mathbb{R}^d\backslash\{\mathbf{0}\}$. Moreover, $B \subset \overline{\mathbb{R}}^d_0$ is relatively compact (or bounded) if and only if $B \cap \mathbb{R}^d$ is bounded away from $\mathbf{0}$ (i.e., $\mathbf{0} \notin \overline{B \cap \mathbb{R}^d}$) in $\mathbb{R}^d$.

The vector $\mathbf{X}$ with values in $\mathbb{R}^d$ is *regularly varying* with index $\alpha > 0$ and *spectral measure* $\sigma$ on the Borel $\sigma$-field of the unit sphere $\mathbb{S}^{d-1} = \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| = 1\}$ if there exists a sequence of constants $a_n \to \infty$ such that

$$n\,P(|\mathbf{X}| > t\,a_n\,, \widetilde{\mathbf{X}} \in \cdot) \xrightarrow{w} t^{-\alpha}\,\sigma(\cdot)\,, \quad t > 0\,,$$

where $\xrightarrow{w}$ denotes weak convergence and, as before, $\widetilde{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$ for $\mathbf{x} \neq \mathbf{0}$. It is always possible to choose $(a_n)$ such that $P(|\mathbf{X}| > a_n) \sim n^{-1}$, and then $\sigma$ is a probability measure. Equivalently, $\mathbf{X}$ is regularly varying if there exists a sequence $a_n \to \infty$ (which can be chosen as above) and a non-null Radon measure $\mu$ on $\mathcal{B}(\overline{\mathbb{R}}^d_0)$ such that $\mu(\overline{\mathbb{R}}^d\backslash\mathbb{R}^d) = 0$ and

$$n\,P(a_n^{-1}\mathbf{X} \in \cdot) \xrightarrow{v} \mu(\cdot)\,,$$

where $\xrightarrow{v}$ denotes vague convergence on the Borel $\sigma$-field $\mathcal{B}(\overline{\mathbb{R}}^d_0)$.

Regular variation on $\mathbb{D}$ is naturally expressed in terms of $\widehat{w}$-convergence of boundedly finite measures on $\overline{\mathbb{D}}_0$; for details on $\widehat{w}$-convergence and its relationship with vague and weak convergence we refer to Appendix A2.6 in Daley and Vere-Jones [6], cf. also Kallenberg [16] and Resnick [21, 22]. A boundedly finite measure assigns finite mass to bounded sets. A sequence of boundedly finite measures $(m_n)$ on a complete separable metric space $\mathbb{E}$ converges to the measure $m$ in the $\widehat{w}$-topology, $m_n \xrightarrow{\hat{w}} m$, if $m_n(B) \to m(B)$ for every bounded Borel set $B$ with $m(\partial B) = 0$. Equivalently, $m_n \xrightarrow{\widehat{w}} m$ refers to

$$m_n(f) = \int_{\mathbb{E}} f dm_n \to \int_{\mathbb{E}} f dm = m(f) \tag{2.1}$$

for all bounded continuous functions $f$ on $\mathbb{E}$ which vanish outside a bounded set. If the state space $\mathbb{E}$ is locally compact ($\overline{\mathbb{R}}^d_0$ is locally compact while $\overline{\mathbb{D}}_0$ is not), then a boundedly finite measure is called a Radon measure, and $\widehat{w}$-convergence coincides with vague convergence and we write $m_n \xrightarrow{v} m$. Finally we notice that if $m_n \xrightarrow{\hat{w}} m$ and $m_n(\mathbb{E}) \to m(\mathbb{E}) < \infty$, then $m_n \xrightarrow{w} m$.

We say that the random field $X$ with values in $\mathbb{D}$ (and its distribution) are *regularly varying* with index $\alpha > 0$ and *spectral measure* $\sigma$ on $\mathbb{S}_{\mathbb{D}}$, if there exists a sequence of constants $a_n \to \infty$ such that

$$n\,P(|X|_\infty > t\,a_n\,,\widetilde{X}\in\cdot)\stackrel{w}{\to} t^{-\alpha}\,\sigma(\cdot)\,,\quad t>0\,, \tag{2.2}$$

where $\stackrel{w}{\to}$ denotes weak convergence on the Borel $\sigma$-field $\mathcal{B}(\mathbb{S}_{\mathbb{D}})$. One can always choose $(a_n)$ such that $P(|X|_\infty > a_n)\sim n^{-1}$, and then $\sigma$ is a probability measure on $\mathbb{S}_{\mathbb{D}}$. The convergence in (2.2) is equivalent to

$$n\,P(a_n^{-1}\,X\in\cdot)\stackrel{\widehat{w}}{\to} m(\cdot)\,, \tag{2.3}$$

where $\stackrel{\widehat{w}}{\to}$ denotes $\widehat{w}$-convergence on the Borel $\sigma$-field $\mathcal{B}(\overline{\mathbb{D}}_0)$ and $m$ is a finitely bounded measure with the property that $m(\overline{\mathbb{D}}_0\backslash\mathbb{D})=0$; see Hult and Lindskog [13] for a proof of the equivalence between (2.3) and (2.2).

We will often make use of the following useful result by Hult and Lindskog [13] proved for $d=1$. The proof for $d>1$ is analogous and therefore omitted. The result characterizes a regularly varying random field in terms of the finite-dimensional distributions and the modulus of continuity. Write, for an $x\in\mathbb{D}$, $\delta>0$ and a set $A\subset[0,1]^d$,

$$\begin{aligned} w''(x,\delta) &= \sup_{\mathbf{s}_1\le\mathbf{s}\le\mathbf{s}_2\,,|\mathbf{s}_2-\mathbf{s}_1|\le\delta}\min(|x(\mathbf{s})-x(\mathbf{s}_1)|,|x(\mathbf{s})-x(\mathbf{s}_2)|)\,,\\ w(x,A) &= \sup_{\mathbf{s}_1,\mathbf{s}_2\in A}|x(\mathbf{s}_1)-x(\mathbf{s}_2)|\,. \end{aligned}$$

**Lemma 2.1.** *The random field $X$ with values in $\mathbb{D}$ is regularly varying if and only if there exist a sequence $(a_n)$ satisfying $n\,P(|X|_\infty > a_n)\to 1$ and a collection of Radon measures $m_{\mathbf{s}_1,\dots,\mathbf{s}_k}$, $\mathbf{s}_i\in[0,1]^d$, $i=1,\dots,k$, $k\ge1$, not all of them being the null measure, with $m_{\mathbf{s}_1,\dots,\mathbf{s}_k}(\overline{\mathbb{R}}^k\backslash\mathbb{R}^k)=0$, such that the following conditions hold:*

(1) *The following relation holds:*

$$n\,P(a_n^{-1}(X(\mathbf{s}_1),\dots,X(\mathbf{s}_k))\in\cdot)\stackrel{v}{\to} m_{\mathbf{s}_1,\dots,\mathbf{s}_k}(\cdot)\,, \tag{2.4}$$

*for all $\mathbf{s}_i\in[0,1]^d$, $i=1,\dots,k$, $k\ge1$, where $\stackrel{v}{\to}$ refers to vague convergence on the Borel $\sigma$-field $\mathcal{B}(\overline{\mathbb{R}}^k_0)$.*

(2) *For any $\epsilon,\eta>0$ there exist $\delta\in(0,0.5)$ and $n_0$ such that for $n\ge n_0$,*

$$n\,P(w''(X,\delta)>a_n\epsilon)\ \le\ \eta\,, \tag{2.5}$$

$$n\,P(w(X,[0,1]^d\backslash[\delta,1-\delta]^d)>a_n\epsilon)\ \le\ \eta\,. \tag{2.6}$$

*The measures $m_{\mathbf{s}_1,\dots,\mathbf{s}_k}$, $\mathbf{s}_i\in[0,1]^d$, $i=1,\dots,k$, $k\ge1$, determine the limiting measure $m$ in the definition of regular variation of $X$.*

2.2. **Regular variation, point process convergence and convergence of maxima.** Next we connect regular variation on $\mathbb{D}$ with the weak convergence of the point processes

$$N_n=\sum_{i=1}^n\varepsilon_{a_n^{-1}X_i}\,,\quad n\ge1\,,$$

and the maxima $a_n^{-1}\max_{i=1,\dots,n}X_i$, where the $X_i$'s are iid copies of a regularly varying random field $X$ with values in $\mathbb{D}$ and $\varepsilon_x$ is Dirac measure at $x$.

**Lemma 2.2.** *Let $X,X_1,X_2,\dots$ be an iid sequence of $\mathbb{D}$-valued random fields. In items* (2) *and* (3) *we assume in addition that $X$ has non-negative sample paths.*

(1) *The field $X$ is regularly varying with index $\alpha>0$ and limiting measure $m$ as in* (2.3) *if and only if $N_n\stackrel{d}{\to}N$ in $M_p(\overline{\mathbb{D}}_0)$, the space of point measures with state space $\overline{\mathbb{D}}_0$ equipped with the $\widehat{w}$-topology, where $N$ is a Poisson random measure with mean measure $m$* (PRM($m$)).

(2) *If the relation*

$$a_n^{-1} \max_{t=1,\dots,n} X_t \xrightarrow{d} Z \tag{2.7}$$

*holds in* $\mathbb{D}$ *for some non-degenerate random field* $Z$ *then* $X$ *is regularly varying on* $\mathbb{D}$ *for some positive* $\alpha$.

(3) *Conversely, assume that* $X$ *is regularly varying with index* $\alpha > 0$. *Then* (2.7) *holds for some* $Z$ *in the sense of the finite-dimensional distributions. Moreover, if a* $\mathbb{C}$*-valued version of* $Z$ *exists then the convergence of the finite-dimensional distributions in* (2.7) *can be extended to convergence in* $\mathbb{D}$.

*Proof.* (1) This result follows by an adaptation of Proposition 3.21 in Resnick [22]. While this proposition applies to weak convergence of point processes with a locally compact state space, our state-space $\overline{\mathbb{D}}_0$ is not locally compact. However, the proof (which only involves Laplace functionals of the underlying point processes) remains valid if one changes from vague convergence used in [22] to $\widehat{w}$-convergence as described above; see Daley and Vere-Jones [6], Chapter 9 and Appendix A2.6. The proof of (1) in the case $d = 1$ and for non-negative càdlàg $X$ on $[0, 1]$ can also be found in Theorem 2.4 of de Haan and Lin [11]. (The proof is given under the assumption that $\alpha = 1$ which does not restrict generality.)
(2) and (3) The proof follows by an adaptation of the proof in Theorem 2.4 in [11] who consider the case of non-negative $X$ on $[0, 1]$. The extension to $d > 1$ does not provide additional difficulties. □

A consequence of Lemma 2.2 (and indeed of finite-dimensional extreme value theory, see Resnick [22], Section 5.4) is that regular variation of $X$ on $\mathbb{D}$ with index $\alpha$ implies that for any choice of $\mathbf{s}_i \in [0, 1]^d$, $i = 1, \dots, k$, $k \geq 1$,

$$a_n^{-1} \left( \max_{t=1,\dots,n} X_t(\mathbf{s}_i) \right)_{i=1,\dots,k} \xrightarrow{d} (Z(\mathbf{s}_i))_{i=1,\dots,k} \,. \tag{2.8}$$

The distribution of $(Z(\mathbf{s}_i))_{i=1,\dots,k}$ is a multivariate extreme value distribution with Fréchet marginals with index $\alpha$ and exponent measure $m_{\mathbf{s}_1,\dots,\mathbf{s}_k}$ which is described in part (1) of Lemma 2.1. By part (1) of the lemma it also follows that the point process convergence $N_n \xrightarrow{d} N$ in $M_p(\overline{\mathbb{D}}_0)$ for some PRM$(m)$, $N$, implies (2.8). However, tightness of the sequence $(a_n^{-1} \max_{t=1,\dots,n} X_t)$ in $\mathbb{D}$ and the tightness condition for regular variation given by (2.5) and (2.6) are in general not equivalent in $\mathbb{D}$. An assumption such as continuity of the limit $Z$ in (2.7) is in general needed. A counterexample showing that regular variation of $X$ on $\mathbb{D}$ does not imply (2.7) was kindly communicated to us by Yohann Gentric.

2.3. **Regular variation of products of random variables.** We will often make use of a simple result on the products of independent random variables, which we will refer to as Breiman's result. See Breiman [4], cf. Basrak et al. [1] for a proof and some multivariate extensions.

**Lemma 2.3.** *Assume* $\xi, \eta$ *are non-negative random variables,* $\eta$ *is regularly varying with index* $\alpha > 0$ *and one of the following conditions holds:*

(1) $0 < E\xi^{\alpha+\delta} < \infty$ *for some* $\delta > 0$.
(2) $0 < E\xi^{\alpha} < \infty$ *and* $P(\eta > x) \sim c\,x^{-\alpha}$ *as* $x \to \infty$ *for some* $c > 0$.

*Then* $P(\xi\eta > x) \sim E\xi^{\alpha}\, P(\eta > x)$ *as* $x \to \infty$.

The proof of the result under condition (2) is difficult to find in the literature, but it follows easily by intersecting the event $\{\xi\eta > x\}$ with the events $\{\xi > \epsilon x\}$ and $\{\xi \leq \epsilon x\}$ for $\epsilon > 0$ sufficiently small and by observing that $P(\xi > x) = o(x^{-\alpha})$.

## 3. Applications to max-stable random fields

3.1. **Preliminaries.** The class of max-stable random fields provides a good collection of examples of regularly varying random fields. It is common (see Resnick [22], de Haan and Lin [11]) to assume that all one-dimensional marginals of a max-stable process are Fréchet with index 1. Of course, the marginals of the process $X$ can be transformed to obtain marginals with any other extreme value distribution. For this reason, we confine our discussion to the case $\alpha = 1$.

Following de Haan [10], a random field $X$ on $[0,1]^d$ is called *max-stable* with unit Fréchet marginals (i.e., $P(X(\mathbf{s}) \le x) = \mathrm{e}^{-x^{-1}}$, $x > 0$, for every $\mathbf{s} \in [0,1]^d$), if for iid copies $X_i$ of $X$ and every $k \ge 1$,

$$k\,X \stackrel{d}{=} \max_{i=1,\ldots,k} X_i\,, \tag{3.9}$$

where $\stackrel{d}{=}$ denotes equality of the finite-dimensional distributions. Condition (3.9) is equivalent to the existence of a field $Y$ with regularly varying finite-dimensional distributions with index $\alpha = 1$ such that for iid copies $Y_i$ of $Y$ and a suitable sequence $(a_n)$ of positive constants,

$$a_n^{-1} \max_{i=1,\ldots,n} Y_i \stackrel{d}{\to} X\,, \tag{3.10}$$

where $\stackrel{d}{\to}$ represents convergence of the finite-dimensional distributions. The finite-dimensional distributions of $X$ have the following canonical form,

$$P(X(\mathbf{s}_1) \le y_1, \ldots, X(\mathbf{s}_k) \le y_k) = \exp\left\{ -\int_0^1 \left( \max_{i=1,\ldots,k} \frac{f_i(x)}{y_i} \right) dx \right\}\,, \tag{3.11}$$

for some suitable choice of non-negative $L^1$ functions $f_i$ which have integral 1, i.e., $\int_0^1 f_i(y)\,dy = 1$, see Resnick [22], Proposition 5.11. It follows that the finite-dimensional distributions of $X$ are regularly varying with index $\alpha = 1$. Moreover, from (3.10), we have the relation

$$n\,P(a_n^{-1}(Y(\mathbf{s}_i))_{i=1,\ldots,k} \in ([0,y_1] \times \cdots \times [0,y_k])^c) \to \int_0^1 \left( \max_{i=1,\ldots,k} \frac{f_i(x)}{y_i} \right) dx\,,$$

which identifies the measure $m_{\mathbf{s}_1,\ldots,\mathbf{s}_k}$ in (2.4).

According to the defining property (3.9), max-stability is only a property of the finite-dimensional distributions of the field $X$. In what follows, this notion is strengthened to requiring that $X$ satisfies (3.9) and assumes values in $\mathbb{D}$. Interestingly, with this additional assumption that $X$ lives in $\mathbb{D}$, regular variation of $X$ on $\mathbb{D}$ is automatic. This is the content of the following lemma.

**Lemma 3.1.** *Assume that $X$ is a max-stable process with values in $\mathbb{D}$. Then $X$ is regularly varying on $\mathbb{D}$ with index 1.*

*Proof.* The proof follows from part (2) of Lemma 2.2 by taking $Y_i = X_i$ for iid copies of the $\mathbb{D}$-valued max-stable field $X$. Equality of the finite-dimensional distributions of $X$ and $n^{-1}\max_{t=1,\ldots,n} X_i$ implies equality in distribution in $\mathbb{D}$ which in turn yields weak convergence in $\mathbb{D}$. Hence $X$ is regularly varying on $\mathbb{D}$ with index 1. □

**Remark 3.2.** For a max-stable field $X$ with unit Fréchet marginals there exists a unique measure $m$ such that $n\,P(n^{-1}X \in \cdot) \stackrel{\widehat{w}}{\to} m$ and $m(tB) = t^{-1}m(B)$ for any bounded set $B$ and $t > 0$. Moreover, for any $\mathbf{s}_i \in [0,1]^d$ and $y_i \ge 0$, $i = 1,\ldots,k$, $k \ge 1$,

$$P(X(\mathbf{s}_1) \le y_1\,, \ldots, X(\mathbf{s}_k) \le y_k) = \mathrm{e}^{-m(A^c)}\,, \tag{3.12}$$

where

$$A = \{z \in \overline{\mathbb{D}}_0 : z(\mathbf{s}_i) \le y_i\,, i = 1,\ldots,k\}\,. \tag{3.13}$$

The measure $m$ is often referred to as the *exponent measure* of $X$ and uniquely determines the finite-dimensional distributions of $X$, i.e., $\mathrm{e}^{-m(A^c)}$ coincides with the right-hand side of (3.11).

3.2. **A representation of a max-stable random field.** In this section, we construct a max-stable $\mathbb{D}$-valued random field $X$ with unit Fréchet marginals and we show that it is regularly varying on $\mathbb{D}$ with index 1.

To start with, consider a unit rate Poisson process on $(0,\infty)$. An increasing enumeration of the points of the process is denoted by $(\Gamma_i)_{i\geq 1}$. Consider an iid sequence $Y, Y_1, Y_2, \ldots$, of random fields on $[0,1]^d$ with values in $\mathbb{D}$ independent of $(\Gamma_i)$. Moreover, assume that $0 < EY^+(\mathbf{s}) < \infty$ for all $\mathbf{s}\in[0,1]^d$, where $a^+$ is the positive part of the real number $a$. Define the càdlàg random field

$$X(\mathbf{s}) = \sup_{j\geq 1}\Gamma_j^{-1}\,Y_j(\mathbf{s}) = \sup_{j\geq 1}\Gamma_j^{-1}\,Y_j^+(\mathbf{s})\,,\quad \mathbf{s}\in[0,1]^d\,. \tag{3.14}$$

Notice that the second equality is due to the fact that $0 < EY^+(\mathbf{s}) < \infty$ and therefore $Y_j(\mathbf{s}) > 0$ infinitely often with probability 1 for every $\mathbf{s}$. In view of (3.14) we assume that $Y(\mathbf{s})$ is positive a.s. and $EY(\mathbf{s}) < \infty$. For $\mathbf{s}\in[0,1]^d$, the random variable $X(\mathbf{s})$ is well defined by virtue of the strong law of large numbers $\Gamma_j/j \stackrel{\text{a.s.}}{\to} 1$ and since $Y_j(\mathbf{s})/j \stackrel{\text{a.s.}}{\to} 0$ by the Borel-Cantelli lemma.

Representation (3.14) was introduced by Schlather [25] as a model for max-stable random fields. We show that (3.14) yields a representation of any max-stable field in $\mathbb{D}$.

**Theorem 3.3.** *The $\mathbb{D}$-valued random field $X$ is max-stable if and only if $X$ has representation (3.14) for some iid sequence $(Y_i)$ of $\mathbb{D}$-valued random fields such that $Y>0$ a.s. and $E|Y|_\infty<\infty$. In either case, $X$ is regularly varying on $\mathbb{D}$ with index 1 and spectral measure $\sigma$ given by*

$$\sigma(S) = E\left(|Y|_\infty I_S(\widetilde{Y})\right)\Big/E|Y|_\infty\,,\quad S\in\mathcal{B}(\mathbb{S}_{\mathbb{D}})\,. \tag{3.15}$$

*Proof.* We first show that $X$ given by (3.14) with $E|Y|_\infty<\infty$ is in $\mathbb{D}$. Since $E|Y|_\infty<\infty$, it follows from the strong law of large numbers and the Borel-Cantelli lemma that $|Y_j|_\infty/\Gamma_j \stackrel{\text{a.s.}}{\to} 0$. Therefore $\sup_{1\leq j\leq n}\Gamma_j^{-1}Y_j$ converges a.s. as $n\to\infty$ in the uniform topology to $X$ given in (3.14) which is finite a.s.

Next we show that the random field (3.14) is max-stable in the sense of (3.9). Consider the point process $N=\sum_{j=1}^\infty \varepsilon_{\Gamma_j^{-1}Y_j}$ with state space $\overline{\mathbb{D}}_0$. Using standard arguments, the log-Laplace functional of $N$ is given by

$$\log E\exp\{-N(f)\} = -E\left(\int_0^\infty (1-\mathrm{e}^{-f(sY)})\,s^{-2}\,ds\right)\,,$$

where $f$ is a bounded continuous function on $\overline{\mathbb{D}}_0$ with bounded support. Define a measure $m$ on the Borel $\sigma$-field of $\overline{\mathbb{D}}_0$ by

$$m(\{x\in\overline{\mathbb{D}}_0 : |x|_\infty>t\,,\widetilde{x}\in S\}) = E|Y|_\infty\,t^{-1}\,\sigma(S)\,,\quad t>0\,,\quad S\in\mathcal{B}(\mathbb{S}_{\mathbb{D}})\,,$$

where the probability measure $\sigma$ is given by (3.15). Recall that for $z\in\overline{\mathbb{D}}_0$ and $s>0$ we have $f(sz)=f((s\,|z|_\infty,\widetilde{z}))$. It follows that the log-Laplace functional is equal to

$$-E\left(\int_0^\infty (1-\mathrm{e}^{-f(s\,|Y|_\infty,\widetilde{Y})})\,s^{-2}\,ds\right) = -E|Y|_\infty\int_{\mathbb{S}_{\mathbb{D}}}\int_0^\infty (1-\mathrm{e}^{-f(t,\theta)})\,t^{-2}\,dt\,\sigma(d\theta)\,.$$

This is the log-Laplace functional of PRM($m$) on $\overline{\mathbb{D}}_0$, hence $N$ is PRM($m$). For any $\mathbf{s}_i\in[0,1]^d$, $i=1,\ldots,k$, and $\mathbf{y}\in\mathbb{R}_+^k$ consider the finite-dimensional set $A\subset\overline{\mathbb{D}}_0$ given by (3.13). Then we have

$$P(N(A^c)=0) = P(X(\mathbf{s}_1)\leq y_1\,,\ldots,X(\mathbf{s}_k)\leq y_k) = \mathrm{e}^{-m(A^c)}\,.$$

Since by definition of $m$, $m(tB)=t^{-1}m(B)$ for any bounded set $B$ and $t>0$, we conclude from Remark 3.2, in particular equation (3.12), that $m$ is the exponent measure of a max-stable random field. This proves max-stability of $X$.

The regular variation with index 1 of the max-stable random field $X$ given in (3.14) follows from Lemma 3.1. The spectral measure (3.15) was calculated in the course of the proof above.

Now we prove the converse. Assume that $X$ is max-stable with unit Fréchet marginals. As such it has an exponent measure $\widetilde{m}$ with corresponding spectral (probability) measure $\widetilde{\sigma}$. Let $(Y_j)$ be an iid sequence of positive $\mathbb{S}_{\mathbb{D}}$-valued random fields with distribution $\widetilde{\sigma}$. We will show that $X$ has the same distribution as the $\mathbb{D}$-valued random field

$$X^* = c \sup_{j\ge 1} \Gamma_j^{-1}\, Y_j\,,$$

where

$$c = m(\{x \in \overline{\mathbb{D}}_0 : |x|_\infty > 1\})\,.$$

By the direct part of the proof, $X^*$ is well defined, max-stable, regularly varying with index 1 and has exponent measure given by

$$m(\{x \in \overline{\mathbb{D}}_0 : |x|_\infty > t\,, \widetilde{x} \in S\}) = c\,t^{-1}\,\widetilde{\sigma}(S)\,, \quad t>0\,, \quad S \in \mathcal{B}(\mathbb{S}_{\mathbb{D}})\,.$$

This means that the exponent measures $m$ and $\widetilde{m}$ coincide. Hence the max-stable fields $X$ and $X^*$ have the same finite-dimensional distributions. This concludes the proof. □

**Remark 3.4.** The condition $E|Y|_\infty < \infty$ can be verified in general circumstances. A prime example is a $\mathbb{C}$-valued centered Gaussian random field which we can interpret as a mean zero Gaussian random element with values in a separable Banach space. Then it is well known that the tail $P(|Y|_\infty > y)$ decays exponentially fast, in particular $E(|Y|_\infty^p) < \infty$ for all $p > 0$, see Landau and Shepp [17], Marcus and Shepp [20], cf. Ledoux and Talagrand [18]. Representation (3.14) is advantageous for simulating max-stable fields as advocated by Schlather [25]. In Figure 3.5 we show two realizations of max-stable fields on $[0,1]^2$ based on the representation (3.14), where $Y$ is an isotropic Gaussian random field with exponential and Gaussian covariance functions, respectively. These fields were generated in the software package R, using the RandomFields package written by Schlather.

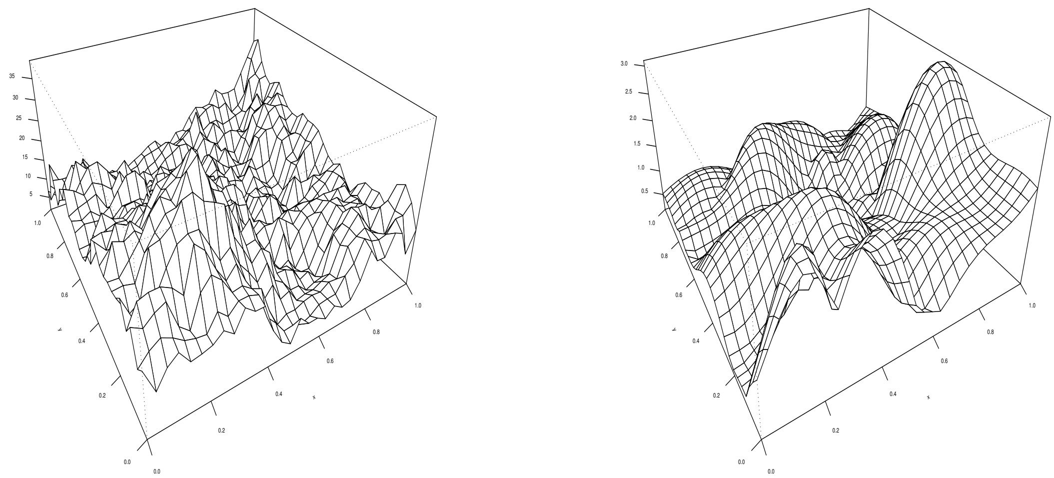

**Figure 3.5.** *Two realizations of a max-stable field using the representation* (3.14), *where $Y$ is an isotropic Gaussian random field on* $[0,1]^2$ *with exponential* (*left*) *and Gaussian* (*right*) *covariance functions.*

**Remark 3.6.** It follows by direct calculation, see Section 4.1 below, that $\Gamma_1^{-1}Y$ is regularly varying with index 1 and has the same limit measure $m$ on $\overline{\mathbb{D}}_0$ as $X = \sup_{j\geq 1}\Gamma_j^{-1}Y_j$. This means that the extreme behavior of a max-stable random field $X$ is determined only by the first term in the supremum. This can also be seen from the fact that for every $\epsilon > 0$,

$$n\,P\Big(n^{-1}\Big|\sup_{j\geq 2}\Gamma_j^{-1}Y_j\Big|_\infty > \epsilon\Big) \to 0\,,\quad n\to\infty\,. \tag{3.16}$$

Indeed, we have

$$n\,P\Big(n^{-1}\Big|\sup_{j\geq 2}\Gamma_j^{-1}Y_j\Big|_\infty > \epsilon\Big) \leq n\sum_{j\geq 2}P\left(\Gamma_j^{-1}\,|Y|_\infty > \epsilon\, n\right)$$

$$= n\int_0^\infty \left(\sum_{j=1}^\infty P(\Gamma_j\leq (\epsilon n)^{-1}y) - P(\Gamma_1\leq(\epsilon n)^{-1}y)\right)\,P(|Y|_\infty\in dy)$$

$$= n\int_0^\infty \Big(\frac{y}{\epsilon n} - (1-\mathrm{e}^{-y/(\epsilon n)})\Big)\,P(|Y|_\infty\in dy)\,. \tag{3.17}$$

Observe that $f_n(y) = n[y/(\epsilon n) - (1-\mathrm{e}^{-y/(\epsilon n)})] \leq c\,y$ for some $c>0$, all $y>0$. Moreover, $f_n(y)\to 0$ as $n\to\infty$ for every $y>0$. Since $E|Y|_\infty < \infty$ by assumption, a dominated convergence argument yields that for every $\epsilon>0$ the right-hand side in (3.17) converges to zero. Combining the arguments above, we conclude that (3.16) holds.

## 4. Examples of regularly varying random fields

In this section we consider some more examples of regularly varying $\mathbb{D}$-valued random fields $X$. In Section 3 we have already studied the class of max-stable random fields which constitute an important family of $\mathbb{D}$-valued regularly varying random fields.

### 4.1. **A simple multiplicative field.**

Let $Y$ be a càdlàg random field and suppose that $\eta$ is a non-negative regularly varying random variable with index $\alpha>0$, independent of $Y$. Assume that $\eta$ and $\xi = |Y|_\infty$ satisfy the conditions of Breiman's Lemma 2.3. For example, the assumptions on $\xi$ are satisfied for Gaussian $Y$. Define the $\mathbb{D}$-valued random field

$$X(\mathbf{s}) = \eta\,Y(\mathbf{s})\,,\quad \mathbf{s}\in[0,1]^d\,.$$

An application of Lemma 2.3 yields that $X$ is regularly varying on $\mathbb{D}$ with index $\alpha$. Indeed, for any Borel set $S\subset\mathbb{S}_{\mathbb{D}}$ and $t>0$,

$$n\,P\left(|\eta Y|_\infty > a_n\,t, \widetilde{Y}\in S\right) \to t^{-\alpha}E\left(|Y|_\infty^\alpha\, I_S(\widetilde{Y})\right)\Big/E|Y|_\infty^\alpha = t^{-\alpha}\,\sigma(S)\,,\quad t>0\,,$$

where $(a_n)$ is chosen such that $P(|X|_\infty > a_n)\sim n^{-1}$. The right-hand side of this relation has the form given in (2.2). Using Breiman's result, one can easily calculate asymptotic expressions related to the finite-dimensional distributions of $X$. For example, for positive $y_i$, $i=1,\ldots,k$,

$$\begin{aligned} n\,P(a_n^{-1}(|X(\mathbf{s}_1)|,\ldots,|X(\mathbf{s}_k)|)\in(y_1,\infty)\times\cdots\times(y_k,\infty)) &= n\,P\left(a_n^{-1}\eta\min_{i=1,\ldots,k}(|Y(\mathbf{s}_i)|/y_i)>1\right)\\ &\to E\left(\min_{i=1,\ldots,k}(|Y(\mathbf{s}_i)|/y_i)^\alpha\right)\,.\end{aligned}$$

In addition, we may conclude that condition (2) of Lemma 2.1 is satisfied for $X$.

We also mention that, if $Y$ has mean zero and finite second moment and $\eta$ has finite second moment, then $X$ and $Y$ have the same correlation structure.

Despite its simplicity, the multiplicative model serves as an approximation to the large values of some important regularly varying random fields. Those include the max-stable fields (see Remark 3.6), but also the s$\alpha$s random fields considered in Section 4.4.

4.2. **Regularly varying Lévy fields.** We consider a $\mathbb{D}$-valued random field $X$ which has independent and stationary increments and for $\mathbf{s} \in [0,1]^d$ the log-characteristic function of $X(\mathbf{s})$ is given by

$$\log E\mathrm{e}^{\,i\,t\,X(\mathbf{s})} = -|[\mathbf{0},\mathbf{s}]| \int_{\overline{\mathbb{R}}_0} \left(\mathrm{e}^{\,i\,t\,y} - 1 - i\,t\,y\,I_{[-1,1]}(y)\right)\,\nu(dy)\,,$$

where $\nu$ is a Lévy measure on $\overline{\mathbb{R}}_0$, satisfying $\int_{\overline{\mathbb{R}}_0}(1\wedge y^2)\,\nu(dy) < \infty$ and $|A|$ is the Lebesgue measure of any set $A$. Following standard theory for Lévy processes (see Sato [24]), we call $X$ a *Lévy random field*. We may and do assume that $X$ has càdlàg sample paths. For fixed $\mathbf{s}$, it follows from Hult and Lindskog [13] that $X(\mathbf{s})$ is regularly varying with limit measure $|[0,\mathbf{s}]|\mu$ for some Radon measure $\mu$ on $\overline{\mathbb{R}}_0$ if and only if the Lévy measure $\nu$ is regularly varying in the sense that for some sequence of constants $a_n \to \infty$,

$$n\,\nu(a_n\,\cdot) \stackrel{v}{\to} \mu \quad \text{on } \overline{\mathbb{R}}_0.$$

Therefore we assume that $X(\mathbf{s})$ is regularly varying with index $\alpha$ for some $\mathbf{s} \in (0,1]^d$. Following the ideas in Hult and Lindskog [13] in the case $d=1$, the càdlàg random field $X$ is regularly varying on $\mathbb{D}$ and the limit measure $m$ in (2.3) can be identified as $m = (\mathbb{LEB} \times \mu) \circ T^{-1}$, where $\mathbb{LEB}$ denotes Lebesgue measure and $T : [0,1]^d \times \overline{\mathbb{R}}_0 \to \overline{\mathbb{D}}_0$ is given by $T(\mathbf{t},x) = x\,I_{[\mathbf{t},\mathbf{1}]}(\mathbf{s})$, $\mathbf{s} \in [0,1]^d$. We conclude that the following property of $m$ in spherical coordinates holds. Let $\theta$ have distribution on $\{1,-1\}$ given by

$$P(\theta = 1) = \mu((1,\infty))/\mu(\{x \in \overline{\mathbb{R}} : |x| > 1\} = 1 - P(\theta = -1)\,,$$

independent of $\mathbf{U}$ which has a uniform distribution on $(0,1)^d$. The spectral measure is then given by

$$\sigma(\cdot) = P\left(\theta(I_{[\mathbf{U},\mathbf{1}]}(\mathbf{s}))_{\mathbf{s}\in[0,1]^d} \in \cdot\right)\,.$$

Hence, for $y > 0$,

$$\frac{m(\{x \in \overline{\mathbb{D}}_0 : |x|_\infty > y\,, \widetilde{x} \in \cdot\,\})}{m(\{x \in \overline{\mathbb{D}}_0 : |x|_\infty > 1\})} = y^{-\alpha}\,\sigma(\cdot)\,.$$

4.3. **Regularly varying Ornstein-Uhlenbeck processes.** Consider an Ornstein-Uhlenbeck process $X$ on $[0,1]$ driven by a regularly varying Lévy process $L$, i.e., a Lévy field with $d=1$, see Section 4.2. It has the stochastic integral representation

$$X(s) = \int_0^s \mathrm{e}^{\,-\lambda\,(s-y)}\,L(dy)\,, \quad s \in [0,1]\,,$$

where $\lambda > 0$ is a constant. It follows from Hult and Lindskog [13], Example 24, that if $L$ is regularly varying with index $\alpha > 0$, then $X$ is regularly varying on $\mathbb{D}$ with the same index and its spectral measure $\sigma$ on $\mathbb{S}_{\mathbb{D}}$ is given by

$$\sigma(\cdot) = P(\theta(\mathrm{e}^{\,-\lambda(s-U)}I_{[U,1]}(s))_{s\in[0,1]} \in \cdot)\,,$$

where $\theta$ and $U$ are as defined in Section 4.2. This example can be extended to filter functions $f(s,y)$ more general than the exponential function $f(s,y) = \mathrm{e}^{\,-\lambda(s-y)}$ (see Hult and Lindskog [13]) as well as to certain classes of predictable integrand processes (see Hult and Lindskog [14]). In particular, for special choices of the function $f$ and regularly varying Lévy processes $L$ one gets regularly varying continuous-time ARMA (CARMA) processes; see for example Brockwell [5].

4.4. **Regularly varying s$\alpha$s series.** In this section we consider the random field

$$X = \sum_{i=1}^{\infty} r_i \, \Gamma_i^{-1/\alpha} \, Y_i \,, \tag{4.1}$$

where $(\Gamma_i)$ is an increasing enumeration of the points of a unit rate Poisson process on $(0,\infty)$, independent of the sequences $(r_i)$ and $(Y_i)$, and $\alpha \in (0,2)$. Here $(Y_i)$ is an iid sequence of $\mathbb{D}$-valued random fields and $(r_i)$ is an iid Rademacher sequence, i.e., $P(r_i = \pm 1) = 0.5$. If $E(|Y(\mathbf{s}_i)|^\alpha) < \infty$, $\mathbf{s}_i \in [0,1]^d$, $i = 1, \ldots, k$, it follows from the theory of $\alpha$-stable processes that the $\mathbb{R}^k$-valued infinite series $(X(\mathbf{s}_1), \ldots, X(\mathbf{s}_k))$ converges a.s. and represents an s$\alpha$s-stable random vector; see Samorodnitsky and Taqqu [26], Chapter 3. In particular, the finite-dimensional distributions are regularly varying with index $\alpha$.

We will always assume that the infinite series in (4.1) converges a.s. in $\mathbb{D}$. Necessary and sufficient conditions for the a.s. convergence of (4.1) in $\mathbb{D}$ or in the space $\mathbb{C}$ in terms of distributional characteristics of $Y_i$ are known in some special cases. We discuss some of them.

**Example 4.1.** Assume that $Y_1$ assumes values in $\mathbb{C}$ and $0 < E(|Y_1|_\infty^\alpha) < \infty$ for some $\alpha \in (0,2)$. It follows from the reasoning in Ledoux and Talagrand [18], Chapter 5, in particular Corollary 5.5, that the infinite series (4.1) represents a symmetric $\alpha$-stable (s$\alpha$s) random field with values in $\mathbb{C}$ and every s$\alpha$s random field with values in $\mathbb{C}$ has such a series representation. It is also shown on p. 135 in [18] that

$$t^\alpha \, P\left( \Big| \sum_{i=2}^{\infty} r_i \, \Gamma_i^{-1/\alpha} \, Y_i \Big|_\infty > t \right) \to 0 \,, \quad t \to \infty \,, \tag{4.2}$$

and that $X = \Gamma_1^{-1/\alpha} \, r_1 \, Y_1 + R$ is regularly varying on $\overline{\mathbb{C}}_0$ (see pp. 134-136 in [18]) with the straightforward interpretation of $\overline{\mathbb{C}}_0$. Choosing $(a_n)$ such that

$$P(\Gamma_1^{-1/\alpha} |Y_1|_\infty > a_n) \sim E|Y_1|_\infty^\alpha \, a_n^{-\alpha} \sim n^{-1} \,,$$

and following the argument in [18], for any Borel set $S \in \mathcal{B}(\mathbb{S}_{\mathbb{C}})$ which is a continuity set with respect to the limiting measure,

$$\begin{aligned} n \, P\left( |X|_\infty > a_n \, t \,, \widetilde{X} \in S \right) &\sim n \, P\left( \Gamma_1^{-1/\alpha} |Y_1|_\infty > a_n \, t \,, \widetilde{Y}_1 \in S \right) \\ &\to t^{-\alpha} \frac{E\left( |Y_1|_\infty^\alpha I_S(\widetilde{Y}_1) \right)}{E(|Y_1|_\infty^\alpha)} \,. \end{aligned} \tag{4.3}$$

Hence $X$ is regularly varying with index $\alpha$ and spectral measure given on the right-hand side. Notice that this measure is only determined by the distribution of the first term in the series representation, and this is completely analogous to the case of max-stable random fields, see Remark 3.6. □

For $Y_i$ with values in $\mathbb{D}$, such general results about the a.s. convergence of the series (4.1) are not readily available. Indeed, the proof relies on the fact that the $\Gamma_i^{-1/\alpha} r_i Y_i$'s are random elements in a separable Banach space such as $\mathbb{C}$. However, results by Rosiński [23] in the case $d = 1$, in particular his Theorem 5.1, indicate that special cases of (4.1) with $\mathbb{D}$-valued $Y_i$'s represent s$\alpha$s Lévy motion on $[0,1]$. An exception is the case $\alpha \in (0,1)$:

**Example 4.2.** Assume $\alpha \in (0,1)$. We have for $m, h \geq 0$,

$$\Big| \sum_{i=m}^{m+h} \Gamma_i^{-1/\alpha} \, r_i \, Y_i \Big|_\infty \leq \sum_{i=m}^{m+h} \Gamma_i^{-1/\alpha} \, |Y_i|_\infty$$

If $E(|Y|_\infty^\alpha) < \infty$ then the right-hand side converges to 0 a.s. as $m, h \to \infty$. Hence the left-hand side is a Cauchy sequence with respect to the uniform topology. Since $\mathbb{D}$ can be made a complete separable metric space (after completing the $J_1$-metric) and a.s. convergence in the uniform sense implies a.s. convergence in the $J_1$-sense, we conclude that the infinite series (4.1) converges a.s. in $\mathbb{D}$. Hence $X$ is an element of $\mathbb{D}$. Adapting the argument on pp. 124–127 in Ledoux and Talagrand [18], we conclude that

$$n\,P\left(a_n^{-1}\sum_{i=2}^{\infty}\Gamma_i^{-1/\alpha}|Y_i|_\infty>\epsilon\right)=o(1)\,,\quad \epsilon>0\,.$$

This proves that $X$ inherits its tail behavior from the first term in the series (4.1), hence it is regularly varying on $\mathbb{D}$. □

## 5. Regular variation of linear combinations of random fields

5.1. **Regular variation.** In this section we prove regular variation of the linear processes $\sum_{i=1}^k \psi_i\,Z_i$, where $(Z_i)$ is an iid sequence of regularly varying random fields with values in $\mathbb{D}$. We start with a result for the truncated series.

**Lemma 5.1.** *Assume that $Z_1$ is regularly varying with index $\alpha$ and limiting measure $m_Z$, $\psi_i$, $i=1,\ldots,k$, are deterministic functions in $\mathbb{D}$ with $\min_{i=1,\ldots,k}|\psi_i|_\infty>0$. Then $\sum_{i=1}^k\psi_i\,Z_i$ is regularly varying with index $\alpha$ and limiting measure*

$$\mu^{(k)}=\sum_{i=1}^k m_Z\circ\psi_i^{-1}\,,$$

*where $\psi_i^{-1}(B)=\{x\in\overline{\mathbb{D}}_0:\psi_i\,x\in B\}$.*

**Proof.** For the sake of illustration we focus on the case $k=2$; the general case $k>2$ following from an inductive argument. We first note that $\psi_i Z_1$ is regularly varying. This follows by a direct application of Lemma 2.1 which yields

$$n\,P(a_n^{-1}\psi_i\,Z_1\in\cdot)\stackrel{\widehat{w}}{\to}\nu_i=m_Z\circ\psi_i^{-1}\,,\quad i=1,2\,.$$

Hence $Y_i=\psi_i Z_i$ are independent regularly varying random elements with values in $\mathbb{D}$. Next we show that $Y_1+Y_2$ is regularly varying. Since $Y_1,Y_2$ are independent it follows from standard regular variation theory (see Resnick [21, 22] or Hult and Lindskog [13]) that

$$n\,P(a_n^{-1}(\mathbf{Y}_1,\mathbf{Y}_2)\in(d\mathbf{u},d\mathbf{v}))\stackrel{v}{\to}\nu_{1:\mathbf{s}_1,\ldots,\mathbf{s}_k}(d\mathbf{u})\,\varepsilon_{\mathbf{0}}(d\mathbf{v})+\nu_{2:\mathbf{s}_1,\ldots,\mathbf{s}_k}(d\mathbf{v})\,\varepsilon_{\mathbf{0}}(d\mathbf{u})\,,$$

where $\varepsilon_{\mathbf{0}}$ is Dirac measure concentrated at $\mathbf{0}\in\mathbb{R}^k$, $\nu_{i:\mathbf{s}_1,\ldots,\mathbf{s}_k}$ are the restrictions of the measures $\nu_i$ as defined in (2.4) and

$$\mathbf{Y}_i=(Y_i(\mathbf{s}_1),\ldots\,Y_i(\mathbf{s}_k))\,,\quad i=1,2\,.$$

Here $\stackrel{v}{\to}$ denotes vague convergence on the Borel $\sigma$-field $\mathcal{B}(\overline{\mathbb{R}}_0^{2k}$. It follows from a multivariate version of Breiman's result (see Basrak et al. [1]) that linear transformations of $(\mathbf{Y}_1,\mathbf{Y}_2)$ are regularly varying. Hence $\mathbf{Y}_1+\mathbf{Y}_2$ is regularly varying with index $\alpha$ and limiting measure given by $\mu^{(2)}$ defined above.

Finally, we verify the tightness conditions (2.5) and (2.6) in Lemma 2.1. Notice that

$$\begin{aligned} w''(Y_1+Y_2,\delta) &\le w''(Y_1,\delta)+w''(Y_2,\delta)\,,\\ w(Y_1+Y_2,[0,1]^d\backslash[\delta,1-\delta]^d) &\le w(Y_1,[0,1]^d\backslash[\delta,1-\delta]^d)+w(Y_2,[0,1]^d\backslash[\delta,1-\delta]^d)\,,\end{aligned}$$

implying (2.5) and (2.6) for $Y_1+Y_2$ by the corresponding relations for $Y_1$ and $Y_2$. This proves the lemma. □

**Lemma 5.2.** *Assume that $Z_1$ is regularly varying with index $\alpha$ and limiting measure $m_Z$ and $(\psi_i)$ is a sequence of deterministic functions in $\mathbb{D}$ with $\min_i |\psi_i|_\infty > 0$ and*

$$\sum_{i=1}^\infty |\psi_i|_\infty^{\min(1,\alpha-\epsilon)} < \infty$$

*for some $\epsilon \in (0,\alpha)$. Then the infinite series $X = \sum_{i=1}^\infty \psi_i Z_i$ converges a.s. in $\mathbb{D}$. Moreover, $X$ is regularly varying with index $\alpha$ and limiting measure*

$$\mu = \sum_{i=1}^\infty m_Z \circ \psi_i^{-1} .$$

**Proof.** For fixed $m \geq 1$, write $X^{(m)} = \sum_{i=1}^m \psi_i Z_i$. The infinite series defining $X(\mathbf{s})$ is bounded by

$$|X(\mathbf{s})| \leq \sum_{i=1}^\infty |\psi_i|_\infty \, |Z_i|_\infty .$$

The right-hand side is a.s. convergent as a consequence of regular variation of $|Z_1|_\infty$ and the summability conditions on $(|\psi_i|_\infty)$; see Davis and Resnick [7]. Hence the infinite series $X(\mathbf{s})$ converges a.s. for every $\mathbf{s}$. Moreover, $|X^{(m)} - X|_\infty \to 0$ a.s. By virtue of this fact and since uniform convergence in $\mathbb{D}$ implies convergence in the Skorokhod metric, we conclude that the limiting random function $X$ is an element of $\mathbb{D}$.

Now we turn to regular variation of $X$. By Lemma 5.1, $X^{(m)}$ is regularly varying with index $\alpha$ for every $m \geq 1$. By the characterization (2.1) of $\widehat{w}$-convergence, it suffices to show that for any bounded continuous $f$ with support vanishing outside a bounded set,

$$n\, Ef(X/a_n) = \int f(x)\, [n\, P(a_n^{-1} X \in dx)] \to \int f(x)\, \mu(dx) .$$

Note that for any $m \geq 1$,

$$n\, Ef(X^{(m)}/a_n) \to \sum_{i=1}^m \int f(\psi_i\, x)\, m_Z(dx) = \int f(x)\, \mu^{(m)}(dx) .$$

Also, as $m \to \infty$,

$$\sum_{i=1}^m \int f(\psi_i\, x)\, m_Z(dx) \to \sum_{i=1}^\infty \int f(\psi_i\, x)\, m_Z(dx) = \int f(x)\, \mu(dx) . \tag{5.1}$$

This can be seen as follows. Suppose the support of $f$ is contained in the set $\{x : |x|_\infty > c\}$ and $K = \max_{x\in\mathbb{D}} |f(x)|$. Then

$$\begin{aligned}
\left| \sum_{i=m+1}^\infty \int f(\psi_i\, x)\, m_Z(dx) \right| &\leq K \sum_{i=m+1}^\infty m_Z(\{x : |\psi_i\, x|_\infty > c\}) \\
&\leq K \sum_{i=m+1}^\infty m_Z(\{x : |\psi_i|_\infty |x|_\infty > c\}) \\
&= K\, m_Z\{x : |x|_\infty > c\}) \sum_{i=m+1}^\infty |\psi_i|_\infty^\alpha \to 0 , \quad m \to \infty .
\end{aligned}$$

To complete the proof we show that

$$\lim_{m\to\infty} \limsup_{n\to\infty} n\, E \left| f(X/a_n) - f(X^{(m)}/a_n) \right| = 0 . \tag{5.2}$$

Set

$$w'(x,\epsilon) = \sup\{|f(x)-f(y)| : y\in \mathbb{D}\backslash\{0\}\,, d(x,y)<\epsilon\}\,,$$

where $d$ is the metric on $\overline{\mathbb{D}}_0$ induced by the Skorokhod topology. Note that $d(X^{(m)}/a_n, X/a_n) \le a_n^{-1}|X^{(m)}-X|_\infty$. Hence we may conclude that

$$\begin{aligned} & n\,E\left|f(X/a_n)-f(X^{(m)}/a_n)\right| \\ \le\; & n\,E\left[w'(X^{(m)}/a_n,\epsilon)\,I_{[0,\epsilon]}(|X-X^{(m)}|_\infty/a_n)\left(I_{(c,\infty)}(|X^{(m)}|_\infty/a_n)+I_{(c,\infty)}(|X|_\infty/a_n)\right)\right] \\ & +n\,K\,P(|X-X^{(m)}|_\infty > a_n\epsilon)\,. \end{aligned} \tag{5.3}$$

For $\epsilon$ small, the first term may be bounded by

$$\begin{aligned} & n\,E\left[w'(X^{(m)}/a_n,\epsilon)\,\left(I_{(c,\infty)}(|X^{(m)}|_\infty/a_n)+I_{(c-\epsilon,\infty)}(|X^{(m)}|_\infty/a_n)\right)\right] \\ \le\; & 2\,nE\left[w'(X^{(m)}/a_n,\epsilon)\,I_{(c-\epsilon,\infty)}(|X^{(m)}|_\infty/a_n)\right]\,. \end{aligned}$$

Using the $\widehat{w}$-convergence, the limit of this expression is

$$2\int w'(x,\epsilon)\,I_{(c-\epsilon,\infty)}(|x|_\infty)\,\mu^{(m)}(dx) \stackrel{m\to\infty}{\to} 2\int w'(x,\epsilon)\,I_{(c-\epsilon,\infty)}(|x|_\infty)\,\mu(dx) \stackrel{\epsilon\to 0}{\to} 0\,.$$

Finally, as to the second term in (5.3), we have

$$|X-X^{(m)}|_\infty \le \sum_{i=m+1}^\infty |\psi_i|_\infty\,|Z_i|_\infty\,,$$

which is regularly varying on $(0,\infty]$; see e.g. Embrechts et al. [8], Lemma A3.26. Hence

$$\lim_{m\to\infty}\limsup_{n\to\infty} n\,P(|X-X^{(m)}|_\infty > \epsilon\,a_n) \le \epsilon^{-\alpha}\lim_{m\to\infty}\sum_{i=m+1}^\infty |\psi_i|_\infty^\alpha = 0\,.$$

This completes the proof. □

5.2. **Point process convergence.** In this section we use the results about the regular variation of the linear combinations for showing point process convergence of the scaled linear process $(X_t)$ defined in (1.1).

**Proposition 5.3.** *For $m\ge 1$ fixed, consider the sequence of point processes*

$$I_n = \sum_{t=1}^n \varepsilon_{a_n^{-1}(Z_t,\ldots,Z_{t-m+1})}$$

*defined on $(\overline{\mathbb{D}}_0)^m$. Then $I_n \stackrel{d}{\to} I$ where $\stackrel{d}{\to}$ denotes convergence in distribution of point processes on the space $\widehat{M}((\overline{\mathbb{D}}_0)^m)$ and*

$$I = \sum_{i=1}^\infty \left[\varepsilon_{(P_i,0,\ldots,0)}+\varepsilon_{(0,P_i,0,\ldots,0)}+\cdots+\varepsilon_{(0,\ldots,0,P_i)}\right]\,.$$

*The space $\widehat{M}((\overline{\mathbb{D}}_0)^m)$ consists of the point measures on $(\overline{D}_0)^m$ endowed with the topology generated by $\widehat{w}$-convergence, and $\sum_{i=1}^\infty \varepsilon_{P_i}$ is $\mathrm{PRM}(m_Z)$ on $\overline{\mathbb{D}}_0$.*

**Proof.** Consider the class $\mathcal{S}$ of bounded sets of the form

$$B=\{(x_1,\dots,x_m)\in(\overline{\mathbb{D}}_0)^m\}:(|x_i|_\infty,\widetilde{x}_i)\in B_i\times C_i\,,i=1,\dots,m\}\,,$$

where $C_i\subset\mathbb{S}_{\mathbb{D}}$, $\sigma_Z(\partial C_i)=0$, and $B_i=(b_i,c_i]$ or $B_i=[0,c_i]$, $0\le b_i<c_i\le\infty$, $i=1,\dots,m$. It is easy to verify that this class of sets is a DC-semiring in the sense of Kallenberg [16]. Moreover, since $B\in\mathcal{S}$ is bounded away from $\mathbf{0}$, either $B=B_1\times\cdots\times B_m$ has empty intersection with all the coordinate axes or intersects only one axis in an interval. That is, with $\mathbf{e}_i$ being the basis element with $i$th component equal to 1 and the rest zero,

$$B_1\times\cdots\times B_m\cap\{y\mathbf{e}_i:y\ge 0\}=\emptyset\quad\text{for } i=1,\dots,m\,, \tag{5.4}$$

or

$$B_1\times\cdots\times B_m\cap\{y\mathbf{e}_i:y\ge 0\}=\begin{cases}\{0\}\times\cdots\times\{0\}\times B_j\times\{0\}\cdots\times\{0\} & i=j\,,\\ \emptyset & i\ne j\,.\end{cases} \tag{5.5}$$

In the latter case, we must have $B_i=[0,c_i]$ for $i\ne j$. We next show that

$$\widetilde{I}_n(B)-I_n(B)\stackrel{P}{\to}0\quad\text{for all } B\in\mathcal{S}\text{ such that } P(I(\partial B)=0)=1\,, \tag{5.6}$$

where

$$\widetilde{I}_n(B)=\sum_{t=1}^n\sum_{i=1}^m\varepsilon_{a_n^{-1}Z_t\mathbf{e}_i}\,.$$

For continuity sets $B$ satisfying (5.4), $\widetilde{I}_n(B)=0$ a.s. and

$$EI_n(B)\le n\,P(a_n^{-1}(|Z_m|,\dots,|Z_1|)\in B)=n\prod_{i=1}^m P(a_n^{-1}|Z_1|\in B_i)\to 0\,.$$

The limit is zero since in order for $B$ to be a continuity set, $b_i>0$ for at least two values of $i$. Hence (5.6) follows. For $B$ satisfying (5.5), so that $0\in B_i$ for all $i\ne j$ and $b_j>0$, we have $I_n(B)\le\widetilde{I}_n(B)$ and

$$\begin{aligned}P(\widetilde{I}_n(B)-I_n(B)>\epsilon)&\le P\left(\bigcup_{t=1}^n\{a_n^{-1}|Z_{t-j}|\in B_j\,,a_n^{-1}|Z_{t-i}|\notin B_i\text{ for some } i\ne j\}\right)\\&\le n\sum_{i\ne j}P(|Z_1|>a_n\,b_j\,,|Z_2|>a_n\,c_i)\\&\le n\sum_{i\ne j}P(|Z_1|>a_n\,b_j)\,P(|Z_2|>a_n\,c_i)\to 0\,.\end{aligned}$$

This proves that (5.6) holds for all $B\in\mathcal{S}$, as was to be shown.

To complete the proof, it suffices to show (see Daley and Vere-Jones [6], Corollary 9.1.VIII) that for any continuity sets $S_1,\dots,S_k\in\mathcal{S}$,

$$(I_n(S_1)\,,\dots,I_n(S_k))\stackrel{d}{\to}(I(S_1)\,,\dots,I(S_k))\,.$$

However, in view of (5.6), this will be implied by

$$(\widetilde{I}_n(S_1)\,,\dots,\widetilde{I}_n(S_k))\stackrel{d}{\to}(I(S_1)\,,\dots,I(S_k))\,.$$

The proof of this result follows by an application of the continuous mapping theorem; see e.g. the proof of Theorem 2.2 in Davis and Resnick [7]. This proves the proposition. □

**Proposition 5.4.** *For each $m \geq 1$, the sequence of point processes*

$$N_n^{(m)} = \sum_{t=1}^{n} \varepsilon_{a_n^{-1} X_t^{(m)}} \xrightarrow{d} N^{(m)} = \sum_{i=1}^{\infty} \sum_{j=0}^{m} \varepsilon_{\psi_j P_i},$$

*where $X_t^{(m)} = \sum_{j=0}^{m} \psi_j Z_{t-j}$ is the finite order moving average process, and $\min_{j=0,\dots,m} |\psi_j|_\infty > 0$.*

**Proof.** By the characterization of weak convergence on $\widehat{M}(\overline{\mathbb{D}}_0)$, we need to show that $N_n^{(m)}(f) \xrightarrow{d} N^{(m)}(f)$ for all continuous bounded functions $f$ that vanish off a bounded set. But $N_n^{(m)}(f) = I_n(f \circ T)$, where $T : (\overline{\mathbb{D}}_0)^{m+1} \to \overline{\mathbb{D}}_0$ is the mapping $T(\mathbf{u}) = \sum_{j=0}^{m} \psi_j \, u_j$. The composition function is continuous on the support $E$ of the point process $I$ in Proposition 5.3, i.e.,

$$\begin{aligned} E &= \big\{\overline{\mathbb{D}}_0 \times \{0\} \times \cdots \times \{0\}\big\} \cup \big\{\{0\} \times \overline{\mathbb{D}}_0 \times \{0\} \times \cdots \times \{0\}\big\} \cup \cdots \\ &\quad \cup \big\{\{0\} \times \cdots \times \{0\} \times \overline{\mathbb{D}}_0\big\}. \end{aligned}$$

To see this, suppose $\mathbf{u}^{(n)} \to \mathbf{u}$ with respect to the $J_1$-metric $d$ in $(\overline{\mathbb{D}}_0)^{m+1}$. If the limit vector is in $E$ then $u_j \neq 0$ for some $j$ and $u_i = 0$ for $i \neq j$. It follows that $|u_i^{(n)}|_\infty \to 0$ for all $i \neq j$. Hence

$$\begin{aligned} d\left(\sum_{i=0}^{m} \psi_i \, u_i^{(n)}, \sum_{i=0}^{m} \psi_i \, u_i\right) &= d\left(\sum_{i=0}^{m} \psi_i \, u_i^{(n)}, \psi_j \, u_j\right) \\ &\leq d\left(\sum_{i=0}^{m} \psi_i \, u_i^{(n)}, \psi_j \, u_j^{(n)}\right) + d\left(\psi_j \, u_j^{(n)}, \psi_j \, u_j\right). \end{aligned}$$

The first term converges to zero since $|\sum_{i=0 \neq j}^{m} \psi_i \, u_i^{(n)}|_\infty \to 0$ while the second term converges to zero since $\max_j |\psi_j|_\infty < \infty$.

In addition, the continuous mapping $f \circ T$ has bounded support. Suppose the support of $f$ is contained in $\{x : |x|_\infty > c\}$ for some $c > 0$. Then the support of $f \circ T$ is contained in the set

$$\left\{\mathbf{u} : \left|\sum_{i=0}^{m} \psi_i \, u_i\right|_\infty > c\right\} \subset \left\{\mathbf{u} : \sum_{i=0}^{m} |\psi_i|_\infty \, |u_i|_\infty > c\right\},$$

which is a bounded set on $(\overline{\mathbb{D}}_0)^m$.

Now, applying the characterization of $\widehat{w}$-convergence, it follows that

$$I_n(f \circ T) = N_n^{(m)} \xrightarrow{d} N^{(m)}(f) = I(f \circ T),$$

which proves the proposition. □

We are now ready to state and prove the main result of this section which extends the above result to the infinite moving average case.

**Theorem 5.5.** *Under the assumptions of Proposition 5.4,*

$$N_n = \sum_{t=1}^{n} \varepsilon_{a_n^{-1} X_t} \xrightarrow{d} N = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \varepsilon_{\psi_j \, P_i}.$$

**Proof.** To transfer the point process convergence result of Proposition 5.4 onto $N_n$, it suffices to show, by Theorem 4.2 in Billingsley [3], that for any $\eta > 0$,

$$\lim_{m \to \infty} \limsup_{n \to \infty} P\left(\widetilde{\rho}\left(N_n^{(m)}, N_n\right) > \eta\right) = 0 \tag{5.7}$$

and

$$\sum_{i=1}^{\infty}\sum_{j=0}^{m}\varepsilon_{\psi_j\,P_i}\xrightarrow{d}\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\varepsilon_{\psi_j\,P_i}\,, \tag{5.8}$$

where $\widetilde{\rho}$ is a metric on $\widehat{M}(\overline{\mathbb{D}}_0)$. Relation (5.8) is immediate; see e.g. (5.1). For (5.7), we show that

$$\lim_{m\to\infty}\limsup_{n\to\infty}P\left(\sum_{t=1}^{n}\left|f(X_t/a_n)-f(X_t^{(m)}/a_n)\right|>\eta\right)=0\,, \tag{5.9}$$

for every bounded continuous function $f$ which vanishes off a bounded set. By the form of the metric $\widetilde{\rho}$, it will then follow that (5.7) holds. The probability in (5.9) is bounded by

$$\eta^{-1}\,n\,E\left|f(X_1/a_n)-f(X_1^{(m)}/a_n)\right|\,,$$

which converges to zero by first letting $n\to\infty$ and then $m\to\infty$, in view of (5.2). This proves the theorem. □

5.3. **Some applications.**

**Example 5.6.** Let $A\subset[0,1]^d$ be a Borel set and define $f_A:\mathbb{D}\to\mathbb{R}$ by

$$f_A(x)=\int_A x(\mathbf{s})\,d\mathbf{s}\,,\quad x\in\mathbb{D}\,.$$

Although we only consider simple averages here, one could also study averages relative to a kernel function given by $\int_A K(\mathbf{t}-\mathbf{s})\,x(\mathbf{s})d\mathbf{s}$. The functional $f_A$ is continuous with respect to the $J_1$-metric. Indeed, consider a sequence $(x_n)$ in $\mathbb{D}$ converging to $x$ in $\mathbb{D}$ in the $J_1$-sense. Then there exist continuous bijections $\lambda_n:[0,1]^d\to[0,1]^d$ which are increasing in every component and such that $\lambda_n(\mathbf{0})=\mathbf{0}$, $\lambda_n(\mathbf{1})=\mathbf{1}$, and

$$d(x_n,x)=\sup_{\mathbf{s}\in[0,1]^d}|x_n(\lambda_n(\mathbf{s}))-x(\mathbf{s})|\vee|\lambda_n(\mathbf{s})-\mathbf{s}|\to0\,,\quad n\to\infty\,.$$

Then

$$\begin{aligned}|f_A(x_n)-f_A(x)| &= \left|\int_A(x_n(\lambda_n(\mathbf{s}))-x(\mathbf{s}))\,d\mathbf{s}+\int_A(x_n(\mathbf{s})-x_n(\lambda_n(\mathbf{s})))\,d\mathbf{s}\right|\\ &\le |A|\,d(x_n,x)+\int_{A_c}|x_n(\mathbf{s})-x_n(\lambda_n(\mathbf{s}))|\,d\mathbf{s}\,,\end{aligned}$$

where $A_c$ denotes the set of continuity points of $x$ in $A$. A dominated convergence argument ensures that the right-hand side converges to zero as $n\to\infty$. Hence $f_A$ is a continuous mapping.

For $\epsilon>0$, note that

$$\{x\in\overline{\mathbb{D}}_0:|f_A(x)|>\epsilon\}\subseteq\{x\in\overline{\mathbb{D}}_0:|x|_\infty|A|>\epsilon\}\,.$$

Hence $f_A(x)$ transforms bounded sets in $\overline{\mathbb{R}}\backslash\{0\}$ into bounded sets in $\overline{\mathbb{D}}_0$ provided $A$ has positive Lebesgue measure. Thus, if $X$ is a regularly varying random field with index $\alpha>0$ and limiting measure $m$ in $\overline{\mathbb{D}}_0$, and if $A\subset[0,1]^d$ has positive Lebesgue measure, then the continuous mapping theorem for regularly varying $\mathbb{D}$-valued random fields ensures that $f_A(X)$ is regularly varying as well with limiting measure $m\circ f_A^{-1}$. In particular, if $X_t$ is a linear random field as defined in (1.1), then for $y > 0$,

$$\begin{aligned} n\,P\left(a_n^{-1}\int_A X_t(\mathbf{s})\,d\mathbf{s} > y\right) &\to \sum_{i=0}^{\infty}(m_Z\circ\psi_i^{-1}\circ f_A^{-1})(y,\infty) \\ &= \sum_{i=0}^{\infty} m_Z(\{x\in\overline{\mathbb{D}}_0 : \int_A \psi_i(\mathbf{s})x(\mathbf{s})\,d\mathbf{s} > y\}) \\ &= y^{-\alpha}\sum_{i=0}^{\infty} m_Z(\{x\in\overline{\mathbb{D}}_0 : \int_A \psi_i(\mathbf{s})x(\mathbf{s})\,d\mathbf{s} > 1\})\,. \end{aligned}$$

Further, if $\psi_i(\mathbf{s}) = \psi_i$ for some constants $\psi_i$ and all $\mathbf{s}\in A$, then we obtain

$$y^{-\alpha}\left[m_Z(\{x\in\overline{\mathbb{D}}_0 : \int_A x(\mathbf{s})\,d\mathbf{s} > 1\})\sum_{i=0}^{\infty}(\psi_i)_+^{\alpha} + m_Z(\{x\in\overline{\mathbb{D}}_0 : \int_A x(\mathbf{s})\,d\mathbf{s} < -1\})\sum_{i=0}^{\infty}(\psi_i)_-^{\alpha}\right].$$

The same result can be derived by observing that $\int_A X_t(\mathbf{s})\,d\mathbf{s}$ has representation as a one-dimensional linear process with iid regularly varying noise $\int_A Z_i(\mathbf{s})\,d\mathbf{s}$. Results of this kind can be found e.g. in Embrechts et al. [8], Appendix 3.3.

If we further specify the noise $Z$ to be a regularly varying Lévy random field as considered in Section 3.2, the latter expression further simplifies. For example, for a random vector $\mathbf{V}$ with uniform distribution on $(0,1)^d$, $\theta$ independent of $\mathbf{V}$ with distribution as described in Section 3.2,

$$\begin{aligned} m_Z(\{x\in\overline{\mathbb{D}}_0 : \int_A x(\mathbf{s})\,d\mathbf{s} > 1\}) &= \alpha\int_1^{\infty}\int_{[0,1]^d} P(\theta I_{[\mathbf{v},\mathbf{1}]\cap A} > 1)\,d\mathbf{v}\,\theta^{-\alpha-1}\,d\theta \\ &= \alpha\int_1^{\infty}\int_{[0,1]^d} I_{[\mathbf{v},\mathbf{1}]\cap A}\,d\mathbf{v}\,\theta^{-\alpha-1}\,d\theta \\ &= |A|\,. \end{aligned}$$

**Example 5.7.** In this example, we consider the limiting distribution of the space-time maxima $\max_{t=1,\dots,n}|X_t|_\infty$. ¿From Theorem 5.5 and the continuous mapping theorem we have

$$N_n^{(1)} = \sum_{i=1}^{\infty}\varepsilon_{a_n^{-1}|X_t|_\infty} \xrightarrow{d} N^{(1)} = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\varepsilon_{|\psi_j P_i|_\infty}\,.$$

Hence

$$\begin{aligned} P(a_n^{-1}\max_{t=1,\dots,n}|X_t|_\infty \le y) &= P(N_n^{(1)}((y,\infty]) = 0) \\ &\to P(N^{(1)}((y,\infty]) = 0) \\ &= P(\sup_{i\ge 1,\,j\ge 0}|\psi_j P_i|_\infty \le y) = G(y)\,. \end{aligned} \tag{5.10}$$

In order to get explicit formulas for the limit distribution, we assume that $\psi_j(\mathbf{s}) = \psi_j$ for some constants $\psi_j$ and all $j\ge 0$. Then

$$G(y) = P(\sup_i |P_i|_\infty \le y/\sup_j|\psi_j|)\,.$$

The points $P_i$, $i = 1,2,\dots$, constitute a PRM on $\overline{\mathbb{D}}_0$ with mean measure $m_Z$, hence the points $|P_i|_\infty$, $i = 1,2,\dots$, constitute a PRM on $(0,\infty]$ with mean measure of $(z,\infty]$ given by

$$m_Z(\{x\in\overline{\mathbb{D}}_0 : |x|_\infty > z\}) = z^{-\alpha}m_Z(\{x\in\overline{\mathbb{D}}_0 : |x|_\infty > 1\}) = c_Z\,z^{-\alpha}\,.$$

Hence

$$G(y) = \mathrm{e}^{-c_Z\,(\sup_i |\psi_i|/y)^\alpha}\,.$$

A straightforward calculation shows that

$$n\,P(|X|_\infty > a_n y) \sim y^{-\alpha}\,c_Z \sum_{j=0}^\infty |\psi_j|^\alpha\,.$$

The extremal index of the process $(|X_t|_\infty)$ can then be read off from $G$ as

$$\max_{j=0,1,\ldots} |\psi_j|^\alpha / \sum_{j=0}^\infty |\psi_j|^\alpha\,. \tag{5.11}$$

We refer to Leadbetter et al. [19] and Embrechts et al. [8], Section 8.1, for the definition and properties of the extremal index of a strictly stationary sequence. We also mention that the extremal index of $(|X_t|_\infty)$ coincides with the extremal index of the absolute value sequence of a one-dimensional linear process $Y_t = \sum_{j=0}^\infty \psi_j \eta_{t-j}$, where $(\eta_j)$ is an iid sequence of real-valued regularly varying random variables with index $\alpha > 0$. Alternatively, the value (5.11) coincides with the extremal index of the sequence $\sum_{j=0}^\infty |\psi_j|\,|Z_{t-j}|_\infty$. These facts about the extremal index of a linear process follow from Davis and Resnick [7].

It is a rather surprising fact that the extremal indices of the sequences $(|X_t|_\infty)$, $(\sum_{j=0}^\infty |\psi_j|\,|Z_{t-j}|_\infty)$ and $(|X_t(\mathbf{s})|)$, $\mathbf{s} \in [0,1]^d$, coincide. A particular consequence is that the extremal index (5.11) can be estimated from the time series of observations $|X_t(\mathbf{s})|$ at any site $\mathbf{s}$.

**Example 5.8.** A result analogous to (5.10) can be obtained for the sequence of maxima at a finite number of sites $\mathbf{s}_i \in [0,1]^d$. We illustrate the case with two distinct sites $\mathbf{s}_1$ and $\mathbf{s}_2$. Then the continuous mapping theorem yields

$$\sum_{i=1}^\infty \varepsilon_{(X_i(\mathbf{s}_1),X_i(\mathbf{s}_2))/a_n} \stackrel{d}{\to} \sum_{i=1}^\infty \sum_{j=0}^\infty \varepsilon_{(\psi_j(\mathbf{s}_1)P_i(\mathbf{s}_1),\psi_j(\mathbf{s}_2)P_i(\mathbf{s}_2))}$$

We conclude that

$$\begin{aligned}
& P(a_n^{-1} \max_{t=1,\ldots,n} X_t(\mathbf{s}_1) \le y_1\,, a_n^{-1} \max_{t=1,\ldots,n} X_t(\mathbf{s}_2) \le y_2) \\
\to\ & P(\sup_{i\ge 1,\, j\ge 0} \psi_j(\mathbf{s}_1)P_i(\mathbf{s}_1) \le y_1\,, \sup_{i\ge 1,\, j\ge 0} \psi_j(\mathbf{s}_2)P_i(\mathbf{s}_2) \le y_2) \\
=\ & P(\sup_{j\ge 0} \psi_j(\mathbf{s}_1) \sup_{i\ge 1} P_i(\mathbf{s}_1) \le y_1\,, \sup_{j\ge 0} \psi_j(\mathbf{s}_2) \sup_{i\ge 1} P_i(\mathbf{s}_2) \le y_2) = p(y_1,y_2)\,.
\end{aligned}$$

We assume that both $\Psi_i = \sup_{j\ge 0} \psi_j(\mathbf{s}_i)$, $i=1,2$, are positive. Then

$$\begin{aligned}
-\log p(y_1,y_2) &= m_Z(\{x \in \overline{\mathbb{D}}_0 : (x(\mathbf{s}_1),x(\mathbf{s}_2)) \notin [0,y_1/\Psi_1] \times [0,y_2/\Psi_2]\}) \\
&= (\Psi_1/y_1)^\alpha\, m_Z(\{x \in \overline{\mathbb{D}}_0 : x(\mathbf{s}_1) > 1\}) + (\Psi_2/y_2)^\alpha\, m_Z(\{x \in \overline{\mathbb{D}}_0 : x(\mathbf{s}_2) > 1\}) \\
&\quad - m_Z(\{x \in \overline{\mathbb{D}}_0 : x(\mathbf{s}_1) > y_1/\Psi_1\,, x(\mathbf{s}_2) > y_2/\Psi_2\})\,.
\end{aligned}$$

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