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Well-posedness for Semi-relativistic Hartree Equations of Critical Type

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Abstract We prove local and global well-posedness for semi-relativistic, nonlinear Schrödinger equations $i\partial_t u = \sqrt{-\Delta + m^2}u + F(u)$ with initial data in $H^s(\mathbb{R}^3)$, $s \geqslant 1/2$. Here $F(u)$ is a critical Hartree nonlinearity that corresponds to Coulomb or Yukawa type self-interactions. For focusing $F(u)$, which arise in the quantum theory of boson stars, we derive global-in-time existence for small initial data, where the smallness condition is expressed in terms of the L^2-norm of solitary wave ground states. Our proof of well-posedness does not rely on Strichartz type estimates. As a major benefit from this, our method enables us to consider external potentials of a quite general class.

Keywords Well-posedness · Cauchy problem · Semi-relativistic Hartree equation · Boson stars

Mathematics Subject Classifications (2000) 35Q40 · 35Q55 · 47J35

1 Introduction

In this paper we study the Cauchy problem for nonlinear Schrödinger equations with kinetic energy part originating from special relativity. That is, we consider the initial value problem for

$$i\partial_t u = \sqrt{-\Delta + m^2}\, u + F(u), \qquad (t, x) \in \mathbb{R}^{1+3}, \tag{1}$$

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where $u(t,x)$ is complex-valued, $m \geqslant 0$ denotes a given mass parameter, and $F(u)$ is some nonlinearity. Here the operator $\sqrt{-\Delta + m^2}$ is defined via its symbol $\sqrt{\xi^2 + m^2}$ in Fourier space.

Such "semi-relativistic" equations have (though not Lorentz covariant in general) interesting applications in the quantum theory for large systems of self-interacting, relativistic bosons. Equation (1) arises, for instance, as an effective description of *boson stars*, see, e. g., [4, 5, 12], where $F(u)$ is a focusing Hartree nonlinearity given by

$$F(u) = \left(\frac{\lambda}{|x|} * |u|^2 \right) u, \tag{2}$$

with some constant $\lambda < 0$ and $*$ as convolution. Motivated by this physical example with focusing self-interaction of Coulomb type, we address the Cauchy problem for (1) and a class of Hartree nonlinearities including (2). In fact, we shall prove well-posedness for initial data $u(0,x) = u_0(x)$ in $H^s = H^s(\mathbb{R}^3)$, $s \geqslant 1/2$; see Theorems 1–3 below.

Let us briefly point out a decisive feature of the example cited in (2) above. Apart from its physical relevance, the nonlinearity given by (2) leads to an L^2-*critical* equation as indicated by the fact that the coupling constant λ has to be dimensionless. In consequence of this, L^2-smallness of the initial datum enters as a sufficient condition for global-in-time solutions. More precisely, we derive for $u_0 \in H^s$, $s \geqslant 1/2$, the following criterion implying global well-posedness

$$\int_{\mathbb{R}^3} |u_0(x)|^2 \, dx < \int_{\mathbb{R}^3} |Q(x)|^2 \, dx. \tag{3}$$

This condition holds irrespectively of the parameter $m \geqslant 0$ in (1); see Theorem 2 below. Here $Q \in H^{1/2}$ is a positive solution (ground state) for the nonlinear equation

$$\sqrt{-\Delta}\, Q + \left(\frac{\lambda}{|x|} * |Q|^2 \right) Q = -Q, \tag{4}$$

which gives rise to solitary wave solutions, $u(t,x) = e^{it} Q(x)$, for (1) with $m = 0$. In fact, it can be shown that criterion (3) guaranteeing global-in-time solutions in the focusing case is optimal in the sense that there exist solutions, $u(t)$, with $\|u_0\|_2^2 > \|Q\|_2^2$, which blow up within finite time; see [7]. Physically, this blow-up phenomenon indicates "gravitational collapse" of a boson star whose mass exceeds a critical value.

Furthermore, criterion (3) can be linked with established results as follows. First, it is reminiscent to a well-known condition derived in [16] for global well-posedness of nonrelativistic Schrödinger equations with focusing, local nonlinearity (see also [13] for Hartree nonlinearities). Second, criterion (3) is in accordance with a sufficient stability condition proved in [12] for the related time-independent problem (i. e., a static boson star); see [6] for more details concerning known results on Hartree equations.

We now give an outline of our methods. The proof of well-posedness presented below does *not* rely on Strichartz (i. e., space-time) estimates for the propagator, $e^{-it\sqrt{-\Delta+m^2}}$, but it employs sharp estimates (e. g., Kato's inequality (17) below) to derive local Lipschitz continuity of L^2-critical nonlinearities of Hartree type. Local well-posedness then follows by standard methods for abstract evolution equations. Furthermore, global well-posedness is derived by means of a priori estimates and conservation of charge and energy whose proof requires a regularization method.

This paper is organized as follows.

- In Section 2 we introduce a class of critical Hartree nonlinearities including (2). First, we state Theorems 1 and 2 that establish local and global well-posedness in energy space $H^{1/2}$ for this class of nonlinearities. In Theorem 3 we extend these results to H^s, for every $s \geqslant 1/2$. Finally, external potentials are included, i. e., we consider

$$i\partial_t u = \left(\sqrt{-\Delta + m^2} + V\right) u + F(u), \tag{5}$$

  where $V : \mathbb{R}^3 \to \mathbb{R}$ is given. In Theorem 4 we state local and global well-posedness for (5) with initial datum $u(0, x) = u_0(x)$ in the appropriate energy space. Assumption 1 imposed below on V is considerably weak and implies that $\sqrt{-\Delta + m^2} + V$ defines a self-adjoint operator via its form sum.
- The main results (i. e., Theorems 1–4) are proved in Section 3.
- Appendices A and B contain some useful facts about fractional derivatives, a discussion of ground states, and some details of the proofs.

Notation

Throughout this text, the symbol $*$ stands for convolution on $\mathbb{R}^3$, i. e.,

$$(f * g)(x) := \int_{\mathbb{R}^3} f(x-y)g(y)\, dy,$$

and $L^p(\mathbb{R}^3)$, with norm $\|\cdot\|_p$ and $1 \leqslant p \leqslant \infty$, denotes the usual Lebesgue L^p-space of complex-valued functions on $\mathbb{R}^3$. Moreover, $L^2(\mathbb{R}^3)$ is associated with the scalar product defined by

$$\langle u, v\rangle := \int_{\mathbb{R}^3} \overline{u}(x)v(x)\, dx.$$

For $s \in \mathbb{R}$ and $1 \leqslant p \leqslant \infty$, we introduce fractional Sobolev spaces (see, e.g., [1]) with their corresponding norms according to

$$H^{s,p}(\mathbb{R}^3) := \left\{u \in \mathcal{S}'(\mathbb{R}^3) : \|u\|_{H^{s,p}} := \|\mathcal{F}^{-1}[(1+\xi^2)^{s/2}\mathcal{F}u]\|_p < \infty\right\},$$

where $\mathcal{F}$ denotes the Fourier transform in $\mathcal{S}'(\mathbb{R}^3)$ (space of tempered distributions). In our analysis, the Sobolev spaces

$$H^s(\mathbb{R}^3) := H^{s,2}(\mathbb{R}^3),$$

with norms $\|\cdot\|_{H^s} := \|\cdot\|_{H^{s,2}}$, will play an important role.

In addition to the common L^p-spaces, we also make use of local L^p-space, $L^p_{\mathrm{loc}}(\mathbb{R}^3)$, with $1 \leqslant p \leqslant \infty$, and weak (or Lorentz) spaces, $L^p_w(\mathbb{R}^3)$, with $1 < p < \infty$ and corresponding norms given by

$$\|u\|_{p,w} := \sup_{\Omega} |\Omega|^{-1/p'} \int_{\Omega} |u(x)|\, dx,$$

where $1/p + 1/p' = 1$ and Ω denotes an arbitrary measurable set with Lebesgue measure $|\Omega| < \infty$; see, e. g., [11] for this definition of L^p_w-norms. Note that $L^p(\mathbb{R}^3) \subsetneq L^p_w(\mathbb{R}^3)$, for $1 < p < \infty$.

The symbol $\Delta = \sum_{i=1}^3 \partial^2_{x_i}$ stands for the usual Laplacian on $\mathbb{R}^3$, and $\sqrt{-\Delta + m^2}$ is defined via its symbol $\sqrt{\xi^2 + m^2}$ in Fourier space. Besides the operator $\sqrt{-\Delta + m^2}$, we also employ Riesz and Bessel potentials of order $s \in \mathbb{R}$, which we denote by $(-\Delta)^{s/2}$ and $(1 - \Delta)^{s/2}$, respectively; see also Appendix A.

Except for theorems and lemmas, we often use the abbreviations $L^p = L^p(\mathbb{R}^3)$, $L^p_w = L^p_w(\mathbb{R}^3)$, and $H^s = H^s(\mathbb{R}^3)$. In what follows, $a \lesssim b$ always denotes an inequality $a \leqslant cb$, where c is an appropriate positive constant that can depend on fixed parameters.

2 Main Results

We consider the following initial value problem

$$\begin{cases} i\partial_t u = \sqrt{-\Delta + m^2}\, u + \left(\dfrac{\lambda e^{-\mu |x|}}{|x|} * |u|^2 \right) u, \\ u(0, x) = u_0(x), \qquad u : [0, T) \times \mathbb{R}^3 \to \mathbb{C}, \end{cases} \tag{6}$$

where $m \geqslant 0$, $\lambda \in \mathbb{R}$, and $\mu \geqslant 0$ are given parameters. Note that $|\lambda|$ could be absorbed in the normalization of $u(t, x)$, but we shall keep λ explicit in the following; see also [4] for this convention.

Our particular choice of the Hartree type nonlinearities in (6) is motivated by the fact that (6) can be rewritten as the following system of equations

$$\begin{cases} i\partial_t u = \sqrt{-\Delta + m^2}\, u + \Psi u, \\ (\mu^2 - \Delta)\Psi = 4\pi\lambda |u|^2, \qquad u(0, x) = u_0(x), \end{cases} \tag{7}$$

where $\Psi = \Psi(t, x)$ is real-valued and $\Psi(t, x) \to 0$ as $|x| \to \infty$. This reformulation stems from the observation that $e^{-\mu|x|}/4\pi|x|$ is the Green's function of $(\mu^2 - \Delta)$ in $\mathbb{R}^3$; see Appendix A. System (7) now reveals the physical intuition behind (6), i. e., the function $u(t, x)$ corresponds to a "positive energy wave" with instantaneous self-interaction that is either of Coulomb or Yukawa type depending on whether $\mu = 0$ or $\mu > 0$, respectively. To prove well-posedness we shall, however, use formulation (6) instead, and we refer to facts from potential theory only when estimating the nonlinearity.

2.1 Local Well-posedness

Let us begin with well-posedness in energy space, i. e., we assume that $u_0 \in H^{1/2}$ holds in (6). The following Theorem 1 establishes local well-posedness in the strong sense, i. e., we have existence and uniqueness of solutions, their continuous dependence on initial data, and the blow-up alternative. The precise statements is as follows.

Theorem 1 *Let $m \geqslant 0, \lambda \in \mathbb{R}$, and $\mu \geqslant 0$. Then initial value problem (6) is locally well-posed in $H^{1/2}(\mathbb{R}^3)$. This means that, for every $u_0 \in H^{1/2}(\mathbb{R}^3)$, there exist a unique solution*

$$u \in C^0\big([0, T); H^{1/2}(\mathbb{R}^3)\big) \cap C^1\big([0, T); H^{-1/2}(\mathbb{R}^3)\big),$$

and it depends continuously on u_0. Here $T \in (0, \infty]$ is the maximal time of existence, where we have that either $T = \infty$ or $T < \infty$ and $\lim_{t \uparrow T} \|u(t)\|_{H^{1/2}} = \infty$ holds.

Remark Continuous dependence means that the map $u_0 \mapsto u \in C^0(I; H^{1/2})$ is continuous for every compact interval $I \subset [0, T)$.

2.2 Global Well-posedness

The local-in-time solutions derived in Theorem 1 extend to all times, by virtue of Theorem 2 below, provided that either $\lambda \geqslant 0$ holds (corresponding to a repulsive nonlinearity) or $\lambda < 0$ and the initial datum is sufficiently small in L^2.

Theorem 2 *The solution of (6) derived in Theorem 1 is global in time, i. e., we have that $T = \infty$ holds, provided that one of the following conditions is met.*

(1) $\lambda \geqslant 0$.
(2) $\lambda < 0$ *and* $\|u_0\|_2^2 < \|Q\|_2^2$, *where* $Q \in H^{1/2}(\mathbb{R}^3)$ *is a strictly positive solution (ground state) of*

$$\sqrt{-\Delta}\, Q + \left(\frac{\lambda}{|x|} * |Q|^2\right) Q = -Q. \tag{8}$$

Moreover, we have the estimate $\|Q\|_2^2 > \frac{4}{\pi|\lambda|}$.

Remarks

(1) Notice that condition (2) implies global well-posedness for (6) irrespectively of $m \geqslant 0$.

(2) For $a > 0$, the function $Q_a(x) = a^{3/2} Q(ax)$ yields another ground state with $\| Q_a \|_2 = \| Q \|_2$ that satisfies

$$\sqrt{-\Delta}\, Q_a + \left(\frac{\lambda}{|x|} \ast |Q_a|^2 \right) Q_a = -a Q_a. \tag{9}$$

We refer to Appendix B for a discussion of $Q \in H^{1/2}$.

(3) Condition (2) resembles a well-known criterion derived in [16] for global-in-time existence for L^2-critical nonlinear (nonrelativistic) Schrödinger equations.

(4) It is shown in [7] that criterion (3) for having global-in-time solutions in the focusing case is optimal in the sense that there exist solutions, $u(t)$, with $\|u_0\|_2^2 > \|Q\|_2^2$, which blow up within finite time.

2.3 Higher Regularity

We now turn to well-posedness of (6) in H^s, for $s \geqslant 1/2$, which is settled by the following result.

Theorem 3 *For every $s \geqslant 1/2$, the conclusions of Theorems* 1 *and* 2 *hold, where $H^{1/2}(\mathbb{R}^3)$ and $H^{-1/2}(\mathbb{R}^3)$ in Theorem* 1 *are replaced by $H^s(\mathbb{R}^3)$ and $H^{s-1}(\mathbb{R}^3)$, respectively.*

Remark For $s = 1$, this result is needed in [4] for a rigorous derivation of (6) with Coulomb type self-interaction (i. e., $\mu = 0$) from many-body quantum mechanics.

2.4 External Potentials

Now we consider the following extension of (6) that arises by adding an external potential:

$$\begin{cases} i\partial_t u = \left(\sqrt{-\Delta + m^2} + V\right) u + \left(\dfrac{\lambda e^{-\mu|x|}}{|x|} \ast |u|^2 0 \right) u, \\ u(0,x) = u_0(x), \qquad u : [0,T) \times \mathbb{R}^3 \to \mathbb{C}, \end{cases} \tag{10}$$

where $m \geqslant 0$, $\lambda \in \mathbb{R}$, $\mu \geqslant 0$ are given parameters, and $V : \mathbb{R}^3 \to \mathbb{R}$ denotes a preassigned function that meets the following condition.

Assumption 1 *Suppose that $V = V_+ + V_-$ holds, where V_+ and V_- are real-valued, measurable functions with the following properties.*

(1) *$V_+ \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$ and $V_+ \geqslant 0$.*

(2) V_- is *$\sqrt{-\Delta}$-form bounded with relative bound less than* 1, *i. e., there exist constants $0 \leqslant a < 1$ and $0 \leqslant b < \infty$, such that*

$$|\langle u, V_- u \rangle| \leqslant a \langle u, \sqrt{-\Delta}\, u \rangle + b \langle u, u \rangle$$

holds for all $u \in H^{1/2}(\mathbb{R}^3)$.

We mention that Assumption 1 implies that $\sqrt{-\Delta + m^2} + V$ leads to a self-adjoint operator on L^2 via its form sum. Furthermore, the energy space given by

$$X := \left\{ u \in H^{1/2}(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)\, |u(x)|^2 \, dx < \infty \right\} \tag{11}$$

is complete with norm $\| \cdot \|_X$, and its dual space is denoted by X^*. We refer to Section 3.4 for more details on $\sqrt{-\Delta + m^2} + V$ and X.

After this preparing discussion, the extension of Theorems 1 and 2 for the initial value problem (10) can be now stated as follows.

Theorem 4 *Let $m \geqslant 0$, $\lambda \in \mathbb{R}$, $\mu \geqslant 0$, and suppose that V satisfies Assumption 1. Then* (10) *is locally well-posed in the following sense. For every $u_0 \in X$, there exists a unique solution*

$$u \in C^0([0, T); X) \cap C^1([0, T); X^*),$$

and it depends continuously on u_0. Here $T \in (0, \infty]$ is the maximal time of existence such that either $T = \infty$ or $T < \infty$ and $\lim_{t \uparrow T} \|u(t)\|_X = \infty$ holds. Moreover, we have that $T = \infty$ holds, if one of the following conditions is satisfied.

(1) $\lambda \geqslant 0$.
(2) $\lambda < 0$ *and* $\|u_0\|_2^2 < (1 - a)\|Q\|_2^2$, *where Q is the ground state mentioned in Theorem* 2 *and* $0 \leqslant a < 1$ *denotes the relative bound introduced in Assumption* 1.

Remarks

(1) To meet Assumption 1 for V_+, we can choose, for example, $V_+(x) = |x|^\beta$, with $\beta \geqslant 0$; or even super-polynomial growth such as $V_+(x) = e^{|x|}$. Note that Assumption 1 for V_- is satisfied (by virtue of Sobolev inequalities), if

$$|V_-(x)| \leqslant \frac{c}{|x|^{1-\varepsilon}} + d$$

holds for some $0 < \varepsilon \leqslant 1$ and constants $0 \leqslant c, d < \infty$. In fact, we can even admit $\varepsilon = 0$ provided that $c < 2/\pi$ holds, as can be seen from inequality (17) below.

(2) Since we avoid using Strichartz estimates in our well-posedness proof below, we only need that V_+ belongs to L^1_{loc}. In contrast to this, compare, for instance, the conditions on V in [17] for deriving Strichartz type estimates for $e^{-it(-\Delta+V)}$ in order to prove local well-posedness for (nonrelativistic) nonlinear Schrödinger equations with external potentials.

3 Proof of the Main Results

In this section we prove Theorems 1–4. Although Theorem 4 generalizes Theorems 1 and 2, we postpone the proof of Theorem 4 to the final part of this section.

3.1 Proof of Theorem 1 (Local Well-posedness)

Let $u_0 \in H^{1/2}$ be fixed. In view of (6) we put

$$A := \sqrt{-\Delta + m^2} \quad \text{and} \quad F(u) := \left(\frac{\lambda e^{-\mu|x|}}{|x|} * |u|^2 \right) u, \tag{12}$$

and we consider the integral equation

$$u(t) = e^{-itA} u_0 - i \int_0^t e^{-i(t-\tau)A} F(u(\tau))\, d\tau. \tag{13}$$

Here $u(t)$ is supposed to belong to the Banach space

$$Y_T := C^0\big([0, T); H^{1/2}(\mathbb{R}^3)\big), \tag{14}$$

with some $T > 0$ and norm $\|u\|_{Y_T} := \sup_{t \in [0,T)} \|u(t)\|_{H^{1/2}}$. The proof of Theorem 1 is now organized in two steps as follows.

Step 1: Estimating the Nonlinearity

We show that the nonlinearity $F(u)$ is locally Lipschitz continuous from $H^{1/2}$ into itself. This is main point of our argument for local well-posedness and it reads as follows.

Lemma 1 *For $\mu \geqslant 0$, the map $J(u) := \big(\frac{e^{-\mu|x|}}{|x|} * |u|^2\big)u$ is locally Lipschitz continuous from $H^{1/2}(\mathbb{R}^3)$ into itself with*

$$\|J(u) - J(v)\|_{H^{1/2}} \lesssim (\|u\|^2_{H^{1/2}} + \|v\|^2_{H^{1/2}}) \|u - v\|_{H^{1/2}},$$

for all $u, v \in H^{1/2}(\mathbb{R}^3)$.

Proof (*Of Lemma 1*) We prove the claim for $\mu = 0$ and $\mu > 0$ in a common way, so let $\mu \geqslant 0$ be fixed. For $s \in \mathbb{R}$, it is convenient to introduce

$$\mathcal{D}^s := (\mu^2 - \Delta)^{s/2}.$$

Note that due to the equivalence

$$\|u\|_2 + \|\mathcal{D}^{1/2} u\|_2 \lesssim \|u\|_{H^{1/2}} \lesssim \|u\|_2 + \|\mathcal{D}^{1/2} u\|_2,$$

it is sufficient to estimate the quantities

$$I := \|J(u) - J(v)\|_2 \quad \text{and} \quad II := \|\mathcal{D}^{1/2}[J(u) - J(v)]\|_2,$$

where I is needed only if $\mu = 0$. Using now the identity

$$J(u) - J(v) = 1/2\left[\left(\frac{e^{-\mu|x|}}{|x|} * (|u|^2 - |v|^2)\right)(u+v) + \right.$$
$$\left. + \left(\frac{e^{-\mu|x|}}{|x|} * (|u|^2 + |v|^2)\right)(u-v)\right]$$

together with Hölder's inequality (which we tacitly apply from now on), we find that

$$\begin{aligned} I \lesssim\ & \left\| \left(\frac{e^{-\mu|x|}}{|x|} * (|u|^2 - |v|^2)\right)(u+v)\right\|_2 + \\ & + \left\| \left(\frac{e^{-\mu|x|}}{|x|} * (|u|^2 + |v|^2)\right)(u-v)\right\|_2 \\ \lesssim\ & \left\| \frac{e^{-\mu|x|}}{|x|} * (|u|^2 - |v|^2)\right\|_6 \|u+v\|_3 + \\ & + \left\| \frac{e^{-\mu|x|}}{|x|} * (|u|^2 + |v|^2)\right\|_\infty \|u-v\|_2. \end{aligned} \tag{15}$$

Observing that $e^{-\mu|x|}|x|^{-1} \in L^3_w$ holds, the first term of right-hand side of (15) can be bounded by means of the weak Young inequality (see, e. g., [11]) as follows

$$\begin{aligned} \left\| \frac{e^{-\mu|x|}}{|x|} * (|u|^2 - |v|^2)\right\|_6 &\lesssim \left\| \frac{e^{-\mu|x|}}{|x|}\right\|_{3,w} \| |u|^2 - |v|^2 \|_{6/5} \\ &\lesssim \|u+v\|_3 \|u-v\|_2. \end{aligned} \tag{16}$$

The second term in (15) can be estimated by noting that

$$\left\| \frac{e^{-\mu|x|}}{|x|} * |u|^2 \right\|_\infty \lesssim \sup_{y\in\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x-y|}\, dx \lesssim \|(-\Delta)^{1/4} u\|_2^2, \tag{17}$$

which follows from the operator inequality $|x-y|^{-1} \leqslant \frac{\pi}{2}(-\Delta_{x-y})^{1/2}$ (see, e. g., [10, Section V.5.4]) and translational invariance, i. e., we use that $\Delta_{x-y} = \Delta_x$ holds for all $y \in \mathbb{R}^3$. Combining now (16) and (17) we find that

$$\begin{aligned} I &\lesssim \|u+v\|_3^2 \|u-v\|_2 + \left(\|u\|_{H^{1/2}}^2 + \|v\|_{H^{1/2}}^2\right) \|u-v\|_2 \\ &\lesssim \left(\|u\|_{H^{1/2}}^2 + \|v\|_{H^{1/2}}^2\right) \|u-v\|_{H^{1/2}}, \end{aligned}$$

where we make use of the Sobolev inequality $\|u\|_3 \lesssim \|u\|_{H^{1/2}}$ in $\mathbb{R}^3$.

It remains to estimate II. To do so, we appeal to the generalized (or fractional) Leibniz rule (see Appendix A) leading to

$$\begin{aligned}
II &\lesssim \left\| \mathcal{D}^{1/2} \left[\left(\frac{e^{-\mu|x|}}{|x|} * (|u|^2 - |v|^2) \right) (u+v) \right] \right\|_2 + \\
&\quad + \left\| \mathcal{D}^{1/2} \left[\left(\frac{e^{-\mu|x|}}{|x|} * (|u|^2 + |v|^2) \right) (u-v) \right] \right\|_2 \\
&\lesssim \left\| \mathcal{D}^{1/2} \left(\frac{e^{-\mu|x|}}{|x|} * (|u|^2 - |v|^2) \right) \right\|_6 \|u+v\|_3 + \\
&\quad + \left\| \frac{e^{-\mu|x|}}{|x|} * (|u|^2 - |v|^2) \right\|_\infty \|\mathcal{D}^{1/2}(u+v)\|_2 + \\
&\quad + \left\| \mathcal{D}^{1/2} \left[\left(\frac{e^{-\mu|x|}}{|x|} * (|u|^2 + |v|^2 \right) \right] \right\|_6 \|u-v\|_3 + \\
&\quad + \left\| \frac{e^{-\mu|x|}}{|x|} * (|u|^2 + |v|^2) \right\|_\infty \|\mathcal{D}^{1/2}(u-v)\|_2.
\end{aligned} \tag{18}$$

By referring to Appendix A, we notice that $\frac{e^{-\mu|x|}}{4\pi|x|} * f$ can be expressed as $\mathcal{D}^{-2} f = (\mu^2 - \Delta)^{-1} f$ in $\mathbb{R}^3$ (here $f \in \mathcal{S}(\mathbb{R}^3)$ is initially assumed, but our arguments follow by density). Thus, the first term of the right-hand side of (18) is found to be

$$\begin{aligned}
\left\| \mathcal{D}^{1/2} \left(\frac{e^{-\mu|x|}}{|x|} * (|u|^2 - |v|^2) \right) \right\|_6 &\lesssim \|\mathcal{D}^{-3/2}(|u|^2 - |v|^2)\|_6 \\
&\lesssim \left\| G^\mu_{3/2} * (|u|^2 - |v|^2) \right\|_6 \\
&\lesssim \left\| G^\mu_{3/2} \right\|_{2,w} \| |u|^2 - |v|^2 \|_{3/2} \\
&\lesssim \|u+v\|_3 \|u-v\|_3,
\end{aligned} \tag{19}$$

where we use weak Young's inequality together with the fact that $\mathcal{D}^{-3/2} f$ corresponds to $G^\mu_{3/2} * f$ with some $G^\mu_{3/2} \in L^2_w(\mathbb{R}^3)$; see (42). The $\| \cdot \|_\infty$-part of the second term occurring in (18) can be estimated by using the Cauchy-Schwarz inequality and (17) once again:

$$\begin{aligned}
\left\| \frac{e^{-\mu|x|}}{|x|} * (|u|^2 - |v|^2) \right\|_\infty &\leqslant \left\| \frac{1}{|x|} * (|u|^2 - |v|^2) \right\|_\infty \\
&\lesssim \sup_{y \in \mathbb{R}^3} \left| \int_{\mathbb{R}^3} \frac{|u(x)|^2 - |v(x)|^2}{|x-y|} \, dx \right| \\
&\lesssim \sup_{y \in \mathbb{R}^3} \left| \left\langle (u(x) + v(x)), \frac{1}{|x-y|} (u(x) - v(x)) \right\rangle \right|
\end{aligned}$$

$$\begin{aligned}&\lesssim \|(-\Delta)^{1/4}(u+v)\|_2\|(-\Delta)^{1/4}(u-v)\|_2\\&\lesssim (\|u\|_{H^{1/2}}+\|v\|_{H^{1/2}})\|u-v\|_{H^{1/2}}\end{aligned}\tag{20}$$

The remaining terms in (18) deserve no further comment, since they can be estimated in a similar fashion to all estimates derived so far. Thus, we conclude that

$$\|J(u)-J(v)\|_{H^{1/2}} \lesssim I + II \lesssim (\|u\|^2_{H^{1/2}}+\|v\|^2_{H^{1/2}})\|u-v\|_{H^{1/2}}$$

and the proof of Lemma 1 is now complete. □

Remarks

(1) The proof of Lemma 1 relies on (17) in a crucial way. Employing just the Sobolev embedding $H^{1/2} \subset L^2 \cap L^3$ (in $\mathbb{R}^3$) together with the (non weak) Young inequality is not sufficient to conclude that $\|\frac{e^{-\mu|x|}}{|x|} * |u|^2\|_\infty < \infty$ whenever $u \in H^{1/2}$.

(2) The proof of Lemma 1 fails for "super-critical" Hartree nonlinearities $J(u) = (|x|^{-\alpha} * |u|^2)u$, where $1 < \alpha < 3$. Thus, the choice $\alpha = 1$ represents a borderline case when deriving local Lipschitz continuity in energy space $H^{1/2}$.

Step 2: Conclusion

Returning to the proof of Theorem 1, we note that A defined in (12) gives rise to a self-adjoint operator L^2 with domain H^1. Moreover, its extension to $H^{1/2}$, which we denote by $A : H^{1/2} \to H^{-1/2}$, generates a C^0-group of isometries, $\{e^{-itA}\}_{t\in\mathbb{R}}$, acting on $H^{1/2}$. Local well-posedness in the sense of Theorem 1 now follows by standard methods for evolution equations with locally Lipschitz nonlinearities. That is, existence and uniqueness of a solution $u \in Y_T$ for the integral equation (13) is deduced by a fixed point argument, for $T > 0$ sufficiently small. The equivalence of the integral formulation (13) and the initial value problem (6), with $u_0 \in H^{1/2}$, as well as the blow-up alternative can also be deduced by standard arguments; see, e. g., [3, 14] for general theory on semilinear evolution equations. Finally, note that $u \in C^1([0,T); H^{-1})$ follows by (6) itself. The proof of Theorem 1 is now accomplished.

3.2 Proof of Theorem 2 (Global Well-posedness)

The first step taken in the proof of Theorem 2 settles conservation of energy and charge that are given by

$$\begin{aligned}E[u] := 1/2 \int_{\mathbb{R}^3} \overline{u}(x)\sqrt{-\Delta+m^2}\,u(x)\,dx +&\\ + 1/4 \int_{\mathbb{R}^3} \left(\frac{\lambda e^{-\mu|x|}}{|x|} * |u|^2\right)(x)\,|u(x)|^2\,dx,&\end{aligned}\tag{21}$$

$$N[u] := \int_{\mathbb{R}^3} |u(x)|^2\,dx,\tag{22}$$

respectively. After deriving the corresponding conservation laws (where proving energy conservation requires a regularization), we discuss how to obtain a-priori bounds on the energy norm of the solution.

Step 1: Conservation Laws

Lemma 2 *The local-in-time solutions of Theorem* 1 *obey conservation of energy and charge, i. e.,*

$$E[u(t)] = E[u_0] \quad \textit{and} \quad N[u(t)] = N[u_0],$$

for all $t \in [0, T)$.

Proof (*Of Lemma 2*) Let u be a local-in-time solution derived in Theorem 1, and let T be its maximal time of existence. Since $u(t) \in H^{1/2}$ holds, we can multiply (6) by $i\bar{u}(t)$ and integrate over $\mathbb{R}^3$. Taking then real parts yields

$$\frac{d}{dt} N[u(t)] = 0 \quad \text{for} \quad t \in [0, T), \tag{23}$$

which shows conservation of charge.

At a formal level, conservation of energy follows by multiplying (6) with $\dot{\bar{u}}(t) \in H^{-1/2}$ and integrating over space, but the paring of two elements of $H^{-1/2}$ is not well-defined. Thus, we have to introduce a regularization procedure as follows; see also, e. g., [2, 8] for other regularization methods for nonlinear (nonrelativistic) Schrödinger equations. Let us define the family of operators

$$\mathcal{M}_\varepsilon := (\varepsilon A + 1)^{-1}, \quad \text{for} \quad \varepsilon > 0, \tag{24}$$

where the operator $A = \sqrt{-\Delta + m^2} \geqslant 0$ is taken from (12). Consider the sequences of embedded spaces

$$\ldots H^{3/2} \hookrightarrow H^{1/2} \hookrightarrow H^{-1/2} \hookrightarrow H^{-3/2} \ldots$$

It is easy to see (by using functional calculus) that the following properties hold.

- (a) For $\varepsilon > 0$ and $s \in \mathbb{R}$, we have that $\mathcal{M}_\varepsilon$ is a bounded map from H^s into H^{s+1}.
- (b) $\|\mathcal{M}_\varepsilon u\|_{H^s} \leqslant \|u\|_{H^s}$ whenever $u \in H^s$ and $s \in \mathbb{R}$.
- (c) For $u \in H^s$ and $s \in \mathbb{R}$, we have that $\mathcal{M}_\varepsilon u \to u$ strongly in H^s as $\varepsilon \downarrow 0$.

We shall use tacitly properties (a)–(c) in the following analysis.

By means of $\mathcal{M}_\varepsilon$ and noting that $E \in C^1(H^{1/2}; \mathbb{R})$, we can compute in a well-defined way for $t_1, t_2 \in [0, T)$ as follows

$$\begin{aligned} E[\mathcal{M}_\varepsilon u(t_2)] - E[\mathcal{M}_\varepsilon u(t_1)] &= \int_{t_1}^{t_2} \langle E'(\mathcal{M}_\varepsilon u), \mathcal{M}_\varepsilon \dot{u} \rangle \, dt \\ &= \int_{t_1}^{t_2} \operatorname{Re} \langle A\mathcal{M}_\varepsilon u + F(\mathcal{M}_\varepsilon u), -i\mathcal{M}_\varepsilon (Au + F(u)) \rangle \, dt \end{aligned}$$

$$
\begin{aligned}
&= \int_{t_1}^{t_2} \operatorname{Im}\Big[\langle A\mathcal{M}_\varepsilon u, \mathcal{M}_\varepsilon A u\rangle + \langle F(\mathcal{M}_\varepsilon u), \mathcal{M}_\varepsilon A u\rangle + \\
&\qquad + \langle A\mathcal{M}_\varepsilon u, \mathcal{M}_\varepsilon F(u)\rangle + \langle F(\mathcal{M}_\varepsilon u), \mathcal{M}_\varepsilon F(u)\rangle\Big]\, dt \\
&=: \int_{t_1}^{t_2} f_\varepsilon(t)\, dt, \qquad (25)
\end{aligned}
$$

where we write $u = u(t)$ for brevity and recall the definition of F from (12). We observe that the first term in $f_\varepsilon(t)$ is the "most singular" part, i. e., if $\varepsilon = 0$ we would have pairing of two $H^{-1/2}$-elements. But for $\varepsilon > 0$ we can use the obvious fact that $\mathcal{M}_\varepsilon A = A\mathcal{M}_\varepsilon$ holds and conclude that

$$
\operatorname{Im}\langle A\mathcal{M}_\varepsilon u, \mathcal{M}_\varepsilon A u\rangle = \operatorname{Im}\langle A\mathcal{M}_\varepsilon u, A\mathcal{M}_\varepsilon u\rangle = 0.
$$

Notice that this manipulation is well-defined, since $A\mathcal{M}_\varepsilon u$ and $\mathcal{M}_\varepsilon A u$ are in $H^{1/2}$ whenever $u \in H^{1/2}$. After some simple calculations, we find $f_\varepsilon(t)$ to be of the form

$$
\begin{aligned}
f_\varepsilon(t) &= \operatorname{Im}\Big[\langle F(\mathcal{M}_\varepsilon u), \mathcal{M}_\varepsilon A u\rangle + \langle A\mathcal{M}_\varepsilon u, \mathcal{M}_\varepsilon F(u)\rangle + \\
&\qquad + \langle F(\mathcal{M}_\varepsilon u), \mathcal{M}_\varepsilon F(u)\rangle\Big] \\
&= \operatorname{Im}\Big[\langle A^{1/2} F(\mathcal{M}_\varepsilon u), A^{1/2}\mathcal{M}_\varepsilon u\rangle + \langle A^{1/2}\mathcal{M}_\varepsilon u, A^{1/2}\mathcal{M}_\varepsilon F(u)\rangle + \\
&\qquad + \langle F(\mathcal{M}_\varepsilon u), \mathcal{M}_\varepsilon F(u)\rangle\Big],
\end{aligned}
$$

Since $\mathcal{M}_\varepsilon u \to u$ strongly in $H^{1/2}$ as $\varepsilon \downarrow 0$, we can infer, by Lemma 1, that

$$
\begin{aligned}
\lim_{\varepsilon\downarrow 0} f_\varepsilon(t) &= \operatorname{Im}\big[\langle A^{1/2}F(u), A^{1/2}u\rangle + \langle A^{1/2}u, A^{1/2}F(u)\rangle + \langle F(u), F(u)\rangle\big] \\
&= \operatorname{Im}(\text{Real Number}) = 0.
\end{aligned}
$$

To interchange the ε-limit with the t-integration in (25), we appeal to the dominated convergence theorem. That is, we seek for a uniform bound on $f_\varepsilon(t)$. In fact, by using the Cauchy–Schwarz inequality and Lemma 1 again we find the following estimate

$$
\begin{aligned}
|f_\varepsilon(t)| &\lesssim |\langle A^{1/2}F(\mathcal{M}_\varepsilon u), A^{1/2}\mathcal{M}_\varepsilon u\rangle| + |\langle A^{1/2}\mathcal{M}_\varepsilon u, A^{1/2}\mathcal{M}_\varepsilon F(u)\rangle| + \\
&\qquad + |\langle F(\mathcal{M}_\varepsilon u), \mathcal{M}_\varepsilon F(u)\rangle| \\
&\lesssim \|A^{1/2}F(\mathcal{M}_\varepsilon u)\|_2 \|A^{1/2}\mathcal{M}_\varepsilon u\|_2 + \\
&\qquad + \|A^{1/2}\mathcal{M}_\varepsilon u\|_2 \|A^{1/2}\mathcal{M}_\varepsilon F(u)\|_2 + \|F(\mathcal{M}_\varepsilon u)\|_2 \|\mathcal{M}_\varepsilon F(u)\|_2 \\
&\lesssim \|u\|_{H^{1/2}}^4 + \|u\|_{H^{1/2}}^6,
\end{aligned}
$$

for all $\varepsilon > 0$. Putting now all together leads to conservation of energy, i. e., we find for all $t_1, t_2 \in [0, T)$ that

$$\begin{aligned} E[u(t_2)] - E[u(t_1)] &= \lim_{\varepsilon \downarrow 0} \big(E[\mathcal{M}_\varepsilon u(t_2)] - E[\mathcal{M}_\varepsilon u(t_1)] \big) \\ &= \lim_{\varepsilon \downarrow 0} \int_{t_1}^{t_2} f_\varepsilon(t)\, dt = \int_{t_1}^{t_2} \lim_{\varepsilon \downarrow 0} f_\varepsilon(t)\, dt = 0. \end{aligned}$$

This completes the proof of Lemma 2. □

Step 2: A Priori Bounds

To fill the last gap towards the global well-posedness result of Theorem 2, we now discuss how to obtain a priori bounds on the energy norm. By the blow-up alternative of Theorem 1, global-in-time existence follows from an a priori bound of the form

$$\|u(t)\|_{H^{1/2}} \leqslant C(u_0). \tag{26}$$

First, let us assume that $\lambda \geqslant 0$ holds. Then, for all $t \in [0, T)$, we find from Lemma 2 and (22) that

$$\|(-\Delta)^{1/4} u(t)\|_2 \lesssim E[u(t)] = E[u_0].$$

This implies together with charge conservation derived in Lemma 2, i. e.,

$$\|u(t)\|_2^2 = N[u(t)] = N[u_0] \tag{27}$$

an a priori estimate (26). Therefore condition (1) in Theorem 2 is sufficient for global existence.

Suppose now a focusing nonlinearity, i. e., $\lambda < 0$ holds, and without loss of generality we assume that $\lambda = -1$ is true (the general case follows by rescaling). Now we can estimate as follows.

$$\begin{aligned} E[u] &= 1/2 \|(-\Delta + m^2)^{1/4} u\|_2^2 - 1/4 \int_{\mathbb{R}^3} \left(\frac{e^{-\mu|x|}}{|x|} * |u|^2 \right)(x)\, |u(x)|^2\, dx \\ &\geqslant 1/2 \|(-\Delta + m^2)^{1/4} u\|_2^2 - 1/4 \int_{\mathbb{R}^3} \left(\frac{1}{|x|} * |u|^2 \right)(x)\, |u(x)|^2\, dx \\ &\geqslant 1/2 \|(-\Delta)^{1/4} u\|_2^2 - \frac{1}{4K} \|(-\Delta)^{1/4} u\|_2^2 \|u\|_2^2 \\ &= \left(1/2 - \frac{1}{4K} \|u\|_2^2 \right) \|(-\Delta)^{1/4} u\|_2^2, \end{aligned} \tag{28}$$

where $K > 0$ is the best constant taken from Appendix B. Thus, energy conservation leads to an a priori bound on the $H^{1/2}$-norm of the solution, if

$$\|u_0\|_2^2 < 2K \tag{29}$$

holds. In fact, the constant K satisfies

$$K = \frac{\|Q\|_2^2}{2} > \frac{2}{\pi},$$

where $Q(x)$ is a strictly positive (ground state) solution of

$$\sqrt{-\Delta}\,Q - \left(\frac{1}{|x|} * |Q|^2\right) Q = -Q; \tag{30}$$

see Appendix B. Going back to (29), we find that

$$\|u_0\|_2^2 < \|Q\|_2^2 \tag{31}$$

is sufficient for global existence for $\lambda = -1$. The assertion of Theorem 2 for all $\lambda < 0$ now follows by simple rescaling. The proof of Theorem 2 is now complete.

3.3 Proof of Theorem 3 (Higher Regularity)

To prove Theorem 3, we need the following generalization of Lemma 1, whose proof is a careful but straightforward generalization of the proof of Lemma 1. We defer the details to Appendix A.1.

Lemma 3 *For $\mu \geqslant 0$ and $s \geqslant 1/2$, the map $J(u) := (\frac{e^{-\mu|x|}}{|x|} * |u|^2)u$ is locally Lipschitz continuous from $H^s(\mathbb{R}^3)$ into itself with*

$$\|J(u) - J(v)\|_{H^s} \lesssim (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|u - v\|_{H^s}$$

for all $u, v \in H^s(\mathbb{R}^3)$. Moreover, we have that

$$\|J(u)\|_{H^s} \lesssim \|u\|_{H^r}^2 \|u\|_{H^s}$$

holds for all $u \in H^s(\mathbb{R}^3)$, where $r = \max\{s - 1, 1/2\}$.

Local well-posedness of (10) in H^s, for $s > 1/2$, can be shown now as follows. We note that $\{e^{-itA}\}_{t\in\mathbb{R}}$, with $A = \sqrt{-\Delta + m^2}$, is a C^0-group of isometries on H^s. Moreover, since the nonlinearity defined in (12), is locally Lipschitz continuous from H^s into itself, local well-posedness in H^s follows similarly as explained in the proof of Theorem 1 for $H^{1/2}$. To show global well-posedness in H^s, we prove by induction and Lemma 3 that an a priori bound on the $H^{1/2}$-norm of solution implies uniform bounds on the H^s-norm on any compact interval $[0, T_*] \subset [0, T)$. This claim follows from (13) and the second inequality stated in Lemma 3 by noting that

$$\begin{aligned}
\|u(t)\|_{H^s} &\leqslant \|e^{-itA}u_0\|_{H^s} + \int_0^t \|e^{-i(t-\tau)A}F(u(\tau))\|_{H^s}\,d\tau \\
&\leqslant \|u_0\|_{H^s} + \int_0^t \|F(u(\tau))\|_{H^s}\,d\tau \\
&\lesssim C_1 + C_2 \int_0^t \|u(\tau)\|_{H^s}\,d\tau,
\end{aligned}$$

holds, provided that $\|u(t)\|_{H^r} \lesssim 1$ for $r = \max\{s-1, 1/2\} < s$. Invoking Gronwall's inequality we conclude that

$$\|u(t)\|_{H^s} \lesssim e^{C_2 T_*}, \qquad \text{for} \quad t \in [0, T_*] \subset [0, T).$$

Induction now implies that an a-priori bound on $\|u(t)\|_{H^{1/2}}$ guarantees uniform bounds $\|u(t)\|_{H^s}$ on any compact interval $I \subset [0, T)$. Thus, the maximal time of existence of an H^s-valued solution coincides with the maximal time of existence when viewed as an $H^{1/2}$-valued solution. Therefore sufficient conditions for global existence for $H^{1/2}$-valued solutions imply global-in-time H^s-valued solutions. This completes the proof of Theorem 3.

3.4 Proof of Theorem 4 (External Potentials)

Let $V = V_+ + V_-$ satisfy Assumption 1 in Section 2. We introduce the quadratic form

$$\mathcal{Q}(u, v) := \langle u, \sqrt{-\Delta + m^2}\, v\rangle + \langle u, V_- v\rangle + \langle u, V_+ v\rangle, \tag{32}$$

which is well-defined on the set (energy space)

$$X := \big\{u \in L^2(\mathbb{R}^3) : \mathcal{Q}(u, u) < \infty\big\}. \tag{33}$$

Note that Assumption 1 also guarantees that $C_0^\infty(\mathbb{R}^3) \subset X$. It easy to show that our assumption on V implies that the quadratic form (32) is bounded from below, i. e., we have $\mathcal{Q}(u, u) \geqslant -M\langle u, u\rangle$ holds for all $u \in X$ and some constant $M \geqslant 0$. By the semi-boundedness of $\mathcal{Q}$, we can assume from now on (and without loss of generality) that

$$\mathcal{Q}(u, u) \geqslant 0 \tag{34}$$

holds for all $u \in X$. Since $\mathcal{Q}(\cdot, \cdot)$ is closed (it is a sum of closed forms), the energy space X equipped with its norm

$$\|u\|_X := \sqrt{\langle u, u\rangle + \mathcal{Q}(u, u)} \tag{35}$$

is complete, and we have the equivalence

$$\|u\|_{H^{1/2}} + \|V_+^{1/2} u\|_2 \lesssim \|u\|_X \lesssim \|u\|_{H^{1/2}} + \|V_+^{1/2} u\|_2. \tag{36}$$

Furthermore, there exists a nonnegative, self-adjoint operator

$$A : D(A) \subset L^2 \to L^2 \tag{37}$$

with $X = D(A^{1/2})$, such that

$$\langle u, Av\rangle = \mathcal{Q}(u, v) \tag{38}$$

holds for all $u \in X$ and $v \in D(A)$; see, e. g., [10]. This operator can be extended to a bounded operator, still denoted by $A : X \to X^*$, where X^* is the dual space of X.

To prove now the assertion about local well-posedness in Theorem 4, we have to generalize Lemma 1 to the following statement.

Lemma 4 *Suppose $\mu \geqslant 0$ and let V satisfy Assumption* 1. *Then the map $J(u) := (\frac{e^{-\mu|x|}}{|x|} * |u|^2)u$ is locally Lipschitz continuous from X into itself with*

$$\|J(u) - J(v)\|_X \lesssim (\|u\|_X^2 + \|v\|_X^2)\|u - v\|_X$$

for all $u, v \in X$.

Proof (*Of Lemma 4*) By (36), it suffices to estimate $\|J(u) - J(v)\|_{H^{1/2}}$ and $\|V_+^{1/2}[J(u) - J(v)]\|_2$ separately. By Lemma 1, we know that

$$\begin{aligned}\|J(u) - J(v)\|_{H^{1/2}} &\lesssim (\|u\|_{H^{1/2}}^2 + \|v\|_{H^{1/2}}^2)\|u - v\|_{H^{1/2}}\\ &\lesssim (\|u\|_X^2 + \|v\|_X^2)\|u - v\|_X.\end{aligned}$$

It remains to estimate $\|V_+^{1/2}[J(u) - J(v)]\|_2$, which can be achieved by recalling (20) and proceeding as follows.

$$\begin{aligned}\|V_+^{1/2}[J(u) - J(v)]\|_2 &\lesssim \left\| V_+^{1/2}\left[\left(\frac{e^{-\mu|x|}}{x} * (|u|^2 - |v|^2)\right)(u+v)\right]\right\|_2 +\\ &\quad + \left\| V_+^{1/2}\left[\left(\frac{e^{-\mu|x|}}{x} * (|u|^2 + |v|^2)\right)(u-v)\right]\right\|_2\\ &\lesssim \left\| \frac{e^{-\mu|x|}}{|x|} * (|u|^2 - |v|^2)\right\|_\infty \|V_+^{1/2}(u+v)\|_2 +\\ &\quad + \left\| \frac{e^{-\mu|x|}}{|x|} * (|u|^2 + |v|^2)\right\|_\infty \|V_+^{1/2}(u-v)\|_2\\ &\lesssim \|u+v\|_{H^{1/2}}\|u-v\|_{H^{1/2}}\|V_+^{1/2}(u+v)\|_2 +\\ &\quad + (\|u\|_{H^{1/2}}^2 + \|v\|_{H^{1/2}}^2)\|V_+^{1/2}(u-v)\|_2\\ &\lesssim \big(\|u\|_X^2 + \|v\|_X^2)\|u-v\|_X.\end{aligned}$$

This completes the proof of Lemma 4. □

Returning to the proof of Theorem 4, we simply note that $\{e^{-itA}\}_{t\in\mathbb{R}}$ is a C^0-group of isometries on X, where $A = \sqrt{-\Delta + m^2} + V$ is defined in the form sense (see above). By Lemma 4, the nonlinearity is locally Lipschitz on X. Thus, local well-posedness now follows in the same way as for Theorem 1.

To establish global well-posedness we have to prove conservation of charge, $N[u]$, and energy, $E[u]$, which is for (10) given by

$$\begin{aligned}E[u] := 1/2\int_{\mathbb{R}^3} \overline{u}(x)\sqrt{-\Delta + m^2}\,u(x)\,dx + 1/2\int_{\mathbb{R}^3} V(x)|u(x)|^2\,dx +\\ +\ 1/4\int_{\mathbb{R}^3}\left(\frac{\lambda e^{-\mu|x|}}{|x|} * |u|^2\right)(x)\,|u(x)|^2\,dx.\end{aligned} \tag{39}$$

As done in Section 3.2, we have to employ a regularization method using the class of operators

$$\mathcal{M}_\varepsilon := (\varepsilon A + 1)^{-1}, \qquad \text{for} \quad \varepsilon > 0, \tag{40}$$

where we assume without loss of generality that $A \geqslant 0$ holds. The mapping $\mathcal{M}_\varepsilon$ acts on the sequence of embedded spaces

$$\ldots X^{+2} \hookrightarrow X^{+1} \hookrightarrow X^{-1} \hookrightarrow X^{-2} \ldots, \tag{41}$$

with corresponding norms given by $\|u\|_{X^s} := \|(1+A)^{s/2}u\|_2$. Note that $X = X^{+1}$ (with equivalent norms) and that its dual space obeys $X^* = X^{-1}$. By using functional calculus, it is easy to show that $\mathcal{M}_\varepsilon$ exhibits properties that are analog to (a)–(c) in Section 3.2.

The rest of the argument for proving conservation of energy carries over from Section 3.2 without major modifications. Finally, we mention that deriving a priori bounds on $\|u(t)\|_X$ leads to a similar discussion as presented in Section 3.2, while noting that we have to take care that V_- has a relative $(-\Delta)^{1/2}$-form bound, $0 \leqslant a < 1$, introduced in Assumption 1. This completes the proof of Theorem 4.

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Appendix A: Fractional Calculus

The following result (generalized Leibniz rule) is proved in [9] for Riesz and Bessel potentials of order $s \in \mathbb{R}$, which are denoted by $(-\Delta)^{s/2}$ and $(1-\Delta)^{s/2}$, respectively. But as a direct consequence of the Milhin multiplier theorem [1], the cited result holds for $\mathcal{D}^s := (\mu^2 - \Delta)^{s/2}$, where $\mu \geqslant 0$ is a fixed constant.

Lemma 5 (Generalized Leibniz Rule) *Suppose that* $1 < p < \infty$, $s \geqslant 0$, $\alpha \geqslant 0$, $\beta \geqslant 0$, *and* $1/p_i + 1/q_i = 1/p$ *with* $i = 1, 2$, $1 < q_i \leqslant \infty$, $1 < p_i \leqslant \infty$. *Then*

$$\|\mathcal{D}^s(fg)\|_p \leqslant c(\|\mathcal{D}^{s+\alpha} f\|_{p_1}\|\mathcal{D}^{-\alpha} g\|_{q_1} + \|\mathcal{D}^{-\beta} f\|_{p_2}\|\mathcal{D}^{s+\beta} g\|_{q_2}),$$

where the constant c depends on all of the parameters but not on f and g.

A second fact we use in the proof of our main result is as follows. For $0 < \alpha < 3$ and $\mu \geqslant 0$, the potential operator $\mathcal{D}^{-\alpha} = (\mu^2 - \Delta)^{-\alpha/2}$ corresponds to $f \mapsto G^\mu_\alpha * f$, with $f \in \mathcal{S}(\mathbb{R}^3)$, and we have that

$$G^\mu_\alpha \in L^{3/(3-\alpha)}_w(\mathbb{R}^3). \tag{42}$$

To see this, we refer to the inequality and the exact formula

$$0 \leqslant G^\mu_\alpha(x) \leqslant G^0_\alpha(x) = \frac{c_\alpha}{|x|^{3-\alpha}}, \qquad \text{for} \quad \mu \geqslant 0 \quad \text{and} \quad 0 < \alpha < 3, \tag{43}$$

with some constant c_α; these facts can be derived from [15, Section V.3.1]. Now (42) follows from $|x|^{-\sigma} \in L^{3/\sigma}_w(\mathbb{R}^3)$ whenever $0 < \sigma < 3$. Another observation used in Section 2 is the well-known explicit formula

$$G^\mu_2(x) = \frac{e^{-\mu|x|}}{4\pi|x|}. \tag{44}$$

That is, $(\mu^2 - \Delta)$ in $\mathbb{R}^3$ has the Green's function $\frac{e^{-\mu|x|}}{4\pi|x|}$ with vanishing boundary conditions.

A.1 Proof of Lemma 3

Proof (*Of Lemma 3*) We only show the second inequality derived in Lemma 3, since the first one can be proved in a similar way.

Let $\mu \geqslant 0$ and $s \geqslant 1/2$. We put $\mathcal{D}^\alpha := (\mu^2 - \Delta)^{\alpha/2}$ for $\alpha \in \mathbb{R}$. By the generalized Leibniz rule and (17), we have that

$$\begin{aligned}\|\mathcal{D}^s J(u)\|_2 &\lesssim \|\mathcal{D}^s[(\mathcal{D}^{-2}|u|^2)u]\|_2 \\ &\lesssim \|\mathcal{D}^{s-2}|u|^2\|_{p_1}\|u\|_{q_1} + \|\mathcal{D}^{-2}|u|^2\|_\infty \|\mathcal{D}^s u\|_2 \\ &\lesssim \|\mathcal{D}^{s-2}|u|^2\|_{p_1}\|u\|_{q_1} + \|u\|^2_{H^{1/2}}\|u\|_{H^s}, \end{aligned} \tag{45}$$

where $1/p_1 + 1/q_1 = 1/2$ with $1 < p_1, q_1 \leqslant \infty$. The first term of the right-hand side of (45) can be controlled as follows, where we introduce $r = \max\{s-1, 1/2\}$.

(1) For $1/2 \leqslant s < 3/2$, we choose $p_1 = 3/s$ and $q_1 = 6/(3-2s)$ which leads to

$$\begin{aligned}\|\mathcal{D}^{s-2}|u|^2\|_{3/s}\|u\|_{6/(3-2s)} &\lesssim \|G^\mu_{2-s}\|_{3/(1+s),w}\| |u|^2\|_{3/2}\|u\|_{H^s} \\ &\lesssim \|u\|^2_{H^{1/2}}\|u\|_{H^s} \lesssim \|u\|^2_{H^r}\|u\|_{H^s},\end{aligned}$$

where we use the weak Young inequality, as well as Sobolev's inequality $\|u\|_{6/(3-2s)} \lesssim \|u\|_{H^s}$ in $\mathbb{R}^3$, and (42) once again.

(2) For $s \geqslant 3/2$, we choose $p_1 = 6$ and $q_1 = 3$. This yields

$$\begin{aligned}\|\mathcal{D}^{s-2}|u|^2\|_6\|u\|_3 &\lesssim \|\mathcal{D}^{s-1}|u|^2\|_2\|u\|_3 \lesssim \|\mathcal{D}^{s-1}u\|_6\|u\|_3^2 \\ &\lesssim \|\mathcal{D}^s u\|_2\|u\|_3^2 \lesssim \|u\|_{H^s}\|u\|^2_{H^r},\end{aligned}$$

while using twice Sobolev's inequality $\|f\|_6 \lesssim \|\mathcal{D}f\|_2$ in $\mathbb{R}^3$.

Putting now all together, we conclude that

$$\begin{aligned}\|J(u)\|_{H^s} &\lesssim \|J(u)\|_2 + \|\mathcal{D}^s J(u)\|_2 \\ &\lesssim \|u\|^2_{H^{1/2}}\|u\|_2 + \|u\|^2_{H^r}\|u\|_{H^s} \lesssim \|u\|^2_{H^r}\|u\|_{H^s}.\end{aligned}$$

□

Appendix B: Ground States

We consider the functional (see also [12])

$$K[u] := \frac{\|(-\Delta)^{1/4}u\|_2^2\|u\|_2^2}{\int_{\mathbb{R}^3}(|x|^{-1} * |u|^2)(x)\,|u(x)|^2\,dx}, \tag{46}$$

which is well-defined for all $u \in H^{1/2}$ with $u \not\equiv 0$. Note that by using (17) we can estimate the denominator in $K[u]$ as follows.

$$\int_{\mathbb{R}^3} \left(\frac{1}{|x|} * |u|^2 \right)(x)\, |u(x)|^2 \, dx \leqslant \left\| \frac{1}{|x|} * |u|^2 \right\|_\infty \|u\|_2^2 \leqslant \frac{\pi}{2} \|(-\Delta)^{1/4} u\|_2^2 \|u\|_2^2, \tag{47}$$

which leads to the bound

$$\frac{2}{\pi} \leqslant K[u] < \infty. \tag{48}$$

Indeed, we will see that the estimate from below is a strict inequality. With respect to the related variational problem

$$K := \inf \left\{ K[u] : u \in H^{1/2}(\mathbb{R}^3),\ u \not\equiv 0 \right\} \tag{49}$$

we can state the following result.

Lemma 6 (Ground States) *There exists a minimizer, $Q \in H^{1/2}(\mathbb{R}^3)$, for (49), and we have the following properties.*

(1) *$Q(x)$ is a smooth function that can be chosen to be real-valued, strictly positive, and spherically symmetric with respect to the origin. It satisfies*

$$\sqrt{-\Delta}\, Q - \left(\frac{1}{|x|} * |Q|^2 \right) Q = -Q, \tag{50}$$

and it is nonincreasing, i. e., we have that $Q(x) \geqslant Q(y)$ whenever $|x| \leqslant |y|$.

(2) *The infimum satisfies $K = \|Q\|_2^2/2$ and $K > 2/\pi$.*

Proof (*Sketch of Proof*) We present the main ideas for the proof of the preceding lemma. That (49) is attained at some real-valued, radial, nonnegative and nonincreasing function $Q(x) \geqslant 0$ can be proved by direct methods of variational calculus and rearrangement inequalities; see also [16] for a similar variational problem for nonrelativistic Schrödinger equations with local nonlinearities. Furthermore, any minimizer, $Q \in H^{1/2}$, has to satisfy the corresponding Euler–Lagrange equation that reads

$$\sqrt{-\Delta}\, Q - \left(\frac{\lambda}{|x|} * |Q|^2 \right) Q = -Q, \tag{51}$$

after a suitable rescaling $Q(x) \mapsto aQ(bx)$ with some $a, b > 0$.

Let us make some comments about the properties of Q. Using an bootstrap argument and Lemma 3 for the nonlinearity, it follows that Q belongs to H^s, for all $s \geqslant 1/2$. Hence it is a smooth function. To see that $Q(x) \geqslant 0$ is strictly positive, i. e., $Q(x) > 0$, we rewrite (51) such that

$$Q = \left(\sqrt{-\Delta} + 1\right)^{-1} W, \tag{52}$$

where $W := (|x|^{-1} * |Q|^2) Q$. By functional calculus, we have that

$$\left(\sqrt{-\Delta} + 1\right)^{-1} = \int_0^\infty e^{-t} e^{-t\sqrt{-\Delta}} \, dt. \tag{53}$$

Next, we notice by the explicit formula for the kernel (in $\mathbb{R}^3$)

$$e^{-t\sqrt{-\Delta}}(x,y) = \mathcal{F}^{-1}\big(e^{-t|\xi|}\big)(x-y) = C \cdot \frac{t}{[t^2 + |x-y|^2]^2},$$

with some constant $C > 0$; see, e.g., [11]. This explicit formula shows that $e^{-t\sqrt{-\Delta}}$ is positivity improving. This means that if $f \geqslant 0$ with $f \not\equiv 0$ then $e^{-t\sqrt{-\Delta}} f > 0$ almost everywhere. Hence $(\sqrt{-\Delta}+1)^{-1}$ is also positivity improving, by (53), and we conclude that $Q(x) > 0$ holds almost everywhere, thanks to (52) and $W \geqslant 0$. Moreover, we know that $Q(x)$ is a nonincreasing, continuous function. Therefore $Q(x) > 0$ holds in the strong sense, i.e., for every $x \in \mathbb{R}^3$.

Finally, to see that (2) holds, we consider the variational problem

$$I_N := \inf\big\{E[u] : u \in H^{1/2}(\mathbb{R}^3),\ \|u\|_2^2 = N\big\}, \tag{54}$$

where $N > 0$ is a given parameter and

$$E[u] = 1/2\|(-\Delta)^{1/4}u\|_2^2 - 1/4 \int_{\mathbb{R}^3} \left(\frac{1}{|x|} * |u|^2\right) |u(x)|^2\, dx.$$

Due to the scaling behavior $E[\alpha^{3/2}u(\alpha\cdot)] = \alpha E[u]$, we have that either $I_N = 0$ or $I_N = -\infty$ holds. By noting that

$$E[u] \geqslant \left(\frac{1}{2} - \frac{N}{4K}\right)\|(-\Delta)^{1/4}u\|_2^2,$$

and the fact that equality holds if and only if u minimizes $K[u]$, we find that $I_N = 0$ holds if and only if $N \leqslant N_c := 2K$. Moreover, $I_N = 0$ is attained if and only if $N = N_c$. Let $\tilde{Q}$ be such a minimizer with $\|\tilde{Q}\|_2^2 = N_c$. Thanks to the proof of part (1), we can assume without loss of generality that $\tilde{Q}$ is real-valued, radial, and strictly positive. Calculating the Euler–Lagrange equation for (54), with $N = N_c$, yields

$$\sqrt{-\Delta}\,\tilde{Q} - \left(\frac{1}{|x|} * |\tilde{Q}|^2\right)\tilde{Q} = -\theta\,\tilde{Q},$$

for some multiplier θ, where it is easy to show that $\theta > 0$ holds. Setting now $Q(x) = \theta^{-3/2}\tilde{Q}(\theta^{-1}x)$, which conserves the L^2-norm, leads to a ground state $Q(x)$ satisfying (51). Thus, we have that

$$K = \|\tilde{Q}\|_2^2/2 = \|Q\|_2^2/2.$$

To prove that $K > 2/\pi$ holds, let us assume $K = 2/\pi$. This implies that the first inequality in (47) is an equality for $u(x) = Q(x) > 0$. But this leads to $(|x|^{-1} * |Q|^2)(x) = \text{const.}$, which is impossible.

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