Faculty of Natural Sciences
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Bachelor's Thesis

Group Cohomology

With a view towards Gorenstein Rings

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Abstract

In this thesis we describe the basic properties of homology and cohomology with coefficients. Induction and coinduction is shown to coincide when the subgroup H of G has finite index. We then proceed with a proof of Shapiro's Lemma.

We study the group algebra over a field and prove Maschke's theorem. The group algebra over a field is a self-injective ring, and we notice that the group algebra of a p-group over a field of characteristic p is a local Gorenstein ring.

Contents

Introduction

This project takes off from basic homological algebra, where the bifunctors $- \otimes_R -$ and $\mathrm{Hom}_R(-,-)$ and their derived functors $\mathrm{Tor}^R(-,-)$ and $\mathrm{Ext}_R(-,-)$ have been introduced.

Firstly, we define the group algebra formed by a group G and a commutative ring. Then group homology and group cohomology is introduced, and we show certain properties of these concepts.

We study the case where H is a subgroup of G. We will define induction and coinduction as ways of handling kH-modules as kG-modules. The *induced module*, $\mathrm{Ind}_H^G A$, will be defined as $kG \otimes_{kH} A$ and the *co-induced module* $\mathrm{Coind}_H^G A$ as $\mathrm{Hom}_{kH}(kG, A)$, and we will prove some statements about the relation between the induced and co-induced modules and the originals.

One main goal of this project is to prove that induction and coinduction coincide when the subgroup H of G has finite index. Another main goal will be to prove Shapiro's Lemma which states that the homology of $\mathrm{Ind}_H^G M$ with coefficients in G is isomorphic to the homology of M with coefficients in H, and, dually, that the cohomology of $\mathrm{Coind}_H^G M$ with coefficients in G is isomorphic to the cohomology of M with coefficients in H.

Having described the general concept of group cohomology, we will take a look at very special case – a case where there is no cohomology! When G is a finite group and k is a field, we will see that the group algebra kG is a self-injective ring. If k is a field of characteristic $p \geq 2$ and G is a p-group, then kG is a local ring and hence a *zero-dimensional local Gorenstein ring*.

One could say that to what is a group cohomological question, we find a ring theoretical answer.

Gorenstein rings have been studied in great detail. Gorenstein Rings are fundamental in modern ring theory and are beginning to play a significant role in algebraic topology. See [Matsumura86], [Avramov97], [Frankild05] and [Dwyer04].

The chapters one and two of this work have sought inspiration from [Weibel94, Chapter 6 sections 1 through 3], [Brown82, Chapter III, sections 0 through 3 and section 5], [Andersen99, chapter 2] and [Iyengar04, Chapter 1]. Chapter three of this work mainly seeks inspiration in [Iyengar04, chapters 2 and 3].

I thank my supervisor Anders Juel Frankild for pointing me towards much of the literature in the bibliography.

1 Group homology and cohomology

We now present an introduction to group homology and cohomology with an algebraic flavour.

1.1 Basic concepts and properties

Definition 1.1.1 (The Group Algebra)**.** Given a group G and a commutative ring k (most often we let k denote the integers $\mathbb{Z}$ or a field), we may define the group algebra kG as the free module $\left\{\sum_{g\in G} r_g g | g \in G\ ,\ r_g \in k\right\}$ over k with the elements of G as basis, equipped with the coordinatewise addition: $\sum_{g\in G} r_g g + \sum_{g\in G} s_g g = \sum_{g\in G}(r_g + s_g)g$ and the multiplication $\sum_{g\in G} r_g g \cdot \sum_{g'\in G} s_{g'} g' = \sum_{g\in G}\sum_{g'\in G} r_g s_{g'} g \cdot g'$ which we may rewrite as

$$\sum_{g\in G} r_g g \cdot \sum_{g'\in G} s_{g'} g' = \sum_{x\in G}\left(\sum_{g\in G}(r_g \cdot s_{g^{-1}x})g\right) x$$

Definition 1.1.2 (Augmentation)**.** Given a group G and a commutative ring k, the augmentation $\varepsilon : kG \to G$ is the function $\sum_{g\in G} r_g g \stackrel{\varepsilon}{\mapsto} \sum_{g\in G} r_g$.

Note 1.1.3. Augmentation is clearly surjective since G is not empty: $r \cdot 1_G \mapsto r$. Augmentation is a k-algebra homomorphism from kG to k since for $r = \sum_{g\in G} r_g g \in kG$ and $s = \sum_{g\in G} s_g g \in kG$ and $\rho \in k$:

i. $\varepsilon(\rho \cdot r) = \varepsilon\left(\rho \cdot \sum_{g\in G} r_g g\right) = \varepsilon\left(\sum_{g\in G}(r_g\rho)g\right) = \sum_{g\in G} r_g\rho = \rho \cdot \sum_{g\in G} r_g = \rho \cdot \varepsilon(r)$.

ii. $\varepsilon(r+s) = \varepsilon\left(\sum_{g\in G} r_g g + \sum_{g\in G} s_g g\right) = \varepsilon\left(\sum_{g\in G}(r_g + s_g)g\right) = \sum_{g\in G}(r_g + s_g) = \sum_{g\in G}(r_g) + \sum_{g\in G}(s_g) = \varepsilon(r) + \varepsilon(s)$.

iii. $\varepsilon(r \cdot s) = \varepsilon\left(\sum_{g\in G} r_g g \cdot \sum_{g\in G} s_g g\right) = \varepsilon\left(\sum_{x\in G}\left(\sum_{g\in G} s_g r_{g^{-1}x}\right) x\right) = \sum_{x\in G}\left(\sum_{g\in G} s_g r_{g^{-1}x}\right) \stackrel{*}{=} \sum_{g\in G}\sum_{\gamma\in G} s_g r_\gamma = \varepsilon(\sum_{g\in G} r_g g) \cdot \varepsilon(\sum_{g\in G} s_g g)$, where the rewriting $*$ is allowed because multiplication by an element of G is a bijection $G \leftrightarrow G$.

Definition 1.1.4 (Augmentation Ideal)**.** For a group G and a commutative ring k we define the *augmentation ideal* I as the kernel of the augmentation homomorphism $kG \xrightarrow{\varepsilon} k$ above. In other words, I consists of the elements whose coefficients add up to zero.

Note 1.1.5. It is clear that I as a k-module is generated by $g - g'$ with $g, g' \in G$. Furthermore, $\{g - 1 | 1 \neq g \in G\}$ is a basis for I over k since $g - g'$ may be written as $(g-1) - (g'-1)$.

Remark 1.1.6 (Functoriality)**.** Given a commutative ring k and a group homomorphism $\varphi : G \to G'$ we may extend φ to a k-homomorphism $k\varphi : kG \to kG'$ by $g \mapsto \varphi(g)$ for $g \in G$ (and hence $\sum_{g\in G} r_g g \mapsto \sum_{g\in G} r_g \varphi(g)$).

The kernel of $k\varphi$ consists of elements $\sum_{g\in G} r_g g$ such that $\sum_{g\in G} r_g \varphi(g) = 0$. As in the case of $G' = 1$ described in (1.1.2), the kernel must be generated by $\{g_1 - g_2 | g_1, g_2 \in \mathrm{Ker}\varphi\}$, and hence by $\{g - 1 | g \in \mathrm{Ker}\varphi\}$.

Example 1.1.7. Let G be a group and let k be a commutative ring. If $N \lhd G$ is a normal subgroup of G, consider the surjection $\pi : G \to G/N$. Extending π to $k\pi : kG \to k(G/N)$, we have again a surjection, and the kernel is generated by $\{n - 1 | n \in N\}$. The augmentation ideal I_N of N, as we know from (1.1.5), is also generated by $\{n - 1 | n \in N\}$. Hence, there is a group homomorphism $k(G/N) \to kG/(I_N kG)$ defined as $[b]_N \mapsto [b]_{I_n kG}$. Since $\kappa[b]_N = [\kappa b]_N \mapsto [\kappa b]_{I_n kG} = \kappa[b]_{I_n kG}$, it is k-linear and therefore, a k isomorphism.

Definition 1.1.8 (Trivial module)**.** Given a group G and a commutative ring k, a kG-module M is said to be *trivial* if $g \cdot m = m$ for all $m \in M$.

Proposition 1.1.9. *A kG-module M is trivial if and only if $x \cdot m = \varepsilon(x)m$ for any $x \in kG$.*

Proof. Say $x = \sum_{g\in G} x_g g$ and assume the above definition. Then

$$\left(\sum_{g\in G} x_g g\right) \cdot m = \sum_{g\in G} x_g (gm) = \sum_{g\in G} x_g m = \left(\sum_{g\in G} x_g\right) m.$$

If instead we assume that $x \cdot m = \varepsilon(x) \cdot m; \forall x \in kG$ then we have $g \cdot m = \varepsilon(g) \cdot m = 1 \cdot m = m$. ☐

Note 1.1.10. We view k as a trivial kG-module. As such, $\varepsilon : kG \to k$ is a kG-module homomorpism since ε is k-linear and since the trivial action of G on k makes ε commute with the action of G: $\varepsilon(g) = \varepsilon(1) = g \cdot \varepsilon(1)$.

Invariants and co-invariants

Definition 1.1.11 (Invariants)**.** Given a group G, a commutative ring k, and a left kG-module M we define the *invariants* $M^G = \{m \in M | g \cdot m = m \forall g \in G\}$.

Note 1.1.12. M^G forms a trivial kG-submodule of M and it is by inclusion the largest trivial module.

Definition 1.1.13 (Co-invariants)**.** Given a left kG-module M we define IM as the kG-module $\{\sum r_m m | r_m \in I, m \in M\}$ of finite sums of products of elements in I and M. Since IM is a kG-submodule of M we may define the *co-invariants* as the quotient $M_G = M/IM$.

Note 1.1.14. With respect to inclusion, M_G is the largest trivial kG-module M/S which is a quotient of M and a submodule of M.

1.2 Homology and Cohomology with coefficients

We introduce the concepts of homology and cohomology with coefficients and prove some general facts about them.

Definition 1.2.1. Let G be a group and k be a commutative ring. We form the tensor product $M \otimes_{kG} N$ from the right kG-module M and left kG-module N as the quotient $M \otimes_k N = \{n \otimes_k m | n \in N, m \in M\}$ modulo the equivalence relation $rm \otimes_k n \sim m \otimes_k rn$

Proposition 1.2.2. *Let k be a commutative ring, G a group and M a left kG-module. Then $k \otimes_{kG} M = M_G$.*

Proof. By definition of I we have a short exact sequence

$$0 \to I \to kG \to k \to 0.$$

Since k, kG and I are all free and hence flat, the following remains exact after the functor $- \otimes M$ has been applied:

$$I \otimes_{kG} M \to kG \otimes_{kG} M \to k \otimes_{kG} M \to 0$$

We have an isomorphism $kG \otimes_{kG} M \to M$ given by $a \otimes m \mapsto a \cdot m$. Composing this isomorphism with the inclusion $I \otimes_{kG} M \overset{\iota \otimes \mathrm{Id}}{\to} kG \otimes_{kG} M$ we obtain an isomorphism $I \otimes_{kG} M \to IM$. Therefore, we have $k \otimes_{kG} M \cong M/IM$ by exactness of

$$\begin{array}{ccccccc} I \otimes_{kG} M & \xrightarrow{\iota \otimes_{kG} M} & kG \otimes_{kG} M & \xrightarrow{\varepsilon \otimes_{kG} M} & k \otimes_{kG} M & \longrightarrow & 0 \\ \downarrow \cong & & \downarrow \cong & & & & \\ IM & & M & & & & \end{array}$$

Since exactness for one yields surjectivity of $\varepsilon \otimes_{kG} M$, and besides implies that the kernel of $\varepsilon \otimes_{kG} M$ is isomorphic to the image via the inclusion of IM in M. □

Definition 1.2.3. Let G be a group and k be a commutative ring. From two left kG-modules, M and N, we form the commutative group $\mathrm{Hom}_{kG}(M, N)$ of kG-homomorphisms from M to N. Furthermore, since kG is abelian, this is a kG-module with the kG-action $(r\varphi)(m) = r \cdot (\varphi(m))$.

Definition 1.2.4. Let k be a commutative ring, G a group and M, N left kG-modules. Consider group of homomorphisms $\mathrm{Hom}_k(M, N)$. We may introduce the *diagonal action* of the group G on $\mathrm{Hom}_k(M, N)$ by

$$(gu)(m) = g \cdot u(g^{-1}m) \qquad g \in G, u \in \mathrm{Hom}_k(M, N) \ ,$$

induced from the actions of G on each of the two modules.

Observation 1.2.5. Since $g\mu = \mu$ if and only if μ commutes with the action of g on M and on $\mathrm{Hom}_k(M, N)$ respectively, we have

$$\mathrm{Hom}_{kG}(M, N) = \mathrm{Hom}_k(M, N)^G$$

Proposition 1.2.6. *For a group G, a commutative ring k and a left kG-module M, we have that* $\mathrm{Hom}_{kG}(k, M) = M^G$

Proof. By definition of I we have a short exact sequence

$$0 \to I \to kG \to k \to 0.$$

Since k and kG are both projective kG-modules, the following remains exact after having applied the functor $\mathrm{Hom}_{kG}(-, M)$

$$0 \to \mathrm{Hom}_{kG}(k, M) \to \mathrm{Hom}_{kG}(kG, M).$$

It is known that we have an isomorphism $\phi : \mathrm{Hom}_{kG}(kG, M) \to M$ given by $\alpha \mapsto \alpha(1)$. Composing $\mathrm{Hom}_{kG}(\varepsilon, M)$ with ϕ we have a map, $\mathrm{Hom}_{kG}(k, M) \to M$, taking any homomorphism $h \in \mathrm{Hom}_{kG}(kG, M)$ to the composition $(h \circ \varepsilon)(1) = h(1)$. To prove the converse, note that a homomorphism $k \to M$ is uniquely determined by its value on 1. Hence

$$\mathrm{Hom}_{kG}(k, M) \cong \{\alpha(1) | \alpha \in \mathrm{Hom}_{kG}(k, M)\} \ .$$

Now let $h \in \mathrm{Hom}_{kG}(k, M)$ and $g \in G$, then $g \cdot (h(1)) = h(g \cdot 1) = h(1)$ since we view k as a trivial kG-module. Therefore

$$\{\alpha(1) | \alpha \in \mathrm{Hom}_{kG}(k, M)\} = M^G \ ,$$

which in other words means that we have showed $\mathrm{Hom}_{kG}(k, M) \cong M^G$. $\square$

Definition 1.2.7. For a group G, a commutative ring k and a kG-module M we define the *homology of G with coefficients in M* as

$$H_*(G, M) = \mathrm{Tor}_*^{kG}(k, M)$$

Where the bifunctor $\mathrm{Tor}_*^R(-,-)$ is the left derived functor of the bifunctor $- \otimes_R -$ mentioned above (1.2.1).

Definition 1.2.8. For a group G, a commutative ring k and a kG-module M we define the *cohomology of G with coefficients in M* as

$$H^*(G, M) = \mathrm{Ext}_{kG}^*(k, M)$$

Where the bifunctor $\mathrm{Ext}_R^*(-,-)$ is the derived of the bifunctor $\mathrm{Hom}_R(-,-)$ mentioned in (1.2.3).

1.3 Finite groups

If G is an infinite group and k is a commutative ring, the group of invariants $(kG)^G$ is zero. When G is finite, this does not hold. When G is finite, $(kG)^G$ is isomorphic to k.

Definition 1.3.1 (Norm)**.** We define the *norm* N of the finite group G as the sum $N = \sum_{g \in G} g \in kG$.

Proposition 1.3.2. *For a finite group G and a commutative ring k the group of invariants $(kG)^G$ is isomorphic to the subgroup $k \cdot N$ of kG, where N denotes the norm of G.*

Proof. Since both left and right multiplication by a given group element $\gamma \in G$, is bijective, $\gamma N = \gamma \sum_{g \in G} g = \sum_{g \in G} \gamma g = \sum_{g \in G} g = N$, and similarly $N\gamma = N$ for all $\gamma \in G$. That is, N belongs to kG^G. It is now clear that $\zeta N = N\zeta = N$ for all $\zeta \in kG$, and hence the norm N is a central element of kG.

Since $N \in kG^G$, we must clearly have $kN \subseteq kG^G$.

The other inclusion arises from the following argument: Let $a = \sum_{g \in G} z_g g \in kG^G$. We now claim that all coefficients z_g equal to z_1, the coefficiant of the neutral element of G.—This follows immediately from $\sum_{g \in G} z_g g = a = a\gamma = \sum_{g \in G} z_g g\gamma$ for all $\gamma \in G$ which implies in particular $z_\gamma \gamma = z_1 \gamma$, and hence $z_\gamma = z_1$ for all $\gamma \in G$. Now we may rewrite a as $z_1 \cdot N$, and since a was an arbitrary element of kG^G, $kG^G \subseteq kN$ □

Remark 1.3.3. By the same argument, if $a = \sum_{g \in G} z_g g \in kG^G$ for an infinite group G, then all z_g must be equal. And since only finitely many of them may differ from zero, they must all be zero.

Definition 1.3.4 (Local ring)**.** Let R be a ring. R is a *local ring* if it has a unique maximal left ideal. $\mathfrak{m}$ is the unique maximal left ideal for R if and only if $\mathfrak{m}$ is the unique maximal right ideal for R, and we call $\mathfrak{m}$ *the maximal ideal* of R. We let $(R, \mathfrak{m})$ *is a local ring* denote that R is a local ring with maximal ideal $\mathfrak{m}$.

Statement 1.3.5. *Let k be field and let G be an abelian group. Then I_G is the maximal ideal of kG.*

Proposition 1.3.6. *Let k be a field of characteristic $p \geq 2$ and let G be a finite group of order p^n. Then kG is a local ring with the augmentation ideal I_G as its maximal ideal.*

Proof. Firstly, note that since the ring kG is a finite algebra (or a finite vector space over k), any descending chain of ideals must terminate. Therefore, the claim is equivalent to saying that I_G is nilpotent.

We will proceed by induction on the power n of the prime p.

For $n = 1$ we have a group of order p. Any group of order p is abelian and hence, by the statement (1.3.5) above, I_G is nilpotent.

Assume proposition (1.3.6) for groups of order p^i, $i < n$. Let G be a group of order p^n. Then the centre Z is non-trivial[1]. Furthermore, the centre is as always a normal subgroup of G. Hence from example (1.1.7) we have an isomorphism

$$k(G/Z) \cong kG/(I_Z kG)$$

By the induction hypothesis $I_{G/Z}$ is nilpotent, say with exponent i, and by (1.3.5), I_Z is nilpotent, say with the exponent j. Now let $g - 1$ be a generator of I_G. $g - 1$ is mapped by the canonical projection to the element $g' - 1$ of $I_{G/Z}$. Via the isomorphism we may also view $g' - 1$ as $[r]_{I_Z kG}$. Because of the nilpotence of $I_{G/Z}$, $(g' - 1)^i = [0]_Z$, and hence $([r]_{I_Z kG})^i = [0]_{I_Z kG}$. That is, $r^i \in I_Z kG$. Say $r^i = \iota kappa$. Since I_Z is nilpotent, $\iota^j = 0$. Hence $(\iota\kappa)^j = \iota^j \kappa^j = 0\kappa = 0$.

Wherefore $(g - 1)^{ij}$ is zero. Since this holds for any generator, it must hold for all elements of I_G

By induction, this holds for any p-group. □

[1][Thorup98, Sætning 7.23]

1.4 Cyclic groups

Let G be a finite cyclic group of order q on generator g. In this section we determine the group homology and cohomology.

Proposition 1.4.1. *Let k be a commutative ring, and let G be a finite cyclic group of order q on generator g. The following is a chain complex,*

$$\cdots \xrightarrow{\cdot N} kG \xrightarrow{\cdot(g-1)} kG \xrightarrow{\cdot N} kG \xrightarrow{\cdot(g-1)} kG \xrightarrow{\varepsilon} k \to 0. \tag{1}$$

is exact, which implies that

$$\cdots \xrightarrow{\cdot N} kG \xrightarrow{\cdot(g-1)} kG \xrightarrow{\cdot N} kG \xrightarrow{\cdot(g-1)} kG \xrightarrow{\varepsilon} k$$

is a free and hence projective resolution of k over kG.

Proof. First, we prove that (1) is at all a chain complex:

The norm of the group G is $N = 1 + g + \ldots + g^{q-1}$ which has the property that $N \cdot (g-1) = g^q - 1 = 0$ in kG and hence (1) is a chain complex.

Now for proving (1) is an exact chain complex, there are three cases

- $\xrightarrow{\cdot(g-1)} kG \xrightarrow{\varepsilon}$;
- $\xrightarrow{\cdot N} kG \xrightarrow{\cdot(g-1)}$;
- $\xrightarrow{\cdot(g-1)} kG \xrightarrow{\cdot N}$,

where we shall show exactness.

- **Exactness at $\xrightarrow{\cdot(g-1)} kG \xrightarrow{\varepsilon}$:** Consider an generator $\gamma - 1$, $\gamma = g^j \in G$, for some $j < q$, which lies in the kernel of ε. Then under multiplication with $g-1$, the element $\sum_{i=0}^{j-1} g^i$ of kG is mapped to

  $$\sum_{i=1}^{j} g^i - \sum_{i=0}^{j-1} g^i = g - 1.$$

  Hence for any element ι in the basis for I_G, we may find an element u_ι such that $u_\iota \cdot (g-1) = \iota$

  Let $l = \sum \kappa_\alpha \iota_\alpha$ be a linear combination of elements of I_G, then $\sum \kappa_\alpha u_{\iota_\alpha}$ is mapped to l under multiplication with $g-1$.

- **Exactness at $\xrightarrow{\cdot N} kG \xrightarrow{\cdot(g-1)}$:** Consider an element $\sum_{i=1}^{q-1} r_i g^i$ of the kernel of $\cdot(g-1)$. For the element to be in the kernel means that $\sum_{i=1}^{q-1} r_i g^i \cdot (g-1) = \sum_{i=1}^{q-1} r_i g^{i+1} + \sum_{i=1}^{q-1}(-r_i) g^i = \sum_{i=1}^{q-1}(r_{i+1} - r_i) g^i = 0$ (where r_q

denotes r_1). Hence we must have $r_{i+1} = r_i$ for all i. Specifically we have $r_i = r_1$ for all i. Now, we simply verify that this is the image of the element $r_i \in kG$ under multiplication by N: $r_1 \cdot \sum_{i=1}^{q-1} g^i = \sum_{i=1}^{q-1} r_1 g^i$.

- **Exactness at** $\xrightarrow{\cdot(g-1)} kG \xrightarrow{\cdot N}$**:** Consider an element $\sum_{i=1}^{q-1} r_i g^i$ of the kernel of $\cdot N$. For the element to be in the kernel means that $\sum_{i=1}^{q-1} r_i g^i \cdot N = \sum_{i=1}^{q-1} r_i g^i \cdot \sum_{i=1}^{q-1} g^i = \sum_{i=1}^{q-1} \left(\sum_{i=1}^{q-1} r_i\right) g^i = 0$, since all coefficients in the factor on the right hand side are 1. Hence $\sum_{i=1}^{q-1} r_i = 0$, or $-\sum_{i=2}^{q-1} r_i = r_1$. We now seek an element $\sum_{i=1}^{q-1} s_i g^i \in kG$ such that $\sum_{i=1}^{q-1} s_i g^i \cdot (g-1) = \sum_{i=1}^{q-1} s_i g^{i+1} - \sum_{i=1}^{q-1} s_i g^i = \sum_{i=1}^{q-1} (s_{i-1} - s_i) g^i = \sum_{i=1}^{q-1} r_i g^i$, which means $s_{i-1} - s_i = r_i$ or in other words $s_{i-1} = s_i + r_i$ for all $i = 2, \dots, q$, (interpreting s_q and r_q as s_1 and r_1 respectively).

  Hence we have found a recursive formula for the coefficients s_i. Now we may take s_{q-1} to be $0 \in k$, but any fixed element of k would do. Set $s_{n-1} = s_n + r_n = s_{q-1} + \sum_{n \leq N < q} r_N$. Then we have $s_1 = s_{q-1} + \sum_{N=2}^{q-1} r_N = s_{q-1} - r_1$, and hence $s_{q-1} = s_1 + r_1 = s_q + r_q$ regardless of our initial choice of s_{q-1}.

  It is indeed intuitive that we may choose s_{q-1} freely since, as we see, it only alters our choice by addition of an element in the kernel (we know the kernel is equal to $\{\sum_{i=1}^{q-1} \kappa \cdot g^i | \kappa \in k\}$).

□

Definition 1.4.2. Let M be a kG-module. We may define ${}_N M$ and MN as ${}_N M = \{m \in M | Nm = 0\}$ and $MN = \{m \cdot N | m \in M\}$.

Observation 1.4.3. Applying $- \otimes_{kG} M$ to (1), we get

$$\cdots \xrightarrow{\cdot N} kG \otimes_{kG} M \xrightarrow{\cdot(g-1)} kG \otimes_{kG} M \xrightarrow{\cdot N} kG \otimes_{kG} M \xrightarrow{\cdot(g-1)} kG \otimes_{kG} M \xrightarrow{\varepsilon} k \otimes_{kG} M \to 0 \,,$$

which is equivalent to

$$\cdots \xrightarrow{\cdot N} M \xrightarrow{\cdot(g-1)} M \xrightarrow{\cdot N} M \xrightarrow{\cdot(g-1)} M \xrightarrow{\varepsilon} 0 \to 0 \,.$$

Taking homology we get

$$H_n(G, M) = \begin{cases} M_G & n = 0 \\ M^G / NM & n \text{ odd} \\ {}_N M / IM & n > 0 \text{ even} \end{cases}$$

If instead we apply $\mathrm{Hom}_{kG}(-, M)$ to (1), we get

$$0 \leftarrow \mathrm{Hom}_{kG}(k, M) \leftarrow \mathrm{Hom}_{kG}(kG, M) \leftarrow \mathrm{Hom}_{kG}(kG, M) \leftarrow \cdots ,$$

which is equivalent to

$$0 \leftarrow 0 \leftarrow M \leftarrow M \leftarrow M \leftarrow M \leftarrow \cdots .$$

Taking cohomology we get

$$H_n(G, M) = \begin{cases} M^G & n = 0 \\ {}_N M/IM & n \text{ odd} \\ M^G/NM & n > 0 \text{ even} \end{cases}$$

1.5 Free group and free product

In this section we will define a free group and the free product of two groups.

We will prove some nice properties of the group algebra over a free group which imply that if G is a free group, $H_n(G, M)$ and $H^n(G, M)$ vanish for $n > 1$. Therefore, in the study of group homology and group cohomology over a free group, specialists often omit specifying the above n, but write Tor for Tor_1 and Ext for Ext^1.

After defining the free product we will show that the augmentation ideal of the free product P of F_1 and F_2 over the commutative ring k is isomorphic to the symmetric direct sum of kP tensored with the augmentation ideal of F_1 over kF_1 and kP tensored with the augmentation ideal of F_2 over kF_2.

Definition 1.5.1 (Free Group)**.** A group G is *free* if there exists a subset $S \subseteq G$ such that for any $g \in G$ we may write g as a product of finitely many of the elements of $S \cup S^{-1}$, where S^{-1} denotes $\{s^{-1} | s \in S\}$. The product has to be unique up to the relation $\sim$ given by $w_1 w_2 \sim w_1 u^{-1} u w_2$ for any unit u and any w_1, w_2 finite products of elements in S. In the situation above we say G is a free group generated by S.

A product in G of n elements from S is called a *word of length n*. By convention the empty word or the empty product is $1 \in G$.

Theorem 1.5.2 (The Augmentation Ideal of a Free Group)**.** *In section (1.1.5) we showed that the augmentation ideal I as a free k-module has the basis* $\{g - 1 | 1 \neq g \in G\}$. *We now claim that* $\{s - 1 | s \in S\}$ *alone generates I as a free kG-module.*

Proof. We will proceed by induction on word length. Since we know I is generated by $\{g - 1 | 1 \neq g \in G\}$ over kG, it will be enough to show that $g - 1$ is generated by $\{s - 1 | s \in S\}$ for all g.

The only word of length 0 is the identity element of addition, 0, which by convention is generated by any set.

Assume $\{s-1|s \in S\}$ generates any element $\eta - 1$ with η of word length n. Let $w = l_1 l_2 \cdot l_{n+1} \in G$ be of length $n+1$. We now consider the two cases conditioned by l_{n+1}. If $w = fs$ with $s \in S$, then we may write $w-1$ as $fs - 1 = (f-1) + f(s-1)$.

If on the other hand $w = fs^{-1}$ with $s \in S$, we may write $w - 1$ as $fs^{-1} - 1 = (f-1) - fs^{-1}(s-1)$. In both cases we have reduced to the sum of $\kappa(s-1)$ and $(f-1)$ with $\kappa \in G$ and where f by assumption has length n. □

Observation 1.5.3. Since the augmentation ideal is free when G is a free group, the following exact sequence is a free resolution of k

$$0 \to I \to kG \xrightarrow{\varepsilon} k \to 0 ,$$

from which it is clear that for any module kG-module M, the modules $H_n(G, M)$ and $H^n(G, M)$ vanish for $n > 1$.

Definition 1.5.4 (Free Product)**.** Let G and Γ be groups. The free product denoted $G * \Gamma$ of G and Γ consists of all finite words $g_1\gamma_1 \dots g_n\gamma_n$, $n \in \mathbb{N}$, $g \in G$, $\gamma \in \Gamma$, where only g_1 and γ_n may possibly be the respective identity elements.

We define the *word length* of an element of the free product as given above to be the natural numer n.

Multiplication is described by

$$g_1\gamma_1 \dots g_n\gamma_n \cdot s_1\sigma_1 \dots s_m\sigma_m = g_1\gamma_1 \dots g_n\gamma_n s_1\sigma_1 \dots s_m\sigma_m$$

with the identity elements (exept g_1 and σ_m) removed and with two neighbouring elements from the same group substituted by their product in that group, recursively, till the word has the form described above.

Note 1.5.5. You might have encountered another definition of free product, namely all words of non-trivial elements, alternating in origin between the two groups, including the empty string which acts as the identity element. These two definitions coincide exept for different understandings of the "word length" of a given element. You go from one word length to the other by multiplying by two and then substracting one for each identity element g_1 or γ_n.

Theorem 1.5.6 (The Augmentation Ideal of a Free Product)**.** *Let G, Γ be groups and let k be a commutative ring. Let I_G, I_Γ and $I_{G*\Gamma}$ denote the augmentation ideals of kG, $k\Gamma$ and $k(G*\Gamma)$ respecively. Then*

$$I_{G*\Gamma} \cong (I_G \otimes_{kG} k(G*\Gamma)) \oplus (I_\Gamma \otimes_{k\Gamma} k(G*\Gamma)) \, .$$

Proof. $k(G*\Gamma)$ is a left kG module. All words starting with the identity element of G form a basis for $k(G*\Gamma)$ as a left kG module. Besides, we know from (1.1.5) that I_G has basis $\{g-1|1 \neq g \in G\}$ over k. Together with the definition of the tensor product, this gives a basis

$$B_G = (g-1)\cdot(1\gamma_1 \ldots g_n\gamma_n), \;\; 1 \neq g, n \in \mathbb{N} \, ,$$

for $I_G \otimes_{kG} k(G*\Gamma)$ over k.

Likewise, $I_\Gamma \otimes_{k\Gamma} k(G*\Gamma)$ has the following basis over k

$$B_\Gamma = (\gamma - 1)\cdot(g_1\gamma_1 \ldots g_n\gamma_n), \;\; 1 \neq \gamma, n \in \mathbb{N} \text{ and } g_1 = 1 \Rightarrow n = \gamma_n = 1 \, .$$

The aim now is to show that any element of $k(G*\Gamma)$ may be written as a sum of elements from these bases, B_G and B_Γ. Since $k(G*\Gamma)$ as a free k-module has the basis $\{w-1|1 \neq w \in G*\Gamma\}$, it is enough to check that elements of the form $w-1$ may be written as such a sum.

We will proceed by induction on word length.

Let $w = g\gamma$ be a word of length 1. We may write $w-1$ as $(g-1)(1\gamma) + 1\gamma - 1 = (g-1)(1\gamma) + (\gamma-1)(e)$. Here e denotes the word $1_G 1_\Gamma$.

Assume $B_G \oplus B_\Gamma$ generates any element $v-1$ with v of word length n. Let $w = g_1\gamma_1 \ldots g_{n+1}\gamma_{n+1}$ be of word length $n+1$. If $g_1 \neq 1$ we may write $w-1$ as $(g_1-1)\cdot(1\gamma_1 \ldots g_{n+1}\gamma_{n+1}) + 1\gamma_1 \ldots g_{n+1}\gamma_{n+1} - 1$. Now we need only deal with $1\gamma_1 \ldots g_{n+1}\gamma_{n+1} - 1$, which we may rewrite as

$$(1-\gamma_1)\cdot(g_2\gamma_2 \cdots g_{n+1}\gamma_{n+1}) + g_2\gamma_2 \cdots g_{n+1}\gamma_{n+1} - 1 \, ,$$

where $g_2 \neq 1$ since

$$g_1\gamma_1 \cdots g_{n+1}\gamma_{n+1} \in G*\Gamma,$$

and where by assumption $g_2\gamma_2 \ldots g_{n+1}\gamma_{n+1} - 1$ may be written as a sum of terms from the bases. Hence we have written w as a sum of terms from B_G and B_Γ, and by induction we may write any word in $G*\Gamma$ as a sum of terms from B_G and B_Γ.

To prove $B_G \cup B_\Gamma$ is linearly independent, consider a sum of non-trivial elements from $B_G \cup B_\Gamma$. We will regard such an element as an element from

$k(G * \Gamma)$ with coefficients from $\{(1-x) | 1_G \neq x \in G \vee 1_\Gamma \neq x \in \Gamma\}$. Now, in a sum of 1 or more non-trivial elements from $B_G \cup B_\Gamma$ the coefficient of the longest word must be non-zero, and hence we have shown that $B_G \cup B_\Gamma$ is linearly independant over k. $\square$

2 kH and kG when H is a subgroup of G

From the inclusion homomorphism of groups $\iota : H \hookrightarrow G$ we obtain a ring homomorphism $\iota : kH \hookrightarrow kG$. This homomorphism is the key to extending all kG-modules to kH-modules, and, less obviously, extending any kH-module to a kG-module.

We start with a more general setup.

2.1 Extension and Co-extension

Definition 2.1.1 (Restriction of scalars)**.** Let $\alpha : R \to S$ be a ring homomorphism. Given an S-module A with the composition $* : S \times A \to A$, we make A an R-module by *restriction of scalars* with the composition $(r, s) \mapsto \alpha(r) * s$. Moreover, S is an R-module with the composition $(s, r) \mapsto s \cdot \alpha(r)$.

Definition 2.1.2 (Extension of scalars)**.** Now, given a left R-module M, we may consider the tensor product $S \otimes_R M$ of the right R-module S and the left R-module M. The *extension of scalars* from R to S is this particular tensor product viewed as an S-module with the composition

$$(s_1, s_2 \otimes m) \mapsto s_1 \cdot s_2 \otimes m \; .$$

Theorem 2.1.3. *Let R and S be rings and let $\alpha : R \to S$ be a ring homomorphism. Given a left R-module M and as S-module N,*

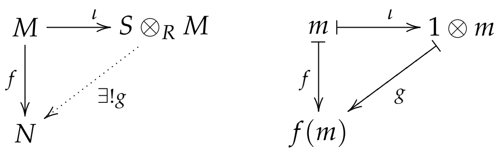
$$\mathrm{Hom}_S(S \otimes_R M, N) = \mathrm{Hom}_R(M, N) \; .$$

Proof. We start by constructing a map $M \xrightarrow{\iota} S \otimes_R M : m \mapsto 1 \otimes m$. We regard $S \otimes_R M$ as an R-module by restriction of scalars, $1 \otimes rm = \alpha(r) \otimes m = \alpha(r)(1 \otimes m)$ and hence $\iota(rm) = \alpha(r)\iota(m)$.

Let $(N, *)$ be an S-module and let f be an R-module map, we wish to prove unique existence of an S-module map $g : S \otimes_R M \to N$ such that $f = g\iota$.

Uniqueness follows from $g(s \otimes m) = s * g(1 \otimes m) = s * g(\iota(m)) = s \cdot f(m)$ and hence the uniquely determined S-module map is $g(s \otimes m) = s * f(m)$.

The following diagrams illustrate this point:



□

Dually, we wish to realise $\mathrm{Hom}_S(N, \mathrm{Hom}_R(S, M)) = \mathrm{Hom}_R(N, M)$ and invent co-extension of scalars and we show the existence of a uniquely detemined map, $\mathrm{Hom}_R(S, M) \xrightarrow{\pi} M$

Definition 2.1.4 (Co-extension of scalars)**.** Regard S as a left R-module by $(r, s) \mapsto \alpha(r) \cdot s$. For any left R-module M we may consider $\mathrm{Hom}_R(S, M)$ as a left S-module with the composition $* : (s_1 * f)(s_2) = f(s_2 \cdot s_1)$. We say that the S-module $\mathrm{Hom}_R(S, M)$ is obtained from the R-module M by *co-extension of scalars*.

Theorem 2.1.5. *Let R and S be rings, and let $\alpha : R \to S$ be a ring homomorphism. For M left R-module and N S-module,*

$$\mathrm{Hom}_S(N, \mathrm{Hom}_R(S, M)) = \mathrm{Hom}_R(N, M)$$

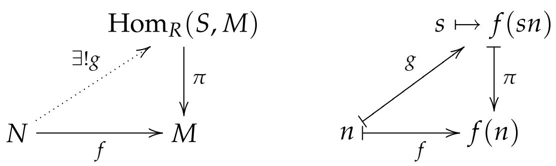
.

Proof. For an R-module M the map $\mathrm{Hom}_R(S, M) \xrightarrow{\pi} M$ sends $f \in \mathrm{Hom}_R(S, M)$ to its value $f(1)$. For $r \in R$ and $f \in \mathrm{Hom}_R(S, M)$ we have $\pi(\alpha(r) * f) = (\alpha(r) * f)(1) = f(1 \cdot \alpha(r)) = r \cdot f(1) = r \cdot \pi(f)$, regarding $\mathrm{Hom}_R(S, M)$ as an R-module by restriction of scalars.

For each R-module N and each R-module map $f : N \to M$ we wish to prove unique existence of an S-module map, $g : N \to \mathrm{Hom}_R(S, M)$ such that $f = \pi g$.

Among the different S-module map conditions, g must satisfy $g(sn) = sg(n)$. Applying π, we get $f(sn) = \pi g(sn) = g(sn)(1) = (sg(n))(1) = (g(n))(s)$, and hence the uniquely determined S-module map must be $g(n) = s \mapsto f(sn)$.

The following diagrams illustrate this point:



□

Note 2.1.6. Combining theorems (2.1.3) and (2.1.5) we get an isomorphism

$$\mathrm{Hom}_S(S \otimes_R M, N) \cong \mathrm{Hom}_S(M, \mathrm{Hom}_R(S, N)) \ ,$$

known as the *hom-tensor adjunction isomorphism*.

There is also a theorem known as the Adjunction Isomorphism:

Statement 2.1.7 (Adjunction Isomorphism)**.** *Let R and S be rings. If B is an R-S-bimodule, C is a right S-module and A is an R-module, then there is a natural isomorphism:*

$$\mathrm{Hom}_S(A \otimes_R B, C) \cong \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C)) \ .$$

A proof is found in [Weibel94, Proposition 2.6.3].

2.2 Induction and Co-induction

For a subgroup H of G we have a ring homomorphism $kH \xrightarrow{k\,\iota} kG$ induced by the inclusion $H \overset{\iota}{\hookrightarrow} G$.

Given a left kH-module M, we may view $kG \otimes_{kH} M$ as a kG-module via the ring homomorphism $k\,\iota$ by extension of scalars. We call this kG-module $\mathrm{Ind}_H^G M$. In this section we will show that $\mathrm{Ind}_H^G M$ is a direct sum of copies of M with as many summands as there are cosets in G/H.

Given a left kH-module M we have the kG-module $\mathrm{Hom}_{kH}(kG, M)$ obtained from M by extension of scalars via the ring homomorphism $k\,\iota$. Later in this section we will show that this is the direct product of copies of M with as many summands as there are cosets in G/H

Given a family of modules, there always is an inclusion of the direct sum of the modules into the direct product of the same modules. This inclusion is an isomorphism when the family of modules is finite.

This applies to Ind and Coind as well: Given a kH-module M we will see that there is an inclusion $\mathrm{Ind}_H^G M \hookrightarrow \mathrm{Coind}_H^G M$ which is an isomorphism whenever there are only finitely many copies of M —that is, whenever the index $[G : H]$ is finite.

Definition 2.2.1. When the ring homomorphism (α) in question is the inclusion $kH \xrightarrow{k\iota} kG$ obtained from the inclusion $H \xrightarrow{\iota} G$, extension and co-extension are called induction and co-induction respectively.

Proposition 2.2.2. *Let H be a subgroup of G and let M be a kH-module. The kG-module $\mathrm{Ind}_H^G M$ is isomorphic to the direct sum $\bigoplus_{b \in G/H} bM$.*

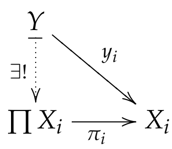
Proof. kG is a free module over kH with one representative for each coset as a basis. Choose such a basis $B \subseteq G$. Then we have $kG \otimes_{kH} M = \bigoplus_{b \in B} b \otimes M = \bigoplus_{b \in B} \{b \otimes m | m \in M\}$, since any element $bh \otimes m \in kG \otimes M$ is equivalent to the element $b \otimes \iota(h)m = b \otimes hm \in kG \otimes M$.

Consider the inclusion $M \hookrightarrow kG \otimes_{kH} M \;:\; m \mapsto 1 \otimes m$ and recall the action $(g, 1 \otimes m) \mapsto g \otimes m$ of G on the subset (now submodule) $1 \otimes_{kH} M$ of $kG \otimes_{kH} M$.

This identifies $\mathrm{Ind}_H^G M = kG \otimes_{kH} M$ as $\bigoplus_{b \in G/H} bM$. ☐

Proposition 2.2.3. *Let H be a subgroup of G and let M be a kH-module. The kG-module $\mathrm{Hom}_{kH}(kG, M)$ is isomorphic to the product $\prod_{b \in G/H} \pi_b M$ where $\pi_b m : kG \to A$ is the kH-map that sends b^{-1} to a and β^{-1} to 0 for all other coset representatives.*

Proof. Firstly, let us remind ourselves that the product $\prod X_i$ of the family of objects $(X_i)_{i \in \alpha}$ consists of an object $\prod X_i$ and canonical projections that is a family of morphisms $\prod X_i \xrightarrow{\pi_i} X_i$ such that for any object Y with morphisms $Y \xrightarrow{y_i} \prod X_i$, there exists a unique morphism $Y \to X$ making the following diagram commute for each i:



Now, to prove $\mathrm{Hom}_{kH}(kG, M)$ is the product $\prod_{b \in G/H} \pi_b M$, let there be given an object Y and morphisms $Y \xrightarrow{y_b} \pi_b M$. We have a uniquely determined map $Y \to \mathrm{Hom}_{kH}(kG, M)$ which has the value of y_b on each basis vector b for kG as a free kH-module.

Since this holds for any Y and family of maps $Y \xrightarrow{y_b} \pi_b M$, $\mathrm{Hom}_{kH}(kG, M)$ is the product $\prod_{b \in G/H} \pi_b M$. ☐

Now, this leads us to the following isomorphism which will be obtained by extension from a map $M \to \mathrm{Hom}_{kH}(kG, M)$.

Theorem 2.2.4. *Let H be a subgroup of the group G and let M be a kH-module. Then if $[G : H] < \infty$*

$$\mathrm{Ind}_H^G M \cong \mathrm{Coind}_H^G M \,.$$

Proof. Consider the kH-map $M \stackrel{\varphi_H}{\to} \mathrm{Hom}_{kH}(kG, M)$ given by

$$\varphi_H(m) = \begin{cases} g \mapsto gm & g \in H \\ g \mapsto 0 & g \notin H \end{cases}$$

This map extends uniquely to a kG-map

$$\varphi : kG \otimes_{kH} M \to \mathrm{Hom}_{kH}(kG, M)$$

or isomorphically

$$\varphi : \bigoplus bM \to \prod \pi_b M.$$

We now verify that φ is the inclusion $\bigoplus bM \to \prod \pi_b M$:

If $gb = \beta h$ which is equivalent to $\beta^{-1} g = hb^{-1}$ in G, we have $g(b \otimes a) = \beta \otimes ha$ and $g(\pi_b a)$ sending β^{-1} to $(\pi_b a)(\beta^{-1} g) = (\pi_b a)(hb^{-1}) = h(\pi_b a)(b^{-1}) = ha$ and the inverse of all other cosets sent to 0.

Hence φ_H extends to the inclusion which for finite index is known to be an isomorphism. □

2.3 Shapiro's Lemma

So far we have introduced $\mathrm{Ind}_H^G M$ and $\mathrm{Coind}_H^G M$ for a left kH-module sM and we have shown how they relate to each other — we have proven them equal in case $[G : H]$ is finite.

Now, for a left kH-module M, we should take a look at how the homologies and cohomologies of $\mathrm{Ind}_H^G M$ and $\mathrm{Coind}_H^G M$ with coefficients in G relate to the homologies and cohomologies of M with coefficients in H.

In this section we prove that the homology of $\mathrm{Ind}_H^G M$ with coefficients in G is isomorphic to the homology of M with coefficients in H and that the cohomology of $\mathrm{Coind}_H^G M$ with coefficients in G is isomorphic to the cohomology of M with coefficients in H (Shapiro's lemma).

As [Weibel94] writes, Shapiro's lemma is a fundamental tool when for computing the group homology and group cohomology.

Theorem 2.3.1 (Shapiro's Lemma)**.** *Let H be a subgroup of G and M be an H-module, then*

$$H_*(G; \mathrm{Ind}_H^G M) \cong H_*(H; M) \text{ and } H^*(G; \mathrm{Coind}_H^G M) \cong H^*(H; M)\ .$$

Proof. Any projective right kG-module resolution $P_\bullet \to k$ is also a projective kH-module resolution.

We have a homology of chain complexes $P_\bullet \otimes_{kG} (kG \otimes_{kH} M) \cong P_\bullet \otimes_{kH} M$, where the left hand side leads to

$$\mathrm{H}^i(P_\bullet \otimes_{kG} (kG \otimes_{kH} M)) = \mathrm{H}^i(P_\bullet \otimes_{kG} \mathrm{Ind}_H^G M) = \mathrm{Tor}_i^{kG}(k, \mathrm{Ind}_H^G M) ,$$

and the right hand side leads to

$$\mathrm{H}^i(P_\bullet \otimes_{kH} M) = \mathrm{Tor}_i^{kG}(k, M) .$$

To show the other isomorphism, let $L_\bullet \to k$ be a projective *left* kG-module resolution. There is an adjunction isomorphism of cochain complexes (2.1.4) $\mathrm{Hom}_{kG}(L_\bullet, \mathrm{Hom}_{kH}(kG, M)) \cong \mathrm{Hom}_{kH}(L_\bullet, M)$.

Taking cohomology we obtain from the left hand side

$$\mathrm{H}^i(\mathrm{Hom}(L_\bullet, \mathrm{Hom}_{kH}(kG, M))) = \mathrm{H}^i(L_\bullet, \mathrm{Coind}_H^G M) = \mathrm{Ext}^i_{kG}(k, \mathrm{Coind}_H^G M) ,$$

and from the right hand side

$$\mathrm{H}^i(\mathrm{Hom}_{kH}(P_\bullet, M)) = \mathrm{Ext}^i_{kH}(k, M) .$$

□

2.4 A Local Ring Homomorphism

Definition 2.4.1 (Local ring homomorphism)**.** Let $(R_1, \mathfrak{m}_1)$ and $(R_2, \mathfrak{m}_2)$ be local rings. A ring homomorphism $f : R_1 \to R_2$ is said to be a *local ring homomorphism* if $f(\mathfrak{m}_1)$ is contained in $\mathfrak{m}_2$.

Observation 2.4.2. Let k be a field of characteristic $p \geq 2$. As we have seen in (1.3.6), kG is a local ring with maximal ideal I_G when G is a p-group.

If G is a p-group, and H is a subgroup of G, then since the order of H must divide G, H is a p-group as well.

Then we have a homomorphism $k\iota : kH \to kG$ extending the inclusion $H \xrightarrow{\iota} G$.

This is a ring homomorphism from the local ring (kH, I_H, k) to the local ring (kG, I_G, k). We now propose that it is a local ring homomorphism:

Proposition 2.4.3. *Let k be a field of characteristic $p \geq 2$. If G is a p-group and H is a subgroup of G, then the inclusion $k\iota$ is local ring homomorphism from kH to kG.*

Proof. Consider an element $h - 1$ of the basis of I_H. Then $h - 1$ is mapped to $h - 1$ via the inclusion, which clearly belongs to I_G. □

Comment 2.4.4. Local ring homomorphisms have been studied in great detail and play an importaint role in Algebraic Geometry (e.g. [Hartshorne77])

Note 2.4.5. Let k be a field of characteristic p. Assume G is an abelian p-group. Then kG and kH are commutative rings. kG is free (with the coset representatives as a basis) and hence flat over kH. Since kG has finite flat dimension over kH we say that $k\iota$ has finite flat dimension. By [Avramov97, 4.4.2], $k\iota$ has finite Gorenstein dimension.

Since $k\iota$ has finite Gorenstein dimension, [Avramov97, Lemma 6.7] implies that $\mathrm{Hom}_{kH}(kG, kH)$ is a dualising complex for $k\iota$. Shapiro's lemma states that $\mathrm{Hom}_{kH}(kG, kH) \cong_{kH} kG$. These two conditions imply that kG is a dualising complex for $k\iota$. Now, [Avramov97, Theorem 7.8] states that $k\iota$ is a Gorenstein map.

3 Projective Modules [2]

Having introduced the concept of group algebras and their properties, let us concentrate on a very specific case. That is the case, where all modules over kG are projective - in other words, the case where there is no cohomology.

Firstly, note that over a general ring, the category of left modules may be very different to the category of right modules. Though, over the group algebra kG, we may define *Conjugation*, which allows us to regard a left kG-module $(M, *)$ as a right kG-module $(M, \tilde{*})$ by

$$m \tilde{*} g = g^{-1} * m \ .$$

Note 3.0.1. Given a field k and a group G, a left kG-module $(M, *)$ is projective exactly when the right kG-module $(M, \tilde{*})$, obtained by conjugation from $(M, *)$, is projective. A left kG-module $(M, *)$ is injective exactly when the right kG-module $(M, \tilde{*})$, obtained by conjugation from $(M, *)$, is injective.

Definition 3.0.2 (Semi-simple)**.** We define a *semi-simple* ring as a ring over which any module is projective.

[2] This chapter leans heavily on the article [Iyengar04], seeking inspiration mainly within the chapters two and three.

3.1 Semi-simple group algebras

Over a field of characteristic p there is a great difference between group algebras kG for a finite group G, depending upon the prime factorisation of $|G|$.

We have already seen one remarkable difference in (1.3.6), but an even deeper theorem is Maschke's, which states that over the group algebra kG any sequence splits if and only if $|G|$ and $\text{char}(k)$ are coprime.

From Maschke's theorem we deduce that $|G|$ being coprime to $\text{char}(k)$ is enough to conclude that *any* kG-module is projective.

Definition 3.1.1 (Section)**.** Given a ring R and a surjective homomorphism of R-modules, $M \xrightarrow{\pi} N$, a *section* is a map $M \xrightarrow{\sigma} N$, such that $\pi \circ \sigma$ is the identity on N.

Observation 3.1.2. Note that the inverse of $z \cdot 1_k$ exists precisely when z is coprime to the characteristic of k: Any element exept 0 of a field is invertible. $z \cdot 1_k$ is different from 0 precisely when $\text{char}(k)$ does not divide z, and, since $\text{char}(k)$ is a prime, precisely when $\text{char}(k)$ and z are coprime.

Lemma 3.1.3. *Let k be a field and let G be a finite group.*

If $0 \to I \to kG \xrightarrow{\varepsilon} k \to 0$ splits, $|G|$ and $\text{char}(k)$ are coprime.

Proof. Let σ denote the kG-homomorphism $k \to kG$ such that $\varepsilon \circ \sigma$ is the identity. For any $\gamma \in G$, since kG acts trivially on k we have $\sigma(1) = \sigma(\gamma \cdot 1)$, and by kG-linearity, $\sigma(\gamma \cdot 1) = \gamma \cdot \sigma(1)$.

Hence, writing $\sigma(1)$ as $\sum_{g \in G} r_g g \in kG$, we obtain $\sum_{g \in G} r_g g = \sigma(1) = \gamma \cdot \sigma(1) = \sum_{g \in G} r_g \gamma g$, which we may rewrite as $\sum_{g \in G} r_{\gamma^{-1} g} g$. Therefore we must have $r_\gamma = r_1$ for all $\gamma \in G$, and we may write $\sigma(1) = r_1 \sum_{g \in G} g$.

By the argument above, and by $\varepsilon \circ \sigma = \text{Id}_N$ we have

$$1 = \varepsilon \left(r_1 \sum_{g \in G} g \right) = r_1 \sum_{g \in G} \varepsilon(g) = r_1 \sum_{g \in G} 1 = r_1 \cdot (|G| \cdot 1)$$

which shows that $|G| \cdot 1$ has an inverse in k, from which we may conclude that $|G|$ and $\text{char}(k)$ are coprime. □

Theorem 3.1.4 (Maschke's Theorem)**.** *Let k be a field of characteristic q, and let G be a finite group. Any short exact sequence of kG-modules splits if and only if q and $|G|$ are coprime.*

Proof. There are two implications to show:

$\Leftarrow$: Let k be a field of characteristic coprime to $|G|$.

Given a short exact sequence $S : 0 \to L \xrightarrow{\iota} M \xrightarrow{\pi} N \to 0$ of kG-modules, we may in particular view S as a short exact sequence of k-modules via the inclusion $k \hookrightarrow kG$. Since k is a field, all modules over k are vector spaces. Since vector spaces are always free, any sequence splits. Hence, there exists a section $\sigma : N \to M$ which is *k-linear*.

We may extend σ to the map $\tilde{\sigma} : N \to M$ given by $\tilde{\sigma}(n) = |G|^{-1} \sum_{g \in G} g \cdot \sigma(g^{-1}n)$, where $|G|^{-1}$ is the inverse of $|G| \cdot 1$ in k with respect to multiplication. This inverse exists precisely because $|G|$ is coprime to the characteristic of k, by the argument sketched previously.

We check first that $\tilde{\sigma}$ is a homomorphism. A sum $n + n'$ of elements of N is mapped to

$$|G|^{-1} \sum_{g \in G} g\sigma(g^{-1}(n + n')) = |G|^{-1} \sum_{g \in G} g\sigma(g^{-1}n + g^{-1}n')$$

which, because of k-linearity of σ, is

$$\begin{aligned} & |G|^{-1} \textstyle\sum_{g \in G} \left(g\sigma(g^{-1}n) + g\sigma(g^{-1}n')\right) \\ = \quad & |G|^{-1} \textstyle\sum_{g \in G} \left(g\sigma(g^{-1}n)\right) + |G|^{-1} \sum_{g \in G} \left(g\sigma(g^{-1}n')\right). \end{aligned}$$

Let $r = \sum_{g \in G} r_g g \in kG$. A scalar product $r \cdot n$ of r with $n \in N$, is mapped to

$$\begin{aligned} |G|^{-1} \sum_{g \in G} g\sigma\left(g^{-1} \sum_{\gamma \in G} r_\gamma \gamma n\right) &\overset{(1)}{=} |G|^{-1} \sum_{g \in G} g\sigma\left(\sum_{\gamma \in G} r_{g\gamma} (\gamma n)\right) \\ &\overset{(2)}{=} |G|^{-1} \sum_{g \in G} g \sum_{\gamma \in G} \sigma\left(r_{g\gamma} (\gamma n)\right) \\ &\overset{(3)}{=} |G|^{-1} \sum_{g \in G} \sum_{\gamma \in G} g\, r_{g\gamma} \sigma(\gamma n) \\ &\overset{(4)}{=} |G|^{-1} \sum_{g \in G} \sum_{\gamma \in G} r_g g \gamma^{-1} \sigma(\gamma n) \\ &= \sum_{g \in G} r_g g |G|^{-1} \sum_{\gamma \in G} \gamma^{-1} \sigma(\gamma n) = r \cdot \tilde{\sigma}(n), \end{aligned}$$

where (1) is simply operations within N and a rearrangement of the indices, (2) holds because σ respects sums, (3) is from k-linearity of σ and the distributive law of kG, and (4) is a re-writing, counting the summands in an other order.

Now we check that $\pi\tilde{\sigma}$ is the identity. Given an element $n \in N$,

$$\begin{array}{rcl} \pi\tilde{\sigma}(n) & = & \pi|G|^{-1}\sum_{g\in G} g\sigma(g^{-1}n) \\ & = & |G|^{-1}\sum_{g\in G} g\pi\sigma(g^{-1}n) \\ & = & |G|^{-1}\sum_{g\in G} g(g^{-1}n) \\ & = & |G|^{-1}\sum_{g\in G} n \\ & = & |G|^{-1}|G|\cdot n = n\ , \end{array}$$

since $\pi\sigma$ is the identity on N.

⇒: Assume that any short exact sequence splits. Then in particular the sequence $0 \to I \to kG \xrightarrow{\varepsilon} k \to 0$ splits. Therefore by lemma (3.1.3) above, $|G|$ and $q = \mathrm{char}(k)$ are coprime.

□

Theorem 3.1.5. *For a finite group G and a field k the following are equivalent:*

i. kG is semi-simple;

ii. k is projective over kG;

iii. $|G|$ and $\mathrm{char}(k)$ are coprime.

Proof. There are three implications to show:

i ⇒ *ii*: This is clear, since if kG is semi-simple *any* kG-module is projective.

ii ⇒ *iii*: Since k is projective over kG, the sequence $0 \to I \to kG \to k \to 0$ splits[3]. Whence *iii* follows from (3.1.3).

iii ⇒ *i*: Given a kG-module M, we may find a projective kG-module P and a surjection $\pi : P \to M$. From this we may form the short exact sequence $0 \to \ker(\pi) \to P \to M \to 0$. Since $|G|$ and $\mathrm{char}(k)$ are coprime, Maschke's theorem above implies that the sequence splits. We then have $P \cong M \oplus ker(\pi)$. Since a direct summand of a projective is itself projective, M is projective.

□

[3][Foxby04, Lemma 7.10] proves this for commutative rings, general proof is similar.

3.2 Stability properties of projective modules

Let k be a field and let G be a group. Consider the category of kG-modules. Given two kG-modules, their direct sum is again a kG-module, and their tensor product over k is again a kG-module. Since $k \otimes_k M$ is isomorphic to M we may say that k is the neutral element with respect to tensore product over k. Viewing the category of kG-modules as a ring with direct sum acting as sum and with tensor product over k acting as multiplication, we will now show that the subgroup of projective modules form an ideal in this ring.

For a finite group G and a finitely generated module M over kG we will then prove simultaneous projectivity of M, $M \otimes_k M$ and $\mathrm{Hom}_k(M, k) \otimes_k M$, which implies that an even larger class of modules are projective simultaneously.

Theorem 3.2.1. *Let R be a ring, G be a group and let P be a projective RG-module. Then $P \otimes_R X$ and $X \otimes_R P$ are projective over RG.*

Proof. From the Hom-tensor adjunction isomorphism (2.1.6) we have a natural isomorphism

$$\mathrm{Hom}_{RG}(P \otimes_R X, -) \cong \mathrm{Hom}_{RG}(P, \mathrm{Hom}_R(X, -)) \ .$$

Thus, if $\mathrm{Hom}_{RG}(P, -)$ is exact, then $\mathrm{Hom}_{RG}(P, \mathrm{Hom}_R(X, -))$, which is isomorphic to $\mathrm{Hom}_{RG}(P \otimes_R X, -)$, is exact. □

Theorem 3.2.2. *Let G be a group. The following are equivalent:*

i. RG is semi-simple;

ii. R is projective over RG.

Proof. There are two implications to show:

$i \Rightarrow ii$: Is clear.

$ii \Rightarrow i$: Given a module M, we know from the stability of tensor products over R (3.2.1) that $R \otimes_R M$ is projective. We have a R-isomorphism φ : $R \otimes_R M \to M$ given by $\varphi(R, m) = R \cdot m$. If φ is a RG homomorphism, then M must be isomorphic to the projective RG-module $R \otimes_R M$ and hence M itself is projective over RG. We verify: Let $r \in RG$ and $(\kappa, m) \in$

$R \otimes_R M$. We then have

$$\begin{aligned}
\varphi(r \cdot (\kappa \otimes_R xm)) &= \varphi(r \cdot \kappa \otimes_R r \cdot m) \\
&\overset{(1)}{=} \varphi(\kappa \otimes_R r \cdot m) \\
&= \kappa \cdot r \cdot m \\
&\overset{(2)}{=} r \cdot \kappa \cdot m \\
&= r \cdot \varphi(\kappa, m)
\end{aligned}$$

(1) holds since RG acts trivially on R. (2) is true because elements of RG commute with elements of R.

□

Definition 3.2.3. Given a group G and a left kG-module M, we may consider the left kG-module $\mathrm{Hom}_k(M,k)$ as in definition (1.2.4) letting $N = k$. Then the action of G on $\mathrm{Hom}_k(M,k)$ is $(g * f)(m) = g \cdot f(g^{-1} * m) \overset{(1)}{=} f(g^{-1} \cdot m)$ for $g \in G$, $f \in \mathrm{Hom}_k(M,k)$ and $m \in M$, (1) holds because kG acts trivially on k.

Proposition 3.2.4. *Let k be a field, G a group and let M and N be left kG-modules. We note that there are maps $\varphi : M \to \mathrm{Hom}_k(\mathrm{Hom}_k(M,k),k)$ and $\psi : N \otimes_k \mathrm{Hom}_k(M,k) \to \mathrm{Hom}_k(M,N)$ given by $\varphi(m) = (f \mapsto f(m))$ and $\psi(n \otimes_k f) = m \mapsto f(m) \cdot n$. Then*

1) φ and ψ are kG-homomorphisms;

2) φ and ψ are isomorphisms when M is finitely generated over k.

Proof. We check that φ is a homomorphism. Let $g \in G$ and $m \in M$. Firstly, note that for $F \in \mathrm{Hom}_k(\mathrm{Hom}_k(M,k),k)$ and $f \in \mathrm{Hom}_k(M,k)$, we have $((g * F)(f))(m) = F(g^{-1} * f(m)) = F(f(g * m))$. In particular $g * (f \mapsto f(m)) = f \mapsto f(g * m)$. On the other hand, $\varphi(g * m) = (f \mapsto f(g * m))$.

Since we already have k-linearity, we may now conclude kG linearity of φ.

φ is injective since all modules over a field such as k are free: Let M have generators $\beta \in B$ as a k-module. Then if m_1 and m_2 are different linear combinations of $\beta \in B$, there must exist one $b \in B$ such that m_1 and m_2 have different b-coefficients. Hence φ will differ in value on the homomorphism sending b to 1_k and β to 0_k for all other β.

Given a module A over k, we know that A is a vector space over k, since k is a field. Therefore, a homomorphism $A \to k$ must be 1_k on one of the basis elements of A and 0 on all others. Hence, there will be as many homomorphisms as there are basis elements. Therefore, $\mathrm{Hom}_k(\mathrm{Hom}_k(M,k),k)$ has the same rank as M. Thus, since φ is injective, it will be a bijection when M (and hence, $\mathrm{Hom}_k(\mathrm{Hom}_k(M,k),k)$) have finite rank as a vector space over k.

We can show that ψ is a homomorphism: Let $g \in G$ and $n \otimes_k f \in N \otimes_k \mathrm{Hom}_k(M,k)$. Then for $F \in \mathrm{Hom}_k(M,N)$, $(g * F)(m) = g \cdot F(g^{-1} * m)$. Applied to $m \mapsto f(m) \cdot n$, the action of g is $m \mapsto (g \cdot f(g^{-1} * m)) \cdot n$. On the other hand, $\psi(g * (n \otimes_k f)) = \psi((g \cdot n) \otimes_k (g * f)) = m \mapsto ((g * f)(m)) \cdot (g \cdot n) = m \mapsto (f(g^{-1}m) \cdot g) \cdot n = m \mapsto (g \cdot f(g^{-1}m)) \cdot n$. From this and from k-linearity we have kG-linearity of ψ.

Over the field k all modules are free. Let B be a k-basis for M and let A be a basis for N. Any homomorphism $M \to N$ is uniquely determined by its value on the basis vectors. Now, for each basis vector $b \in B$ mapped to $a \in A$ we may construct the preimage as $a \otimes_k f_b$, where f_b maps b to 1_k and all other $\beta \in B$ to 0_k. This map on basis vectors is a bijection, since a basis for $N \otimes_k \mathrm{Hom}_k(M,k)$ is $\{a \otimes_k f_b | a \in Ab \in B\}$. Hence, ψ is bijective. □

Theorem 3.2.5 (Stability Theorem). *When k is a field and G is a finite group and M is a finitely generated kG-module, the following six kG-modules are projective simultaneously:*

$$\begin{array}{lcl} M & , \quad \mathrm{Hom}_k(M,k) \otimes_k M \quad , & \mathrm{Hom}_k(M,M) \\ M \otimes_k M & , \quad M \otimes_k \mathrm{Hom}_k(M,k) \quad \textit{and} & \mathrm{Hom}_k(M,k) \end{array}$$

Proof. When M is projective, so are $M \otimes_k M$, $\mathrm{Hom}_k(M,k) \otimes_k M$, and $M \otimes_k \mathrm{Hom}_k(M,k)$ by the stability of tensor products over k (3.2.1).

If either $M \otimes_k \mathrm{Hom}_k(M,k)$ or $M \otimes_k M$ are projective, then by the same lemma also $M \otimes_k M \otimes_k \mathrm{Hom}_k(M,k)$ must be projective.

If $M \otimes_k M \otimes_k \mathrm{Hom}_k(M,k)$ is projective, then so is $M \otimes_k \mathrm{Hom}_k(M,M)$, since they are isomorphic (3.2.3). M is a direct summand of $M \otimes_k \mathrm{Hom}_k(M,M)$ by the inclusion $\iota : m \mapsto m \otimes_k \mathrm{Id}$ and projection $\pi : m \otimes \alpha \mapsto \alpha(m)$. Since a direct summand of a projective module is projective, M is in this case projective.

We now have simultaneous projectivity of

$$M, \quad \mathrm{Hom}_k(M,k) \otimes_k M, \quad M \otimes_k M \quad \textit{and} \quad M \otimes_k \mathrm{Hom}_k(M,k) \cong \mathrm{Hom}_k(M,M)$$

Applying this to $\mathrm{Hom}_k(M,k)$, we have simultaneous projectivity of $\mathrm{Hom}_k(M,k)$ and $\mathrm{Hom}_k(\mathrm{Hom}_k(M,k),k) \otimes_k \mathrm{Hom}_k(M,k)$. Since M is finitely generated and therefore has finite rank over k, $\mathrm{Hom}_k(\mathrm{Hom}_k(M,k),k) \cong_{kG} M$, and hence

$\mathrm{Hom}_k(M,k)$ is projective if and only if $M \otimes_k \mathrm{Hom}_k(M,k)$ (and all the others) are projective. □

3.3 Projectivity versus injectivity

Projective modules and injective modules are indeed very different. Any ring is projective over itself, but not every ring has even finite injective dimension over itself. It is even more rare, that a ring is injective as a module over itself. We show that in case k is a field and G is a finite group, kG is always injective as a module over itself.

Notation 3.3.1. Let us remind ourselves that by the *length* l of a resolution $R_\bullet$ we mean the smallest number s such that $P_i = 0$ for all $i > s$.

For a ring R and a non-zero R-module M, we define

- The projective dimension of M as the smallest number $p \in \mathbb{N}_0$ such that M has a projective resolution of length p;
- The flat dimension of M as the smallest number $f \in \mathbb{N}_0$ such that M has a flat resolution of length f;
- The injective dimension of M as the smallest number $i \in \mathbb{N}_0$ such that M has a injective resolution of length i.
- The Krull dimension of M as the supremum of $k \in \mathbb{N}_0$ such that M has a ascending chain of prime ideals of length k.

For a ring R and an R-module M, we let $\mathrm{pdim}_R M$, $\mathrm{fdim}_R M$ and $\mathrm{injdim}_R M$ denote projective dimension, flat dimension and injective dimension of M over R respectively.

By $\dim_R M$ or "the dimension of M over R" we mean the Krull dimension of M over R.

Lemma 3.3.2 (The Dimension Lemma)**.** *When G is finite and M is a finitely generated left kG-module,*

$$\mathrm{pdim}_{kG} M = \mathrm{fdim}_{kG} M = \mathrm{injdim}_{kG^{\mathrm{op}}} \mathrm{Hom}_k(M,k) .$$

We start by proving the following little lemma:

Lemmidos 3.3.3. *Let G be a group and k a field. For any finitely generated left kG-module A, the following are equivalent:*

i. A is projective;

ii. A is flat;

iii. $\mathrm{Hom}_k(A, k)$ *is injective.*

Proof. There are three implications to show

$i \Rightarrow ii$: Is known to be true for any module over any ring.

$ii \Rightarrow iii$: We even have $ii \Leftrightarrow iii$ from the discussion in section (2.1.2), where we found a kG-isomorphism, $\mathrm{Hom}_k(- \otimes_{kG} A, k) \cong \mathrm{Hom}_{kG}(-, A)$, obtained by extension of scalars from the inclusion $k \hookrightarrow kG$.

$iii \Rightarrow i$: Let n denote the number of generators of A. Then we have a surjective kG-homomorphism $\pi : (kG)^n \twoheadrightarrow A$. Applying the $\mathrm{Hom}_k(-, k)$ functor we get an inclusion $\mathrm{Hom}_k(\pi, k) : \mathrm{Hom}_k(A, k) \hookrightarrow \mathrm{Hom}_k((kG)^n, k)$. If we assume $\mathrm{Hom}_k(A, k)$ is injective, this last homomorphism is split. Thus, there is a homomorphism $\sigma : \mathrm{Hom}_k((kG)^n, k) \to \mathrm{Hom}_k(A, k)$ such that the composition $\sigma \circ \mathrm{Hom}_k(\pi, k)$ is the identity on $\mathrm{Hom}_k(A, k)$. If again we apply the $Hom_k(-, k)$-functor, we get $\mathrm{Hom}_k(\mathrm{Hom}_k(\pi, k), k) : A \to (kG)^n$, since $\mathrm{Hom}_k(\mathrm{Hom}_k(-, k), k) \cong -$ for finitely generated kG-modules, where $\mathrm{Hom}_k(\mathrm{Hom}_k(\pi, k), k)$ again is split. Being a direct summand of a free (and hence projective) module, A must itself be projective.

□

Proof of Lemma 3.3.2. Let M be a finitely generated left kG-module. Let $F_\bullet \to M$ be a finitely generated flat resolution of M. Then, by the lemmidos above (3.3.3), each F_i is also projective, and hence $F_\bullet$ is also a projective resolution of M. Since in general $\mathrm{fdim}_{kG} M \leq \mathrm{pdim}_{kG} M$, this gives $\mathrm{pdim}_{kG} M = \mathrm{fdim}_{kG} M$.

If $J_\bullet \to \mathrm{Hom}_k(M, k)$ is an injective resolution of $\mathrm{Hom}_k(M, k)$, then by injectivity of J_i, $\mathrm{Hom}_k(J_i, k)$ is flat. This gives us a flat resolution of $\mathrm{Hom}_k(\mathrm{Hom}_k(M, k), k)$ which, in case M is finitely generated, is isomorphic to M. Conversely, given a flat resolution $F_\bullet \to M$, each $\mathrm{Hom}_k(F_\bullet, k)$ is injective (by Adjointness[4]). Hence, $\mathrm{fdim}_{kG} M = \mathrm{injdim}_{kG} M$. □

Theorem 3.3.4. *Let k be a field, G a finite group and M a finitely generated left kG-module. The following are equivalent:*

[4][Foxby04, Thm. 19.17] proves this for commutative rings, general proof is similar

i. M is projective;

ii. $\mathrm{fdim}_{kG} M$ *is finite;*

iii. M is injective;

iv. $\mathrm{injdim}_{kG} M$ *is finite;*

Proof. We structure the proof as the proof of three biimplications.

i ⇔ *iii*: M is projective if and only if $\mathrm{Hom}_k(M,k)$ is projective by (3.2.5). $\mathrm{Hom}_k(M,k)$ is projective if and only if $\mathrm{Hom}_k(\mathrm{Hom}_k(M,k),k)$ is injective (by 3.3.2), and $\mathrm{Hom}_k(\mathrm{Hom}_k(M,k),k)$ is isomorphic to M by (3.2.3). Hence M is projective if and only if M is injective.

ii ⇔ *iv*: Since M is finitely generated, we may find a finitely generated projective resolution $P_\bullet$ of M. Applying $\mathrm{Hom}_k(-,k)$ (and theorem (3.2.5)) we get a projective resolution $\mathrm{Hom}_k(P_\bullet,k)$ of $\mathrm{Hom}_k(M,k)$. Hence the projective dimensions of M and $\mathrm{Hom}_k(M,k)$ coincide. By section (3.3.2), the projective dimension of $\mathrm{Hom}_k(M,k)$ is equal to the injective dimension of $\mathrm{Hom}_k(\mathrm{Hom}_k(M,k),k) \cong M$.

i ⇔ *ii*: If M is projective, then M is flat and hence $\mathrm{fdim}_{kG} M$ is finite.

Let M have finite flat dimension. Then, by (3.3.2), M has finite projective dimension. Since M itself is finitely generated, we may find a finite projective resolution of finitely generated modules. Let

$$0- > P_n \stackrel{\partial_n}{\to} \cdots \stackrel{\partial_1}{\to} P_0 \stackrel{\partial_0}{\to} M \to 0$$

be such a resolution.

If a projective resolution of M has length 0 then M is isomorphic to P_0 and hence projective.

If a projective resolution of M has length $n > 0$, then we may find a projective resoluton of M of length $n-1$ as follows:

Applying $i \Rightarrow iii$ to P_n, we see that P_n is injective, and hence, that ∂_n splits. This allows us to write P_{n-1} as the direct sum $\partial_n(P_n) \oplus P_{n-1}/\partial_n(P_n)$. Since each summand of a projective is projective, we may construct a projective resolution $P_{n-1}/\partial_n(P_n) \to \cdots \stackrel{\partial_0}{\to} M \to 0$ of length $n-1$.

By induction, M has projective dimension 0, or, in other words, M is projective.

□

Statement 3.3.5. *Let k be a field, G a finite group and M a finitely generated* right *kG-module. The following are equivalent:*

i. *M is projective;*

ii. $\mathrm{fdim}_{kG} M$ *is finite;*

iii. *M is injective;*

iv. $\mathrm{injdim}_{kG} M$ *is finite;*

The proof of (3.3.5) is symmetric to the proof in the case where M is a left module.

Proposition 3.3.6. *Given a field k and a finite group G, then kG is an injective kG-module if and only if projectivity and injectivity coincide for finitely generated kG-modules.*

Proof. There are two implications:

If projectivity and injectivity coincide for any finitely generated kG-module, then in particular it will coincide for the free and hence projective module kG generated by one element.

Assume kG is injective as a kG-module. Let a finitely generated module M be given. Then M is a direct summand in a finitely generated free module isomorphic to kG^n for some n. Since kG is an injective module, so is kG^n. Since M is a direct summand of an injective module, M is injective. □

3.4 A Gorenstein Perspective

Statement 3.4.1. *Let $(R, \mathfrak{m}, k)$ be an n-dimensional commutative Noetherian local ring. The following conditions are equivalent:*

i. $\mathrm{injdim}_R R < \infty$;

ii. $\mathrm{injdim}_R R = n$;

iii. $\mathrm{Ext}^i_R(k, R) = 0$ *for* $i \neq n$ *and* $\mathrm{Ext}^n_R(k, R) \cong k$;

iv. $\mathrm{Ext}^i_R(k, R) = 0$ *for some* $i > n$;

v. $\mathrm{Ext}^i_R(k, R) = 0$ *for some* $i < n$ *and* $\mathrm{Ext}^n_R(k, R) \cong k$.

A proof is found at [Matsumura86, Part 6, §18 (theorem 18.1)]

Definition 3.4.2. If the above conditions hold, R is said to be a commutative *n-dimensional local Gorenstein ring*.

Definition 3.4.3. If a ring R has finite injective dimension as a left R-module and R has finite injective dimension as a right R-module, then R is said to be a *Gorenstein ring*.

If R is local and Gorenstein (commutative or not), we say R is a *local Gorenstein ring*.

If the right injective and left injective dimension of a Gorenstein ring R over itself are both n, then we say R is an n-dimensional Gorenstein ring.

Observation 3.4.4 (Self-injective group algebras)**.** Let G be a finite group and k a field. Then kG is a free left kG-module, and hence by the theorem (3.3.4), kG is an *injective* left kG-module. The left injective dimension $\text{injdim}_{kG} kG = 0$. As stated in (3.3.5), this also holds when we view kG as a right kG-module. Thus, also the right injective dimension $\text{injdim}_{kG} kG = 0$.

That means, kG is a zero-dimensional Gorenstein ring.

If G is a p-group where p is the characteristic of k, (kG, I_G, k) is a local ring (1.3.6). That is, we may view kG as a zero-dimensional local Gorenstein ring. In that case, we have showed in (2.4.5) that any inclusion $H \hookrightarrow G$ of a subgroup H of G extends to a Gorenstein map $kH \to kG$.

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