Proposed by A. A. Dzhumadil’daeva, Almaty, Republics Physics and Mathematics School, Almaty, Kazakhstan.

Let \( n!! \) denote the product of all positive integers not greater than \( n \) and congruent to \( n \mod 2 \), and let \( 0!! = (-1)!! = 1 \). Thus \( 7!! = 105 \) and \( 8!! = 384 \). For positive integer, \( n \), find in closed form:

\[
\sum_{i=0}^{n} \binom{n}{i} (2i - 1)!!(2(n - i) - 1)!!
\]

Solution: \( 2^n n! \).

Proof:

We supply the products of odd numbers with the missing even numbers to write the sum

\[
\frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} \frac{(2i)!(2(n - i))!}{i!(n - i)!}
\]

Introducing the notation of a descending factorial:

\[
[x]_n := x(x - 1) \cdots (x - n + 1)
\]

(Of course, \( [x]_0 = 0 \)) we may skip the common factors from the fractions and write

\[
\frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} [2i][2(n - i)]_{n-i}
\]

Now we apply the well known identity

\[
[2i]_i = (-4)^i[-\frac{1}{2}]_i
\]

to rewrite the sum as

\[
(-2)^n \sum_{i=0}^{n} \binom{n}{i} [-\frac{1}{2}]_i [-\frac{1}{2}]_{n-i}
\]

This sum may be recognized as the well known Chu–Vandermonde sum, so it equals

\[
(-2)^n [-\frac{1}{2} - \frac{1}{2}]_n = 2^n n!
\]

In my recent textbook, *Summa Summarum*, A K Peters 2007, we find the Chu–Vandermonde formula as no. 8.2 and the transformation as formula 5.12.