Solution to the problem 11164 from José Luis Díaz-Barrero in Amer. Math. Monthly 112,6 p. 568.

Proof of the formula

\[ \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \sum_{1 \leq i \leq j \leq k} \frac{1}{ij} = \frac{1}{n^2} \]

Introducing harmonic numbers as

\[ H_n = \sum_{k=1}^{n} \frac{1}{k} \]

I shall prefer to write the formula as

\[ \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \sum_{j=1}^{k} \frac{1}{j} H_j = \frac{1}{n^2} \]

Now, summation by parts yields this sum to equal

\[ 1 - \sum_{k=1}^{n} (-1)^{k+1} \binom{n-1}{k} \frac{1}{k+1} H_{k+1} = 1 - \frac{1}{n} \sum_{k=2}^{n+1} (-1)^{k} \binom{n}{k} H_k \]

As we may write the terms as a difference of the function

\[ g(n, k) = (-1)^{k-1} \left( \binom{n-1}{k-1} H_k - \frac{1}{n} \binom{n-1}{k} \right) \]

that is

\[ (-1)^{k} \binom{n}{k} H_k = g(n, k + 1) - g(n, k) \]

the sum becomes telescoping to yield only the last term for \( k = 2 \):

\[ 1 + \frac{1}{n} g(n, 2) = 1 - \frac{1}{n} \left( \left( \binom{n-1}{1} H_2 - \frac{1}{n} \binom{n-1}{2} \right) \right) = \frac{1}{n^2} \]