On the rate of aeolian sand transport

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Abstract: An analytic formula for for rate of aeolian sand transport is derived by introducing a few reasonable approximations in the equations of motion of the sand grains and in the model for the modification of the wind profile by the saltation cloud. The derivation here is simpler and uses fewer approximations than an earlier derivation of the formula.

Résumé: Une formule analytique déterminant l'efficacité du transport éolien du sable est dérivée en introduisant quelques approximations dans un modèle décrivant le processus de saltation. La dérivation dans ce cas est plus simple et on fait moins d'approximations que dans une dérivation précédente de cette formule.

Key words: Flow in the saltation layer, grain mechanics, random variability, saltation

1. Introduction

Within the last 10 - 15 years a number of numerical models of aeolian sand transport have been published ([2], [3], [6], [7], [14], [15], [16]) that incorporate the present understanding of the physics of sand transport by wind, see [4]. An analytic model of sand transport by wind based on the same physics ([13]) resulted in a formula for the transport rate that seems to compare well with transport rate data ([11]); see also the discussion in [8]. The presentation here focuses on the transport rate formula, which is derived in a way that compared to the derivation in [13] is simpler and more direct and which uses fewer approximations. The work presented here is strongly inspired by the work of the late Paul Owen, see [9] and [10].

2. Motion of the grains

All sand grains are assumed to be identical. In particular, they have the same shape, so that the drag on a grain is a function only of the relative speed v between the grain and the air. The drag on a grain at the relative speed v is denoted by $D(v)$. The grain mass is denoted by m.

The equations of motion of a sand grain during saltation are

$$\begin{aligned} \ddot{x} &= H(v)(U(y) - \dot{x}) \\ \ddot{y} + g + H(v)\dot{y} &= 0. \end{aligned} \tag{1}$$

Here $x(t)$ and $y(t)$ denote the horizontal and the vertical position of the grain, respectively, $H(v) = D(v)/(mv)$, and $U(y)$ is the mean wind speed at height y. It is assumed that $(x(0), y(0)) = (0, 0)$ and that the launch velocity $(\dot{x}(0), \dot{y}(0)) = \vec{v}_0 = (v_1^0, v_2^0)$ is random.

The first approximation made is that the drag is assumed to be proportional to the relative speed v:

$$D(v) = \delta v. \tag{2}$$

This approximation was also made by Owen in [9]. It is essential that δ is chosen in such a way that the drag is correct at a typical relative velocity v^* during saltation trajectories. For friction speeds in the range 45 – 60 cm/s, $v^* = 235$ cm/s is a reasonable value for saltating grains of size 300 μm. Using (2), we find that $H(v)$ is constantly equal to t_*^{-1}, where $t_* = m/\delta$ can be interpreted as the response time of a grain to changes in the wind.

As a consequence of (2), the equation for the vertical motion of a grain is decoupled from the horizontal motion. The equation for $y(t)$ is

$$t_*\ddot{y} + v_f + \dot{y} = 0, \tag{3}$$

where $v_f = gt_*$ can be interpreted as an approximate terminal velocity of fall for the grains. The solution of (3) is

$$y(t) = t_*(v_f + v_2^0)(1 - e^{-t/t_*}) - v_f t. \tag{4}$$

The height of the grain trajectory is

$$y^{\text{top}} = t_*(v_2^0 - v_f \ln(1 + v_2^0/v_f)). \tag{5}$$

The equation for the horizontal motion is

$$t_*\ddot{x} = U(y) - \dot{x}, \tag{6}$$

which has the solution

$$x(t) = \int_0^t \left(1 - e^{-(t-s)/t_*}\right) U(y(s))ds + t_* v_1^0 \left(1 - e^{-t/t_*}\right), \tag{7}$$

so that

$$\dot{x}(t) = t_*^{-1} \int_0^t e^{-(t-s)/t_*} U(y(s))ds + v_1^0 e^{-t/t_*}. \tag{8}$$

3. The wind profile in the saltation layer

The theory of how the wind profile is modified by the cloud of saltating grains is based on the concept of grain borne shear stress that was introduced by Owen in [9]. The grain borne shear stress at height y is the part of the shear stress in the grain free logarithmic layer above the saltation layer that at height y is carried by the saltating grains. It can be calculated as the mean momentum extracted from the flow by the grains above the level y, and is given by

$$T(y) = \Phi\overline{[\dot{x}(t_y^2) - \dot{x}(t_y^1)]1_{\{y^{\text{top}}>y\}}}, \tag{9}$$

see [12]. In (9), Φ is the mean mass of grains that leaves one unit of area per time unit, t_y^1 is the time at which a grain passes the level y on its way up, t_y^2 is the time at which the grain passes the level y on its way down. Finally, the symbol $1_{\{y^{\text{top}}>y\}}$ indicates that the mean is only taken over trajectories with a height $y^{\rm top}$ larger than y. The trajectory height is given by (5).

The second approximation made in the theory presented here is that the grain borne shear stress is calculated by means of (8) using the wind profile

$$U(z) = \kappa^{-1} U_*^{\rm cr} \ln(z/z_0), \tag{10}$$

where κ von Kármán's constant, $U_*^{\rm cr}$ is the minimal friction speed at which sand can be transported, and z_0 is the equivalent roughness height. This is a more accurate approximation than the one made in [13]. We write the grain borne shear stress at height y as

$$T(y) = \Phi v_f a(y),$$

where the dimensionless function a is given by

$$a(y) = \left(\overline{[\dot{x}(t_y^2) - \dot{x}(t_y^1)] 1_{\{y^{\rm top} > y\}}}\right) / v_f. \tag{11}$$

The wind profile in the saltation layer derived here is based on eddy viscosity theory. A natural assumption, is that the eddy viscosity is given by

$$\nu(z) = \kappa z \sqrt{U_*^2 - T(z)/\rho} = \kappa z U_* \sqrt{1 - T(z)/(\rho U_*^2)}, \tag{12}$$

as was first proposed by Anderson in [1]. Here U_* is the friction speed in the grain-free logarithmic layer above the saltation layer and κ is von Kármán's constant. The quantity $\sqrt{U_*^2 - T(z)/\rho}$ is an equivalent friction speed corresponding to the air borne shear stress at height y. The expression (12) implies that

$$\frac{dU}{dz} = \frac{U_*}{\kappa z} \sqrt{1 - T(z)/(\rho U_*^2)}. \tag{13}$$

A simpler expression for the wind profile is obtained if we approximate $\sqrt{1-x}$ by $1-x$ for x between zero and one:

$$U(z) = \kappa^{-1} U_* \ln(z/z_0) - \frac{\Phi v_f}{\kappa \rho U_*} b(z), \tag{14}$$

where the dimensionless function b is given by

$$b(z) = \int_{z_0}^{z} y^{-1} a(y) dy. \tag{15}$$

This expression for the wind profile in the saltation layer was obtained in [12] by assuming that the eddy viscosity is $\nu(z) = \kappa z U_*$.

In order to determine how the flux Φ depends on the friction speed, we use Bagnold's focus point. Bagnold found empirically, see [5] p. 59, that at a certain height $\bar{z}$ the wind speed has approximately the same value $\bar{U}$ at all friction speeds, i.e.

$$\kappa^{-1} U_* \ln(\bar{z}/z_0) - \frac{\Phi v_f}{\kappa \rho U_*} b(\bar{z}) = \bar{U} \tag{16}$$

at all friction speeds, so that

$$\begin{aligned}\Phi &= \frac{\rho U_* \left[U_* \ln(\bar{z}/z_0) - \kappa \bar{U}\right]}{b(\bar{z}) v_f} \\ &= \frac{K\rho}{v_f} U_*^2 \left(1 - \frac{U_*^{\rm cr}}{U_*}\right). \end{aligned} \tag{17}$$

Here

$$U_*^{\rm cr} = \frac{\kappa \bar{U}}{\ln(\bar{z}/z_0)} \tag{18}$$

can be interpreted as the critical friction speed, i.e. the smallest friction speed at which sand can be transported, and $K = \ln(\bar{z}/z_0)/b(\bar{z})$ is a constant that is related to the efficiency of the transport mechanism. By inserting (17) in (14) we obtain

$$U(z) = \kappa^{-1} U_* \left[\ln(z/z_0) - K\left(1 - \frac{U_*^{\rm cr}}{U_*}\right) b(z)\right]. \tag{19}$$

4. The transport rate

The transport rate is given by

$$Q = \Phi \overline{x(t_i)}, \tag{20}$$

see [12]. Here t_i is the time of impact, and $x(t_i)$ is the length of a saltation jump. By inserting the wind profile (19) in (7) we find that

$$\begin{aligned} x(t_i) &= \kappa^{-1} U_* \int_0^{t_i} \left(1 - e^{-(t_i - s)/t_*}\right) \ln(y(s)/z_0) ds \\ &\quad - \frac{\Phi v_f}{\kappa \rho U_*} \int_0^{t_i} \left(1 - e^{-(t_i - s)/t_*}\right) b(y(s)) ds \\ &\quad + t_* v_1^0 \left(1 - e^{-t_i/t_*}\right). \end{aligned} \tag{21}$$

Thus the dimensionless transport rate is given by

$$\frac{Qg}{\rho U_*^3} = \frac{K}{t_*}\left[\alpha + \frac{U_*^{\rm cr}}{U_*}\left(\frac{\beta}{U_*^{\rm cr}} - \alpha\right) - \left(\frac{U_*^{\rm cr}}{U_*}\right)^2 \frac{\beta}{U_*^{\rm cr}}\right], \tag{22}$$

where α is the mean jump length of a grain with $v_1^0 = 0$ in the wind profile $\kappa^{-1}(\ln(z/z_0) - Kb(z))$, while β is the mean jump length of a grain in the wind profile $\kappa^{-1} U_*^{\rm cr} K b(z))$. The constants K, α and β can be determined empirically.

5. References

[1] R.S. Anderson, Sediment transport by wind: saltation, suspension and ripples, Ph.D. thesis, Univ. of Washington, 1986.

[2] R.S. Anderson, P.K. Haff, Simulation of eolian saltation, Science 241 (1988) 820 - 823.

[3] R.S. Anderson, P.K. Haff, Wind modification and bed response during saltation of sand in air, Acta Mechanica [Suppl] 1 (1991) 21 - 51.

[4] R.S. Anderson, M. Sørensen, B.B. Willetts, A review of recent progress in our understanding of aeolian sediment transport, Acta Mechanica [Suppl] 1 (1991) 1 - 19.

[5] R.A. Bagnold, The physics of blown sand and desert dunes, Methuen, London, 1941.

[6] I.K. McEwan, B.B. Willetts, Numerical model of the saltation cloud, Acta Mechanica [Suppl] 1 (1991) 53 - 66.

[7] I.K. McEwan, B.B. Willetts, Adaptation of the near surface wind to aeolian sand transport, J. Fluid Mech. 252 (1993) 99 - 115.

[8] I.K. McEwan, B.B. Willetts, On the prediction of bed-load sand transport in air, Sedimentology 41 (1994) 1241 - 1251.

[9] P.R. Owen, Saltation of uniform grains in air, J. Fluid Mech. 20 (1964) 225 - 242.

[10] P.R. Owen, The physics of sand movement, Lecture notes, Workshop on physics of desertification, Trieste 1980.

[11] K.R. Rasmussen, H.E. Mikkelsen, Wind tunnel observations of aeolian transport rates, Acta Mechanica [Suppl] 1 (1991) 135 - 144.

[12] M. Sørensen, Estimation of some aeolian saltation transport parameters from transport rate profiles, International Workshop on the Physics of Blown Sand, Vol. 2 , Dept. of Theor. Statist., Univ. of Aarhus, 1985, pp. 141 - 190.

[13] M. Sørensen, An analytic model of wind-blown sand transport, Acta Mechanica [Suppl] 1 (1991) 67 - 81.

[14] P.-J. Spies, I.K. McEwan, Equilibrium of saltation, Earth Surf. Process. and Landforms 25 (2000) 437 - 453.

[15] P.-J. Spies, I.K. McEwan, Butterfield, G.R., One-dimensional transitional behaviour in saltation, Earth Surf. Process. and Landforms 25 (2000) 505 - 518.

[16] B.T. Werner, A steady-state model of wind-blown sand transport, J. Geol. 98 (1990) 1 - 17.