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Effective dynamics for boson stars

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Abstract

We study solutions close to solitary waves of the pseudo-relativistic Hartree equation describing boson stars under the influence of an external gravitational field. In particular, we analyse the long-time *effective dynamics* of such solutions. In essence, we establish a (long-time) stability result for solutions describing boson stars that move under the influence of an external gravitational field. The proof of our main result tackles difficulties that are absent when deriving similar results on effective solitary wave motions for nonlinear Schrödinger equations or nonlinear wave equations. This is due to the fact that the pseudo-relativistic Hartree equation does not exhibit Galilean or Lorentz covariance.

Mathematics Subject Classification: 35Q55; 37K40; 81Q05

1. Introduction

In this paper, we study boson stars described as solutions of the pseudo-relativistic Hartree equation which, initially, are close to a solitary wave. The *pseudo-relativistic Hartree equation* is the nonlinear evolution equation

$$i\partial_t\psi = (\sqrt{-\Delta + m^2} - m)\psi + V\psi - \left(\frac{1}{|x|} * |\psi|^2\right)\psi, \quad (1.1)$$

where $\psi = \psi(x, t)$ is a complex wave field on space-time, $x \in \mathbb{R}^3$ is a point in space and $t \in \mathbb{R}$ is time. Here the symbol $*$ denotes spatial convolution. The kinetic energy operator $\sqrt{-\Delta + m^2} - m$ is appropriate for describing relativistic quantum particles of mass $m > 0$. This operator is defined by its symbol $\sqrt{k^2 + m^2} - m$ in momentum space. The convolution

kernel, $|x|^{-1}$, represents the Newtonian potential of gravitational 2-body interactions. We use units such that the speed of light and Planck's constant are equal to unity. By rescaling ψ we may set Newton's gravitational constant times m^2 equal to unity.

Equation (1.1) describes a system of gravitating bosonic particles in a regime where effects of special relativity are important, because the particles have velocities close to the speed of light, but retardation effects and space-time curvature can be neglected. As recently shown in [1], equation (1.1) emerges as the correct evolution equation for the mean-field dynamics of many-body quantum systems modelling pseudo-relativistic boson stars. The external potential, $V = V(x)$, accounts for gravitational fields from other stars. In what follows, we will assume that V is a smooth, bounded, slowly varying real-valued function; the precise assumptions are stated in section 3.

Equation (1.1) admits some important conserved quantities, namely, the *mass* of the system (proportional to the number of particles) and its *energy*. These quantities are given by⁴

$$\mathcal{N}(\psi) := \frac{1}{2} \int_{\mathbb{R}^3} |\psi|^2 dx, \quad (1.2)$$

and

$$\mathcal{H}_V(\psi) := \frac{1}{2} \int_{\mathbb{R}^3} (|(-\Delta + m^2)^{1/4} \psi|^2 - m|\psi|^2 + V|\psi|^2) dx - \frac{1}{4} \int_{\mathbb{R}^3} \left(\frac{1}{|x|} * |\psi|^2 \right) |\psi|^2 dx, \quad (1.3)$$

respectively. The momentum, \mathcal{P} , also plays an important role. It is defined by

$$\mathcal{P}(\psi) := -\frac{i}{2} \int_{\mathbb{R}^3} \bar{\psi} \nabla \psi dx \quad (1.4)$$

and is conserved when the external potential V is constant.

Before stating the main result derived in this paper, we briefly summarize known results for equation (1.1) as follows.

- **Well-posedness.** The initial-value problem for the pseudo-relativistic Hartree equation (1.1) is known to be locally well-posed in the Sobolev space $H^{\frac{1}{2}}(\mathbb{R}^3)$, provided $V \in L^3(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ holds. Moreover, it was shown in [3] that initial data $\psi_0 \in H^{\frac{1}{2}}(\mathbb{R}^3)$ such that

$$\mathcal{N}(\psi_0) < N_c, \quad (1.5)$$

where $N_c > 2/\pi$ is some universal constant, give rise to global-in-time solutions of (1.1),

$$\psi \in C^0([0, \infty); H^{\frac{1}{2}}(\mathbb{R}^3)) \cap C^1([0, \infty); H^{-\frac{1}{2}}(\mathbb{R}^3)). \quad (1.6)$$

We refer to [3] for a detailed study of the initial-value problem for (1.1); see also [4] for local well-posedness in H^s with $s < 1/2$.

- **Finite-time blow-up.** In [5] it was proven that finite-time blow-up occurs for solutions of (1.1) whose initial data ψ_0 have sufficiently negative energy $\mathcal{H}_V(\psi_0)$, where $\psi_0 = \psi_0(|x|)$ and $V = V(|x|)$ are both assumed to be radial functions. This blow-up result also shows that the smallness condition (1.5) is almost optimal. In physical terms, this type of singularity formation for solutions of (1.1) is of considerable interest, since it indicates the '*gravitational collapse*' of boson stars whose total mass exceeds some critical value, as represented by the universal constant $N_c > 2/\pi$.

⁴ Note that we used the definition $\mathcal{N}(\psi) = \int |\psi|^2 dx$ in [2].

- **Solitary waves when $V \equiv 0$.** A remarkable feature of equation (1.1), with vanishing external potential $V \equiv 0$, is that it admits travelling *solitary wave solutions* given by

$$\psi(x, t) = e^{it\mu} \varphi_{v,\mu}(x - vt). \tag{1.7}$$

Here $\varphi_{v,\mu} \in H^{\frac{1}{2}}(\mathbb{R}^3)$, $\varphi_{v,\mu} \not\equiv 0$, $\mu \in \mathbb{R}$ and the parameter $v \in \mathbb{R}^3$ corresponds to the travelling velocity. The ansatz (1.7) implies that $\varphi_{v,\mu}$ has to satisfy the nonlinear equation

$$(\sqrt{-\Delta + m^2} - m)\varphi + \mu\varphi + iv \cdot \nabla\varphi - \left(\frac{1}{|x|} * |\varphi|^2\right)\varphi = 0. \tag{1.8}$$

In our companion paper [2], we address the existence and fundamental properties of travelling solitary wave solutions for (1.1) with $V \equiv 0$ and strictly positive mass parameter $m > 0$. More precisely, we consider the following minimization problem

$$\mathcal{E}_{v,0}(\psi) := \mathcal{H}_{V \equiv 0}(\psi) - v \cdot \mathcal{P}(\psi) \rightarrow \min \tag{1.9}$$

subject to the constraint $\mathcal{N}(\psi) = N$. Correspondingly, we refer to minimizers of (1.9) as *boosted ground states* in what follows, and equation (1.8) then arises as an Euler–Lagrange equation for (1.9) with multiplier $-\mu$. The results derived in [2] can be summarized as follows.

- (a) *Existence.* Boosted ground states $\varphi_{v,\mu}$ exist if and only if their velocity v and mass $\mathcal{N}(\varphi_{v,\mu})$ satisfy

$$|v| < 1 \quad \text{and} \quad 0 < \mathcal{N}(\varphi_{v,\mu}) < N_c(v), \tag{1.10}$$

where the constant $N_c(v)$ obeys the bounds $(1 - |v|)N_c \leq N_c(v) \leq N_c(0) \equiv N_c$, with $N_c > 2/\pi$ the same constant as in (1.5). (Note that $|v| < 1$ corresponds to velocities below the speed of light in our units.)

- (b) *Exponential decay.* Any boosted ground state $\varphi_{v,\mu}$ decays exponentially, i.e. we have that

$$|\varphi_{v,\mu}(x)| \leq C e^{-\delta|x|}, \tag{1.11}$$

for some constants $C = C(\varphi_{v,\mu}) > 0$ and $\delta = \delta(\mu, v, m) > 0$.

- (c) *Orbital stability.* Travelling solitary waves (1.8), which arise from boosted ground states $\varphi_{v,\mu}$, are *orbitally stable*⁵. (This follows from their variational characterization; also see [6] for a related result on the orbital stability of ground state solitary waves for nonlinear Schrödinger equations (NLS).)

We remark that equation (1.1) also admits travelling solitary wave solutions when $V \equiv 0$ and $m = 0$. But in this limiting case of vanishing mass parameter, ground state solitary waves exhibit instability due to nearby blow-up solutions; see [5]. In this paper, however, we always assume that $m > 0$ holds in equation (1.1). Also observe that the obtained ground state is not known to be unique modulo phase and translation.

1.1. Setup of problem and sketch of main result

The main goal of the present paper is to extend the stability result for ground state solitary waves mentioned above. More precisely, we provide a detailed long-time description of solutions of (1.1), initially close to a boosted ground state, in the presence of an external potential $V \not\equiv 0$. In particular, the solutions we consider have initial data that satisfy condition (1.5) and hence exist for all time.

⁵ Orbital stability: if an initial condition is close in the $H^{\frac{1}{2}}$ -norm to any ground state then so is the solution at all later times.

In this study, two length scales will play a crucial role. The first one is determined by the external potential and is given by

$$\ell_{\text{exp}} := \|\nabla V\|_{\infty}^{-1}. \tag{1.12}$$

The second length scale is inferred from the requirement that the initial condition, ψ_0 , of (1.1) be close to a ground state, $\varphi_{v,\mu}$. We will project, see section 6, such an initial condition to a point on a manifold of boosted ground states. All ground states are exponentially localized with rate $\delta = \delta(\mu, v)$. The projection singles out one ground state, with an associated length scale given by its exponential decay rate. Thus, the second length scale is defined by

$$\ell_{\text{sol}} := \delta^{-1}. \tag{1.13}$$

In the regime where

$$\varepsilon := \frac{\ell_{\text{sol}}}{\ell_{\text{exp}}} \ll 1, \tag{1.14}$$

we expect that solutions of (1.1) with initial condition close to a ground state $\varphi_{v,\mu}$ behave like relativistic point particles.

We now sketch our main result derived in this paper. Let $(y^{(0)}, v^{(0)}, \vartheta^{(0)}, \mu^{(0)})$ be a point in $\mathbb{R}^3 \times \mathbb{R}^3 \times [0, 2\pi) \times \mathbb{R}_+$, with $|v^{(0)}| \leq r < 1$ for some small $r > 0$ and $\mu^{(0)} \in I \subset \mathbb{R}_+$, where I is some open interval. We consider an initial condition, $\psi_0 \in \tilde{\Gamma}$, such that

$$\|\psi_0 - e^{i\vartheta^{(0)}} \varphi_{v^{(0)}, \mu^{(0)}}(\cdot - y^{(0)})\|_{\tilde{\Gamma}} \leq \varepsilon, \tag{1.15}$$

where $\tilde{\Gamma} \subset H^{\frac{1}{2}}$ is a weighted Sobolev space with norm $\|\cdot\|_{\tilde{\Gamma}}$. We then show that the solution of (1.1) with initial condition ψ_0 remains close to a ground state, for times of order ε^{-1} . More explicitly, we prove that

$$\psi(x, t) = e^{i\vartheta(t)} (\varphi_{v(t), \mu(t)}(x - y(t)) + \xi(x - y(t), t)), \tag{1.16}$$

with $\|\xi\|_{\tilde{\Gamma}} \leq C\varepsilon$, for times $0 \leq t \leq c\varepsilon^{-1}$. Here the constants $c > 0, C > 0$ do not depend on ε , and the time-dependent functions (y, v, ϑ, μ) satisfy the *equations of motion*,

$$\dot{y} = v + \mathcal{O}(\varepsilon^2), \quad \gamma(v, \mu)\dot{v} = -\nabla V(y) + \mathcal{O}(\varepsilon^2), \tag{1.17}$$

where the factor $\gamma(v, \mu)$ is a relativistic correction, and

$$\frac{d}{dt} \mathcal{N}(\varphi_{v,\mu}) = \mathcal{O}(\varepsilon^2), \quad \dot{\vartheta} = \mu - V(y) + \mathcal{O}(\varepsilon^2). \tag{1.18}$$

These results yield a fairly detailed description of the solution, $\psi(x, t)$, up to times of order ε^{-1} . For a precise statement of our main result, see theorem 3.1 and its hypotheses in section 3.

1.2. Relation to other work

Work on boson stars. One of the first studies of self-gravitating scalar bosons can be found in [7]. Important properties of a Hamiltonian describing a relativistic quantum particle in an external potential proportional to $|x|^{-1}$ are proven in [8]; see also [9]. Bosonic matter is analysed in [10], and bosonic black holes are discussed in [11]. In these papers, it is argued that the Chandrasekhar limit for boson stars (with $m \approx 1\text{--}50$ GeV) is approximately the mass of a mountain. Moreover, the intuitive scaling ideas used in [10, 11] are turned into rigorous statements in [12]. There, it is conjectured that the ground state energy of n bosonic particles with relativistic kinetic energy is given by the minimum of the pseudo-relativistic Hartree energy functional, $\mathcal{H}_{V=0}$, in the ‘mean-field limit’. This has subsequently been shown in [13], where it is also proven that there exist minimizers, φ_μ , for $\mathcal{H}_{V=0}(\psi)$ subject to the constraint $\mathcal{N}(\psi) = N < N_c$, and that these minimizers can be chosen to be spherically symmetric.

The constant N_c satisfies the bounds $2/\pi < N_c < 1.4$ and is interpreted as the critical mass for the stability of a boson star (bosonic Chandrasekhar limit mass).

A recent review paper on the mean-field limit of quantum Bose gases is paper [14], which contains rigorous results and a survey of open problems for Bose gases. The mean-field limit for systems of gravitating relativistic bosons is discussed in [1]. The recent history of boson stars is described in more detail above, in the first part of the introduction.

The stability of solitary waves. For infinite-dimensional Hamiltonian PDEs, the stability of solitary waves has been extensively studied during the last few decades. For the reader's orientation, we now put our main result into the context of this vast subject, and we try to emphasize the specific difficulties encountered for the pseudo-relativistic Hartree equation (1.1).

For NLS without external potentials, the stability analysis of solitary waves dates back to [6] the beginning of the 1980s, where orbital (or nonlinear) stability of ground state solitary waves was proven by using variational methods and conservation laws. Later on, a more refined stability result for NLS solitary waves was derived by [15, 16] in terms of so-called modulational stability, showing that (in the absence of external potentials) small perturbations of ground state solitary waves stay close (in energy space) to a ground state solitary wave that moves with constant velocity. The proof given in [15] makes extensive use of the linearization around solitary waves, combined with a Lyapunov-type argument. We also refer to [17] for modulational stability in the context of nonlinear wave equations (NLW).

With regard to stability of multi-solitary wave configurations for NLS without external potentials, we refer to the recent work [18]; also see [19, 20] for the delicate issue of asymptotic stability of such configurations.

For NLS with non-vanishing external potentials, long-time effective dynamics for solitary waves of NLS were derived in [21–24], partly based on the ideas developed for the stability analysis for NLS without external potentials; also see [25, 26] for the Korteweg–de Vries equation (KdV). The basic strategy of these works exploits the Hamiltonian structure of these nonlinear PDE and their symplectic structure.

When adapting the methods used in [23] to the pseudo-relativistic Hartree equation, we encounter, however, several difficulties that are completely absent in the case of NLS, KdV or NLW. In particular, equation (1.1) lacks Galilean (or Lorentz) covariance. The lack of such a symmetry, which is extensively used in the works cited above, causes both technical and conceptual problems. For instance, we are led to a set of modulation equations whose complexity cannot be reduced by appealing to symmetries of the equation under consideration. Also, our analysis involves weighted Sobolev norms and the non-local nature of the pseudo-differential operator $\sqrt{-\Delta + m^2}$ leads to some technical difficulties to be dealt with.

A further problem in our analysis is due to the fact that boosted ground states $\varphi_{v,\mu}$ (i.e. minimizers for (1.9) subject to fixed L^2 -norm) are *not known to be unique* up to phase and translation. In addition, symmetry properties of the function $\varphi_{v,\mu}$ are *a priori* not at our disposal, except when $v = 0$ holds. In this particular case, the ground state $\varphi_{v=0,\mu}$ is known to be radial with respect to some point, owing to rearrangement inequalities; see [13]. To overcome the difficulties mentioned above, we single out a ground state at rest $\varphi_{v=0,\mu}$ and impose a non-degeneracy condition on the linearization around $\varphi_{v=0,\mu}$. Then, by an implicit-function-type argument, we obtain solutions $\varphi_{v,\mu}$ of equation (1.8) for $|v| \ll 1$, which are proven to exhibit certain symmetry properties (such as cylindrical symmetry) that become essential in our analysis. For example, these symmetries guarantee the non-degeneracy of the symplectic form reduced to a manifold of travelling solitary waves.

Finally, we point out that the linearization around solitary waves for (1.1) leads to a matrix operator whose entries are given by pseudo-differential operators; see section 2. In consequence, the analytic study of these operators is even harder than in the context of NLS, where matrix Schrödinger operators occur. There powerful methods such as Sturm–Liouville theory can be used in order to show that certain properties indeed hold true (that no extra degeneracy of the kernel follows and that the ground state is unique). We refer to [27] for a comprehensive study of linearized operators for NLS solitary waves.

1.3. Organization of the paper

This paper is organized as follows. In section 2, we rephrase equation (1.1) as a Hamiltonian evolution equation and discuss its Hamiltonian structure. We also state a fundamental assumption concerning the linearization around ground state solitary waves at rest. In section 3, we state our main theorem and sketch its proof. Sections 4–10 contain numerous auxiliary results used in the proof of our main theorem. The main theorem is proven in section 11. The appendices contain proofs of spectral properties and positivity of a certain linear operator.

Notation. The space of measurable functions, $f : \mathbb{R}^3 \rightarrow \mathbb{C}$, such that $|f|^p$ is integrable, is denoted by L^p and its norm by $\|\cdot\|_p$. For $p = 2$, this space is the Hilbert space of square-integrable functions. The space of n times continuously differentiable functions is denoted by C^n . The usual inhomogeneous Sobolev space is denoted by H^s , with its norm

$$\|u\|_{H^s} := \|(1 - \Delta)^{s/2}u\|_2, \quad (1.19)$$

where $s \in \mathbb{R}$. Moreover, we define the weighted norm $\|\cdot\|_{\Gamma}$ by

$$\|u\|_{\Gamma}^2 := \|u\|_{H^{\frac{1}{2}}}^2 + \varepsilon \| |x|^{1/2}u \|_2^2, \quad (1.20)$$

for $\varepsilon > 0$ given as in (1.14). We also use the notation $d_t := \frac{d}{dt}$.

2. The Hamiltonian nature of equation (1.1)

Equation (1.1) is a Hamiltonian evolution equation on an infinite-dimensional phase space, Γ . In this paper we make extensive use of this fact and of the symplectic structure of the phase space. We therefore begin with a brief review of some basic notions in Hamiltonian dynamics.

The phase space, Γ , where (1.1) is well defined, for bounded V , is the complex Sobolev space $H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{C})$. A point in the phase space is identified with a complex-valued function $\psi \in H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{C})$. The decomposition of ψ into real and imaginary parts,

$$\psi = \psi^{(1)} + i\psi^{(2)}, \quad (2.1)$$

where $\psi^{(1)}$ and $\psi^{(2)}$ are real-valued functions in $H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{R})$, corresponds to the identification

$$\Gamma \cong H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{R}^2). \quad (2.2)$$

Note that $\psi^{(1)}$ and $\psi^{(2)}$ are canonically conjugate variables. In this paper, we use $H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{R}^2)$ as the phase space, and, to distinguish the wave fields in this representation from $\psi \in H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{C})$, we use the boldface notation

$$(\boldsymbol{\psi}^{(1)}, \boldsymbol{\psi}^{(2)}) = \boldsymbol{\psi} \in H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{R}^2). \quad (2.3)$$

The tangent space, $T_{\boldsymbol{\psi}}\Gamma$, to Γ at a point $\boldsymbol{\psi} \in \Gamma$ is given by

$$\{z(x) : z(\cdot) \in H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{R}^2), \boldsymbol{\psi} + z \in \Gamma\}. \quad (2.4)$$

Hence

$$T_\psi\Gamma = H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{R}^2). \tag{2.5}$$

A section of the tangent bundle $T\Gamma$ is a *vector field*, i.e. an assignment of a vector $z_\psi \in T_\psi\Gamma$ to each point $\psi \in \Gamma$ that depends continuously on ψ .

The phase space carries a natural metric, $(\cdot, \cdot)_{L^2(\mathbb{R}^3, \mathbb{R}^2), \psi}$: For $\mathbf{u} = (u_\psi^{(1)}, u_\psi^{(2)})$, $\mathbf{w} = (w_\psi^{(1)}, w_\psi^{(2)})$ in $T_\psi\Gamma$, $\psi \in \Gamma$, we set

$$(\mathbf{u}, \mathbf{w})_2 \equiv (\mathbf{u}, \mathbf{w})_{L^2(\mathbb{R}^3, \mathbb{R}^2), \psi} := \int_{\mathbb{R}^3} (u_\psi^{(1)} w_\psi^{(1)} + u_\psi^{(2)} w_\psi^{(2)}) \, dx. \tag{2.6}$$

Furthermore, Γ carries a complex structure denoted by J : For $z \in T_\psi\Gamma$, we set

$$Jz = (z^{(2)}, -z^{(1)}). \tag{2.7}$$

Of course, Γ is *symplectic* with the symplectic 2-form given by

$$\omega(\mathbf{u}, \mathbf{w}) := \int_{\mathbb{R}^3} (u^{(2)} w^{(1)} - u^{(1)} w^{(2)}) \, dx, \tag{2.8}$$

for \mathbf{u}, \mathbf{w} in $T_\psi\Gamma$. We observe that

$$\omega(\mathbf{u}, \mathbf{w}) = -(\mathbf{u}, J\mathbf{w})_2. \tag{2.9}$$

In what follows we also consider a subspace, $\tilde{\Gamma} \subset \Gamma$, given by

$$\tilde{\Gamma} := \{\psi \in \Gamma : |x|^{1/2}\psi \in L^2\} \tag{2.10}$$

and equipped with the norm

$$\|\mathbf{u}\|_{\tilde{\Gamma}}^2 := \|\mathbf{u}\|_{H^{\frac{1}{2}}}^2 + \varepsilon \| |x|^{1/2}\mathbf{u} \|_2^2, \tag{2.11}$$

where ε is defined in (1.14). By the proof of lemma 3 in [5], we find that if $\psi_0 \in \tilde{\Gamma}$, then $\psi(\cdot, t) \in \tilde{\Gamma}$. See also proposition 7.5.

The Hartree energy functional \mathcal{H}_V , the mass \mathcal{N} and the momentum functional \mathcal{P} (see (1.2)–(1.4)) have the form

$$\mathcal{H}_V(\psi) := \frac{1}{2}(\psi, (\sqrt{-\Delta + m^2} - m)\psi)_2 + \frac{1}{2}(\psi, V\psi)_2 - \frac{1}{4} \left(\frac{1}{|x|} * |\psi|^2, |\psi|^2 \right)_2, \tag{2.12}$$

$$\mathcal{N}(\psi) := \frac{1}{2} \|\psi\|_2^2, \tag{2.13}$$

$$\mathcal{P}(\psi) := \frac{1}{2}(\psi, J\nabla\psi)_2. \tag{2.14}$$

These functionals are well defined on $\Gamma = H^{\frac{1}{2}}(\mathbb{R}^3; \mathbb{R}^2)$.

We claim that (1.1) is the Hamiltonian equation of motion corresponding to the Hamiltonian $\mathcal{H}_V(\psi)$, given in (2.12). This is verified by noting that the equation

$$\partial_t \psi = J\mathcal{H}'_V(\psi) \tag{2.15}$$

is equivalent to (1.1).

If $V \equiv 0$, then the Hamiltonian is invariant under spatial translations $x \mapsto x + a$. The corresponding conserved quantity is the momentum $\mathcal{P}(\psi)$ defined above. For bounded V , \mathcal{P} satisfies an Ehrenfest identity

$$d_t \mathcal{P}(\psi) = -\frac{1}{2}(\psi \nabla V, \psi)_2. \tag{2.16}$$

This identity was shown in [23, appendix A] for the nonlinear Schrödinger equation, but the proof carries over to (1.1).

We define a functional $\mathcal{E}_{v,\mu}$ as

$$\mathcal{E}_{v,\mu}(\psi) := \mathcal{H}_{V=0}(\psi) + \mu \mathcal{N}(\psi) - v \cdot \mathcal{P}(\psi), \quad (2.17)$$

which is given, more explicitly, by

$$\begin{aligned} \mathcal{E}_{v,\mu}(\psi) = & \frac{1}{2}(\psi, (\sqrt{-\Delta + m^2} - m)\psi)_2 + \frac{\mu}{2}\|\psi\|_2^2 - \frac{1}{2}v \cdot (\psi, J\nabla\psi)_2 \\ & - \frac{1}{4} \left(|\psi|^2, \frac{1}{|x|} * |\psi|^2 \right)_2. \end{aligned} \quad (2.18)$$

This functional plays a key role in this paper, and we briefly discuss its properties. The ground states $\varphi_{v,\mu} = (\operatorname{Re} \varphi_{v,\mu}, \operatorname{Im} \varphi_{v,\mu})$, i.e. solutions to equation (1.8), are solutions to

$$\mathcal{E}'_{v,\mu}(\varphi_{v,\mu}) = 0. \quad (2.19)$$

Its Hessian, $L_{v,\mu} := \mathcal{E}''_{v,\mu}(\varphi_{v,\mu})$, is given by the linear symmetric operator:

$$L_{v,\mu} := \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}, \quad (2.20)$$

where, for $\xi \in H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{R})$,

$$L_{11,v,\mu}\xi := \left(\sqrt{-\Delta + m^2} - m + \mu - \frac{1}{|x|} * |\varphi_{v,\mu}|^2 \right) \xi - \left(\frac{2}{|x|} * (\xi \varphi_{v,\mu}^{(1)}) \right) \varphi_{v,\mu}^{(1)}, \quad (2.21)$$

$$L_{12,v,\mu}\xi := -v \cdot \nabla \xi - \left(\frac{2}{|x|} * (\xi \varphi_{v,\mu}^{(2)}) \right) \varphi_{v,\mu}^{(1)}, \quad (2.22)$$

$$L_{21,v,\mu}\xi := v \cdot \nabla \xi - \left(\frac{2}{|x|} * (\xi \varphi_{v,\mu}^{(1)}) \right) \varphi_{v,\mu}^{(2)}, \quad (2.23)$$

$$L_{22,v,\mu}\xi := \left(\sqrt{-\Delta + m^2} - m + \mu - \frac{1}{|x|} * |\varphi_{v,\mu}|^2 \right) \xi - \left(\frac{2}{|x|} * (\xi \varphi_{v,\mu}^{(2)}) \right) \varphi_{v,\mu}^{(2)}. \quad (2.24)$$

We find $\nabla \varphi_{v,\mu}$ and $J\varphi_{v,\mu}$ to be elements of the kernel of $L_{v,\mu}$, $\operatorname{Ker}(L_{v,\mu})$, because $\mathcal{E}_{v,\mu}$ is invariant under translation and gauge transformations. For $v = 0$, it is known [13] that one can choose a ground state φ_μ to be spherically symmetric and positive. For $v = 0$ we introduce the notation $\varphi_{v=0,\mu} = \varphi_\mu$, where $\varphi_\mu := (\varphi_\mu, 0)$. Thus, for $v = 0$ $L_{v,\mu}$ reduces to $L_\mu := \operatorname{diag}(L_{11,\mu}, L_{22,\mu})$. A key assumption underlying our analysis is assumption 2.1.

Assumption 2.1. Let $L_\mu = \operatorname{diag}(L_{11,\mu}, L_{22,\mu})$ be defined as above. We assume that the dimension of the null space of $L_{11,\mu}$ is three, i.e.

$$\dim \operatorname{Ker}(L_{11,\mu}) = 3, \quad \text{for } \mu > 0. \quad (2.25)$$

In [28] this assumption is verified numerically for some $\mu > 0$, following [29].

3. The main theorem

In this section we state our assumptions and the main theorem. We then present an outline of the proof, which is implemented in the remaining sections of this paper.

Given a number $\varepsilon > 0$, we require that the external potential $V \in C^3$ satisfies

$$\sup_x |\partial_x^\alpha V| \leq C\varepsilon^{|\alpha|}, \quad \text{for } |\alpha| \leq 3, \quad (3.1)$$

where α is a multi-index and C is a constant.

The ground state $\varphi_{v,\mu}$ is *not* known to be unique modulo phase transformations and translations. We therefore single out a particular class of solutions to (2.19), also denoted by $\varphi_{v,\mu}$, near a spherically symmetric minimizer $\varphi_\mu := \varphi_{v=0,\mu}$ by the use of an implicit function theorem and assumption 2.1. As we will see, there is a maximal number r_0 and a maximal open interval I_0 , with $0 < r_0 < 1$, such that for $|v| < r_0$ and $\mu \in I_0$, $\varphi_{v,\mu}$ solves (2.19). The construction, as well as several properties of these functions are given in proposition 4.3. For any $r < r_0$ and an open non-empty interval $I \subset I_0$, let $Z(r, I)$ be defined by

$$Z(r, I) := \mathbb{R}^3 \times B_r(0) \times [0, 2\pi) \times I, \tag{3.2}$$

where $B_r(0) = \{v \in \mathbb{R}^3 : |v| < r\}$. Consider the manifold

$$M(Z) := \{e^{-\vartheta J} \varphi_{v,\mu}(\cdot - y) : (y, v, \vartheta, \mu) \in Z\}. \tag{3.3}$$

The *soliton manifold* is defined by $M_0 := M(Z_0)$ with $Z_0 := Z(r_0, I_0)$, where r_0 is the maximal speed and I_0 the maximal frequency interval such that $\varphi_{v,\mu}$ is well defined in the sense of proposition 4.3. Thus, Z_0 is the parameter space for M_0 . Furthermore, we introduce a symbol, ζ , for a point in Z_0 by

$$\zeta := (y, v, \vartheta, \mu), \tag{3.4}$$

and note that each point in M_0 can be described by φ_ζ , where

$$\varphi_\zeta(x) := e^{-\vartheta J} \varphi_{v,\mu}(x - y). \tag{3.5}$$

The tangent space to M_0 at φ_ζ is given by

$$T_{\varphi_\zeta} M_0 := \text{span}(\nabla_y \varphi_\zeta, \nabla_v \varphi_\zeta, J \varphi_\zeta, \partial_\mu \varphi_\zeta). \tag{3.6}$$

We can now state our main theorem.

Theorem 3.1. *Suppose that assumption 2.1 is satisfied. Let r_0, I_0 and $\varphi_{v,\mu}$ be as above. Let the external potential V satisfy (3.1). Then there is an open non-empty interval \tilde{I} , with⁶ $\tilde{I} \Subset I_0$ as well as numbers $0 < \tilde{r} < r_0$ and $\varepsilon > 0$, with ε sufficiently small, such that the following holds. If the initial condition $\psi_0 \in \tilde{\Gamma}$ for (2.15) satisfies*

$$\|\psi_0 - \varphi_{\zeta^{(0)}}\|_{\tilde{\Gamma}} \leq \varepsilon, \tag{3.7}$$

for some $\zeta^{(0)} \in \tilde{Z}$, with $\tilde{Z} := Z(\tilde{r}, \tilde{I})$, and $\|\cdot\|_{\tilde{\Gamma}}$ as defined in (2.11), then the solution ψ of (2.15) is of the form

$$\psi(x, t) = e^{-\vartheta J} (\varphi_{v,\mu}(x - y) + \xi(x - y, t)), \tag{3.8}$$

for times $0 \leq t \leq c/\varepsilon$, where $c > 0$ is some constant independent of ε . Here y, v, ϑ, μ are time-dependent functions and

$$\|\xi\|_{\tilde{\Gamma}} \leq C\varepsilon. \tag{3.9}$$

Moreover, we have that

$$|d_t \mathcal{N}(\varphi_{v,\mu})| \leq C\varepsilon^2, \quad |d_t \mathcal{P}(\varphi_{v,\mu}) + \mathcal{N}(\varphi_{v,\mu}) \nabla V(y)| \leq C\varepsilon^2, \tag{3.10}$$

$$|\dot{y} - v| \leq C\varepsilon^2, \quad |\dot{\vartheta} - \mu + V(y)| \leq C\varepsilon^2, \tag{3.11}$$

also for times $0 \leq t \leq c/\varepsilon$, where $c > 0$ and $C > 0$ are some constants independent of ε .

The requirement that the initial condition be ‘close’ in norm to $M(\tilde{Z})$ can, as in e.g. [23–26], be used to introduce an additional small parameter, ε_0 , to separate the two scales mentioned in the introduction. But to simplify the exposition in the present paper, we assume this distance to be also bounded by ε .

⁶ Here $A \Subset B$ is defined by $\bar{A} \subset B$, \bar{A} compact.

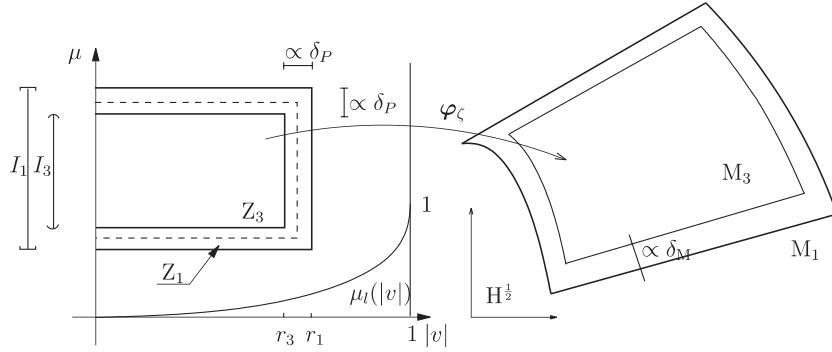


Figure 1. The left figure displays a schematic view of the parameter spaces Z_1 and Z_3 . The function $\mu_l(|v|)$ is defined in proposition 4.3 and represents the lower bound on μ . The dotted line indicates Z_2 ; δ_M and δ_P are the distances between the manifolds and upper bounds on ε , see remark 6.2.

Remark 3.1. We would like to extend the time interval to a result of Nekhoroshev type, i.e. to times of size $C\varepsilon^{-n}$ for any $n > 1$; however, the methods of this paper are not strong enough.

Next, we outline the essential parts of the proof of theorem 3.1. Above, we have introduced r_0, I_0 and \tilde{r}, \tilde{I} . In the process of the proof, we will find a nested sequence of manifolds: there are numbers r_j and open non-empty intervals I_j , for $j = 0, \dots, 3$, with $r_3 := \tilde{r}$ and $I_3 := \tilde{I}$. Here $0 < r_{j+1} < r_j$ and $\tilde{I}_{j+1} \subset I_j$ for $j = 0, 1, 2$. The corresponding parameter domains are $Z_j := Z(r_j, I_j)$ corresponding to soliton submanifolds $M_j := M(Z_j)$. One of the constraints on ε is that it has to be smaller than the distance between the boundaries of the nested sequence of the manifolds. These distances are indicated in figure 1; also see remark 6.2.

Once we derive the properties of the ground state (section 4), we need to derive the finite-dimensional dynamics expressed by (3.10)–(3.11). To do this, we show that the symplectic form is non-degenerate on M_0 by using the symmetry properties of $\varphi_{v,\mu}$ and its derivatives; see section 5. This non-degeneracy of the symplectic form on M_0 is the key fact to show the existence of a skew (or symplectically) orthogonal decomposition of ψ in a tubular neighbourhood $U_\delta(Z_2)$ around M_2 . That is, there is a unique map $\zeta : U_\delta(Z_2) \rightarrow Z_1$ such that

$$\psi(x) = \varphi_{\zeta(\psi)}(x) + e^{-\vartheta(\psi)J} \xi(x - y(\psi), t) \tag{3.12}$$

and

$$\omega(\psi - \varphi_{\zeta(\psi)}, z) = 0, \quad \forall z \in T_{\varphi_{\zeta(\psi)}} M_1. \tag{3.13}$$

This result is proven in section 6.

The existence of the decomposition (3.12)–(3.13) enables us to ‘change variables’ from ψ to (ζ, ξ) , where $\zeta(t) := \zeta(\psi(\cdot, t))$. We also need an upper bound on $|\zeta^{(0)} - \zeta(\psi_0)|$ and on $\|\varphi_{\zeta^{(0)}} - \varphi_{\zeta(\psi_0)}\|_{\tilde{r}}$, that is distances between the orthogonal decomposition (see equation (3.7)) and skew-orthogonal decomposition $\zeta(\psi_0)$ (see, e.g., proposition 6.1 or figure 2). The existence and uniqueness as well as these upper bounds follow by utilizing an implicit function theorem on a suitable quantity, see proposition 6.1.

In section 7, we insert the symplectically orthogonal decomposition into the equation of motion (2.15). We then use the symplectic form to project out the finite-dimensional equations for ζ , which schematically take the form

$$\dot{\zeta} = X(\zeta) + Y(\zeta, \xi). \tag{3.14}$$

We then show that

$$|Y(\zeta, \xi)| \leq C(\varepsilon^2 + \|\xi\|_{H^{\frac{1}{2}}} |\alpha| + \|\xi\|_{H^{\frac{1}{2}}}^2 + \|\xi\|_{H^{\frac{1}{2}}}^3), \tag{3.15}$$

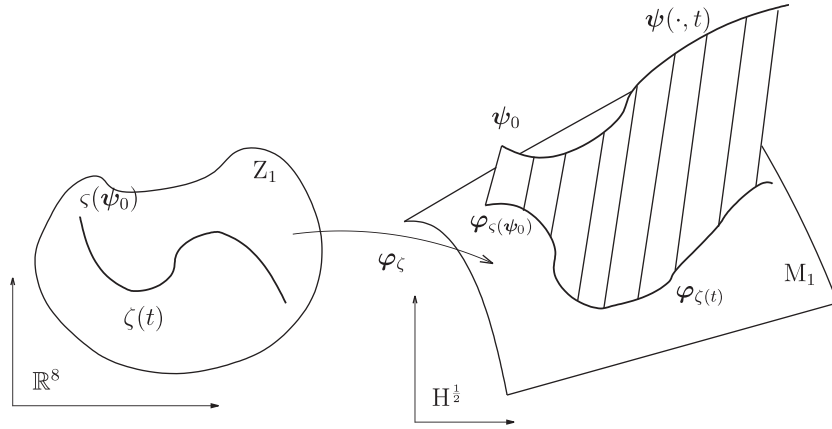


Figure 2. The dynamics of ψ , φ_ζ and $\zeta(t) := \zeta(\psi(\cdot, t))$. The diagonal lines connecting the trajectories of φ_ζ and ψ indicate the skew-orthogonal projection.

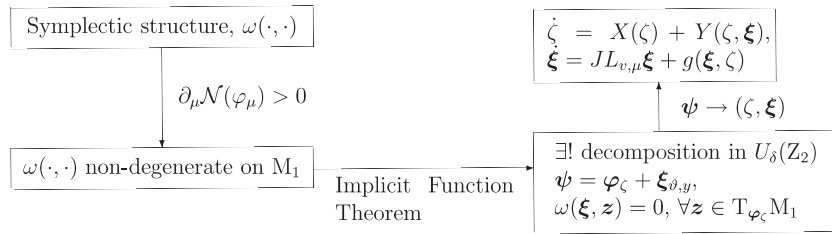


Figure 3. The way to the dynamics of (ζ, ξ) . A schematic representation of sections 5–7. $\xi_{\theta,y}(x, t) := e^{-\theta J} \xi(x + y, t)$, $g(\xi, \zeta)$ is the coupling term. For an explicit form see (7.6).

and $\alpha = (\dot{y} - v, \dot{v}, \mu - \dot{\nu} - V(y), \dot{\mu})$ is bounded by

$$|\alpha| \leq C(\varepsilon + |Y|). \tag{3.16}$$

Finally, \dot{v} and $\dot{\mu}$ are shown to be of size $C\varepsilon$, whereas the full dynamics $|\dot{\zeta} - X(\zeta)|$ is bounded by $C\varepsilon^2$. Thus, by (3.14), we have found the finite-dimensional dynamics, i.e. the equation of motion for ζ .

The equation for the perturbation ξ takes the form

$$\partial_t \xi = JL_{v,\mu} \xi + g(\xi, \zeta). \tag{3.17}$$

The procedure in sections 5–7 of the paper is collected in figure 3.

The next step in the proof of our main theorem is to control the ξ -term. In this analysis two upper bounds on the time interval T_1 and T_2 appear. Here $T_1 > 0$ and $T_2 > 0$ are given functions with the property that T_i^{-1} , $i = 1, 2$, is to leading order proportional to linear combinations of the ‘small parameters’ such as ε , $\|\xi\|_{\mathbb{F}}$ and $|\dot{\zeta} - X(\zeta)|$. Thus T_i is essentially of the size of c/ε . The control of ξ is now done in three steps as follows.

Step 1. We first show, in section 8, that the dynamics of $\mathcal{Q} := (\xi, |x|\xi)_2$ is well behaved and satisfies the inequality

$$\mathcal{Q}(t) \leq \mathcal{Q}(0) + C\|\xi\|_{H^{1/2}}, \tag{3.18}$$

for times $t \leq T_1$ and such that the symplectic decomposition is valid.

Step 2. To control the $H^{\frac{1}{2}}$ -norm of ξ , we introduce a Lyapunov functional, $\mathcal{S}(t)$, which is a linear combination of \mathcal{H}_V , \mathcal{N} and \mathcal{P} at ψ , minus the same quantities at φ_ζ , that is

$$\mathcal{S}(t) := (\mu - V(y))(\mathcal{N}(\psi) - \mathcal{N}(\varphi_\zeta)) + v \cdot (\mathcal{P}(\psi) - \mathcal{P}(\varphi_\zeta)) + \mathcal{H}_V(\psi) - \mathcal{H}_V(\varphi_\zeta). \quad (3.19)$$

In section 9, we bound this functional from below in terms of $\|\xi\|_{H^{\frac{1}{2}}}$, plus small perturbation terms. To do this, we use the spectral properties of $L_{v,\mu}$.

Step 3. In section 10, we bound this functional from above in terms of small quantities, (i.e. powers of $\|\xi\|_{H^{\frac{1}{2}}}$, $|Y|$), for times $t \leq T_2$ and such that the decomposition is valid. To do this, we use that $\mathcal{S}(t)$ is ‘almost conserved’.

In section 11, we combine our estimates on $\|\xi\|_{H^{\frac{1}{2}}}$, Q and $|Y|$ by resolving a nonlinear inequality. This argument in section 11 proves the main theorem.

4. Properties of ground states

In this section, we derive properties of the elements of M_0 .

Let $R_{v,\theta}$ be a rotation of angle θ in x around the v -direction. Let S_v be a reflection in x along the v -direction.

Definition 4.1. Let $v \in \mathbb{R}^3$ with $v \neq 0$. A function $\mathbf{w} = (w^{(1)}, w^{(2)}) : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is called v -symmetric if

$$R_{v,\theta}\mathbf{w} = \mathbf{w}, \quad \text{for all } \theta \in [0, 2\pi), \quad \text{and } S_v\mathbf{w} = (w^{(1)}, -w^{(2)}). \quad (4.1)$$

Analogously for any direction, say \hat{e}_3 , we call a function \hat{e}_3 -symmetric if it satisfies definition 4.1 with v replaced with \hat{e}_3 .

Definition 4.2. Let $f : \mathbb{R} \rightarrow \mathbb{R}$. If f is an even function then we write $f \sim (e)$ and if it is odd $f \sim (o)$.

For functions $\mathbf{w} : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ we use \sim likewise. For example, $\mathbf{w} \sim (eee, eeo)$ means that the first component of \mathbf{w} is even in its three coordinate directions and the second component is even in its first two coordinate directions and odd in the last. We write $\mathbf{w} \sim \mathbf{u}$ to indicate that \mathbf{w} and \mathbf{u} have the same reflection symmetries.

Recall from the introduction that it is known [13] that unboosted ground states are spherically symmetric and can be chosen to be real and positive. It is not known if the ground state is unique up to phase and translation, thus we denote one of them by φ_μ . We then have the following result.

Proposition 4.3. Suppose assumption 2.1 is satisfied and let $m > 0$. Define $\mu_l(|v|) := (1 - \sqrt{1 - |v|^2})m$. Then there is a number $r_0 \in (0, 1)$ and an open non-empty interval $I_0 := (\mu_{l0}, \mu_{h0})$, with $\mu_{l0} > \mu_l(|v|)$ such that for all $\mu \in I_0$ and $|v| \leq r_0 < 1$ we have the following:

- (i) there is a function $(v, \mu) \mapsto \varphi_{v,\mu} \in C^\infty(B_{r_0} \times I_0, H^1)$ that solves (2.19) and $\varphi_{v,\mu} \in H^s$ for all $s \geq 1$. Moreover, we have $\varphi_\mu := \varphi_{0,\mu}$, with $\varphi_\mu = (\varphi_\mu, 0)$ and φ_μ is spherically symmetric and positive;
- (ii) $\varphi_{v,\mu}$ and $\partial_\mu \varphi_{v,\mu}$ are v -symmetric and $\partial_{v_j} \varphi_{v,\mu} \sim J \partial_{x_j} \varphi_{v,\mu}$;
- (iii) $\varphi_{v,\mu}$, $\partial_{x_j} \varphi_{v,\mu}$, $\partial_\mu \varphi_{v,\mu}$ and $\partial_{v_k} \varphi_{v,\mu}$ are pointwise exponentially localized;
- (iv) ‘stability condition’, $\partial_\mu \|\varphi_{v,\mu}\|_2^2 > c$, where c is independent of $v \in \{v : |v| \leq r_0\}$, $\mu \in I_0$. Furthermore φ_μ is a minimizer of $\mathcal{E}_{0,0}$ subject to \mathcal{N} constant;

(v) $L_{v,\mu} := \mathcal{E}''_{v,\mu}(\varphi_{v,\mu})$ has one negative eigenvalue. Moreover, $\dim \text{Ker}(L_{v,\mu}) = 4$; there is a spectral gap between zero and its next spectral point, and the essential spectrum starts at $\mu - \mu_l(|v|) > 0$.

The ground state φ_μ constructed in proposition 4.3(i) is used to define the soliton manifold $M_0 := M(Z_0)$, where $Z_0 = Z(r_0, I_0)$ in (3.3). It is convenient to use $\zeta := (y, v, \vartheta, \mu)$ as a point in Z_0 , and subsequently (as in (3.5))

$$\varphi_\zeta(x) := e^{-\vartheta J} \varphi_{v,\mu}(x - y). \tag{4.2}$$

These ground states φ_ζ satisfy the Euler–Lagrange equation

$$\mathcal{E}'_{v,\mu}(\varphi_\zeta) = 0, \tag{4.3}$$

and taking derivatives of (4.3) with respect to y and ϑ evaluated at $(y, \vartheta) = (0, 0)$ yield

$$L_{v,\mu} \partial_{x_j} \varphi_{v,\mu} = 0, \quad L_{v,\mu} J \varphi_{v,\mu} = 0. \tag{4.4}$$

Thus $\partial_{x_j} \varphi_{v,\mu}$ and $J \varphi_{v,\mu}$ are zero modes to $L_{v,\mu}$. The derivatives of equation (4.3) with respect to v_j and μ , respectively, at $y = 0, \vartheta = 0$, lead to

$$L_{v,\mu} \partial_{v_j} \varphi_{v,\mu} = J \partial_{x_j} \varphi_{v,\mu}, \quad L_{v,\mu} \partial_\mu \varphi_{v,\mu} = -\varphi_{v,\mu}. \tag{4.5}$$

Therefore, $\partial_{v_j} \varphi_{v,\mu}$ and $\partial_\mu \varphi_{v,\mu}$ are zero modes to $(JL_{v,\mu})^2$. The tangent vectors of M_0 are hence in the generalized null space of $JL_{v,\mu}$ (compare with, e.g., [15]). Below, we prove parts (i)–(v) of proposition 4.3 and the completion of the proof can be found in section 4.4.

4.1. Proof of proposition 4.3(i),(ii)

The functions $\varphi_{v,\mu}$ will be constructed as a class of solutions to the Euler–Lagrange equation (2.19), starting from an unboosted, with $v = 0$, minimizer $\varphi_0 := (\varphi_0, 0)$ at frequency μ_0 . Here, φ_0 is a spherically symmetric positive minimizer to $\mathcal{E}_{0,0}$ at constant \mathcal{N} from [13] and [2].

A non-zero velocity v breaks the rotation symmetry of the map $\mathcal{E}'_{v,\mu}$. Without loss of generality we pick a preferred direction, \hat{e}_3 , the unit vector parallel to the x_3 -axis, and choose coordinates so that $v = \tilde{v} \hat{e}_3$. Let $R_{3,\theta}$ be the spatial rotation around \hat{e}_3 of angle θ , S_3 the spatial reflection along \hat{e}_3 and let K be the matrix

$$K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.6}$$

The \hat{e}_3 -symmetric Sobolev space, $H^s_{\hat{e}_3}$, of order s is defined as

$$H^s_{\hat{e}_3} := \{\psi \in H^s(\mathbb{R}^3, \mathbb{R}^2) : R_{3,\theta} \psi = \psi, \forall \theta \in [0, 2\pi), K S_3 \psi = \psi\}, \tag{4.7}$$

where $R_{3,\theta} \psi(x) := \psi(R_{3,\theta} x)$.

Remark 4.4. That $H^s_{\hat{e}_3}$ is a closed subspace of $H^s(\mathbb{R}^3, \mathbb{R}^2)$ follows by noting that $R_{3,\theta}$ and $K S_3$ are bounded operators on $H^s(\mathbb{R}^3, \mathbb{R}^2)$ and that $H^s_{\hat{e}_3} = \bigcap_{\theta \in [0, 2\pi)} \text{Ker}(1 - R_{3,\theta}) \cap \text{Ker}(1 - K S_3)$.

We recast proposition 4.3(i) and the first part of (ii) into proposition 4.5

Proposition 4.5. *Suppose that assumption 2.1 is satisfied. Let φ_0, μ_0 and $H^s_{\hat{e}_3}$, be as above; let $v := \tilde{v} \hat{e}_3$, with $\tilde{v} \in \mathbb{R}$. Then there is an open neighbourhood, $W \subset \mathbb{R} \times \mathbb{R}_+$, with $(0, \mu_0) \in W$, and a unique function $(\tilde{v}, \mu) \mapsto \varphi_{\tilde{v} \hat{e}_3, \mu} \in C^\infty(W, H^1_{\hat{e}_3})$ such that $\varphi_{0 \hat{e}_3, \mu_0} = \varphi_0$ and $\varphi_{\tilde{v} \hat{e}_3, \mu}$ solves $\mathcal{E}'_{\tilde{v} \hat{e}_3, \mu}(\psi) = 0$ for all $(\tilde{v}, \mu) \in W$. In addition, $\varphi_{\tilde{v} \hat{e}_3, \mu}$ belongs to H^s for all $s \geq 1$.*

Remark 4.6. (a) A solution to $\mathcal{E}'_{v,\mu}(\psi) = 0$ when $v \neq 0$ points in an arbitrary direction is obtained by rotating $\varphi_{\tilde{v}\hat{e}_3,\mu}$ in x from \hat{e}_3 to $\hat{v} := v/|v|$. See the proof of corollary 4.7 for details. (b) The Sobolev space of order one of radially symmetric functions, $H^1_{\text{rad}}(\mathbb{R}^3, \mathbb{R})$, is a scalar subspace of $H^1_{\hat{e}_3}$. This ensures the existence and uniqueness of a solution $\varphi_{v=0,\mu} = \varphi_\mu = (\varphi_\mu, 0)$ to $\mathcal{E}'_{0,\mu}(\psi) = 0$, where $\varphi_\mu \in H^1_{\text{rad}}$. (c) $\mathcal{E}_{v,\mu}$ is invariant under translation and change of gauge. Thus, $e^{i\vartheta J} \varphi_{v,\mu}(\cdot + y)$ for any $y \in \mathbb{R}^3$, $\vartheta \in [0, 2\pi)$, is also a solution to (2.19).

Proposition 4.5 proves that $\varphi_{\tilde{v}\hat{e}_3,\mu}, \partial_\mu \varphi_{\tilde{v}\hat{e}_3,\mu} \in H^s_{\hat{e}_3}$. For arbitrary coordinates this implies that $\varphi_{v,\mu}$ and $\partial_\mu \varphi_{v,\mu}$ are v -symmetric. The reflection symmetries of $\partial_{v_j} \varphi_{v,\mu}$ now follow from corollary 4.7

Corollary 4.7. *Suppose that assumption 2.1 is satisfied. Then*

$$\partial_{v_j} \varphi_{v,\mu} \sim J \partial_{x_j} \varphi_{v,\mu}, \quad j = 1, 2, 3. \tag{4.8}$$

The corollary is proved at the end of this subsection.

Lemma 4.8. *We have that*

$$(\tilde{v}, \mu, \psi) \mapsto \mathcal{E}'_{\tilde{v}\hat{e}_3,\mu}(\psi) \in C^\infty(\mathbb{R} \times \mathbb{R} \times H^1_{\hat{e}_3}, H^0_{\hat{e}_3}). \tag{4.9}$$

Proof of proposition 4.5. Let $F(\tilde{v}, \mu, \psi) := \mathcal{E}'_{\tilde{v}\hat{e}_3,\mu}(\psi)$. To find solutions to the equation $F(\tilde{v}, \mu, \psi) = 0$, we use the implicit function theorem in [30], which has three assumptions, which we now verify: that F is C^∞ is shown by lemma 4.8. The equation $F = 0$ has a solution $(0, \mu_0, \varphi_0)$ and $\varphi_0 \in H^1_{\hat{e}_3}$. The last condition is that $F'_\psi(0, \mu_0, \varphi_0) =: L_{\mu_0}$, where $L_{\mu_0} = \text{diag}(L_{11,\mu_0}, L_{22,\mu_0})$ is invertible. We have that $L_{22,\mu_0} \geq 0$ and $L_{22,\mu_0} \varphi_0 = 0$; the zero eigenvalue is non-degenerate since $e^{-tL_{22,\mu_0}}$ is positivity improving. (This follows from the explicit kernel for $e^{-t\sqrt{-\Delta+m^2}}$ and Trotter’s product formula.) The kernel of L_{11,μ_0} is spanned by $\{\partial_{x_j} \varphi_0\}_j$, owing to the kernel assumption. Thus, the kernel of L_{μ_0} is spanned by $\{J\varphi_0, \partial_{x_j} \varphi_0, j = 1, 2, 3\}$; none of these functions are \hat{e}_3 -symmetric. Thus, L_{μ_0} , as a map $H^1_{\hat{e}_3} \subset H^0_{\hat{e}_3} \rightarrow H^0_{\hat{e}_3}$, is invertible.

We conclude, by the implicit function theorem [30] that there is a neighbourhood $W \subset \mathbb{R} \times \mathbb{R}_+$ with $(0, \mu_0) \in W$ and a unique map $(\tilde{v}, \mu) \mapsto \varphi_{\tilde{v}\hat{e}_3,\mu} \in C^\infty(W, H^1_{\hat{e}_3})$ such that $\varphi_{0\hat{e}_3,\mu_0} = \varphi_0$ and $\varphi_{\tilde{v}\hat{e}_3,\mu}$ solves $F(\tilde{v}, \mu, \psi) = 0$.

That $\varphi_{\tilde{v}\hat{e}_3,\mu} \in H^n$ for any $1 \leq n \in \mathbb{N}$ follows from a simple bootstrap argument; see the proof of theorem 3 in [2]. □

Proof of lemma 4.8. The Hardy–Littlewood–Sobolev inequality together with the Sobolev embedding theorem (see, e.g., [31]) shows that $\mathcal{E}'_{v,\mu} \in C(H^1, L^2)$. To see that $\mathcal{E}'_{\tilde{v}\hat{e}_3,\mu}$ preserves \hat{e}_3 -symmetry, let U be either of KS_3 or of $R_{3,\theta}$. Both operations leave $\mathcal{E}_{\tilde{v}\hat{e}_3,\mu}$ invariant, i.e.

$$\mathcal{E}_{\tilde{v}\hat{e}_3,\mu}(U\psi) = \mathcal{E}_{\tilde{v}\hat{e}_3,\mu}(\psi), \quad \psi \in H^1. \tag{4.10}$$

By (Fréchet) differentiation of (4.10)

$$\mathcal{E}'_{\tilde{v}\hat{e}_3,\mu}(U\psi) = U \mathcal{E}'_{\tilde{v}\hat{e}_3,\mu}(\psi). \tag{4.11}$$

Let ψ satisfy $U\psi = \psi$. Equation (4.11) then states

$$\mathcal{E}'_{\tilde{v}\hat{e}_3,\mu}(\psi) = \mathcal{E}'_{\tilde{v}\hat{e}_3,\mu}(U\psi) = U \mathcal{E}'_{\tilde{v}\hat{e}_3,\mu}(\psi), \quad \text{where } \psi \in H^1, \tag{4.12}$$

and hence $\mathcal{E}'_{\tilde{v}\hat{e}_3,\mu} : H^1_{\hat{e}_3} \rightarrow H^0_{\hat{e}_3}$. Repeating the argument for higher order derivatives of (4.10) gives that $\mathcal{E}^{(n)}_{\tilde{v}\hat{e}_3,\mu}(\psi)$ preserves \hat{e}_3 -symmetry.

That $\psi \mapsto \mathcal{E}'_{v,\mu}(\psi)$ is C^1 follows from the Hardy–Sobolev–Littlewood inequality and the Sobolev embedding theorems for H^1 . That is, let $\mathbf{u}, \mathbf{w}, \boldsymbol{\xi}, \boldsymbol{\zeta} \in H^1$, then

$$\begin{aligned} & \sup_{\|\boldsymbol{\xi}\|_{H^1}=1, \|\boldsymbol{\zeta}\|_{H^1}=1} |(\mathcal{E}''_{v,\mu}(\mathbf{u})\boldsymbol{\xi}, \boldsymbol{\zeta})_2 - (\mathcal{E}''_{v,\mu}(\mathbf{w})\boldsymbol{\xi}, \boldsymbol{\zeta})_2| \\ &= \sup_{\|\boldsymbol{\xi}\|_{H^1}=1, \|\boldsymbol{\zeta}\|_{H^1}=1} \left| \left(\frac{1}{|x|} * ((\mathbf{u} + \mathbf{w}) \cdot (\mathbf{u} - \mathbf{w})), \boldsymbol{\xi} \cdot \boldsymbol{\zeta} \right)_2 \right. \\ & \quad \left. + \left(\boldsymbol{\zeta} \cdot (\mathbf{u} - \mathbf{w}), \frac{2}{|x|} * (\mathbf{u} \cdot \boldsymbol{\xi}) \right)_2 + \left(\boldsymbol{\zeta} \cdot \mathbf{w}, \frac{2}{|x|} * ((\mathbf{u} - \mathbf{w}) \cdot \boldsymbol{\xi}) \right)_2 \right| \\ & \leq C(\|\mathbf{w}\|_{H^1}, \|\mathbf{u}\|_{H^1})\|\mathbf{u} - \mathbf{w}\|_{H^1}. \end{aligned} \tag{4.13}$$

Analogously one can show that $\psi \mapsto \mathcal{E}'_{v,\mu}(\psi)$ is C^2 . The polynomial nature of the nonlinearity implies that $\mathcal{E}^{(4)}_{v,\mu}(\psi)$ is a (tri-)linear bounded operator independent of ψ . Thus, $\psi \mapsto \mathcal{E}'_{v,\mu}(\psi)$ is C^∞ in ψ .

The map $(\tilde{v}, \mu) \mapsto \mathcal{E}'_{\tilde{v}\hat{e}_3,\mu}(\psi)$ is linear and hence smooth. Differentiation with respect to either μ or \tilde{v} does not change $\mathcal{E}'_{\tilde{v}\hat{e}_3,\mu}(\psi)$'s symmetries. \square

Proof of corollary 4.7. The momentum term is the only term in $\mathcal{E}_{v,\mu}$ that breaks the rotation symmetry. For an arbitrary rotation

$$\mathcal{E}_{v,\mu}(R\psi) = \mathcal{E}_{R^{-1}v,\mu}(\psi), \tag{4.14}$$

taking the derivative gives the relation

$$R^{-1}\mathcal{E}'_{v,\mu}(R\psi) = \mathcal{E}'_{R^{-1}v,\mu}(\psi). \tag{4.15}$$

Given the \hat{e}_3 -symmetric function $\varphi_{\tilde{v}\hat{e}_3,\mu}$, we find $\varphi_{v,\mu}$ by any rotation, R , that takes \hat{e}_3 to \hat{v} as $\varphi_{v,\mu} = R\varphi_{\tilde{v}\hat{e}_3,\mu}$.

This relation between $\varphi_{v,\mu}$ and $\varphi_{\tilde{v}\hat{e}_3,\mu}$ is the key to show the corollary. Let R_1 be the rotation from $v_1\hat{e}_1 + v_3\hat{e}_3$ to $|v|\hat{e}_3$ given by

$$R_1(\theta) := \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 0 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix}, \tag{4.16}$$

where θ is the angle between v and \hat{e}_3 . We find for $\tilde{v} = |v| > 0$ that

$$\partial_{v_1}\varphi_{v,\mu} = \partial_{v_1}(R_1\varphi_{\tilde{v}\hat{e}_3,\mu}) = \frac{v_1}{\tilde{v}}R_1\partial_{\tilde{v}}\varphi_{\tilde{v}\hat{e}_3,\mu} + (\partial_{v_1}\theta)(\partial_\theta R_1x) \cdot R_1\nabla_x\varphi_{\tilde{v}\hat{e}_3,\mu} \tag{4.17}$$

$$= R_1\left(\frac{v_1}{\tilde{v}}\partial_{\tilde{v}}\varphi_{\tilde{v}\hat{e}_3,\mu} + (\partial_{v_1}\theta)(R_1^{-1}\partial_\theta R_1x) \cdot \nabla_x\varphi_{\tilde{v}\hat{e}_3,\mu}\right). \tag{4.18}$$

At the point $v_1 = 0, v_3 = \tilde{v}$, this simplifies to

$$\partial_{v_1}\varphi_{v,\mu}|_{v=(0,0,\tilde{v})} = -\frac{1}{\tilde{v}}\hat{e}_2 \cdot (x \wedge \nabla_x)\varphi_{\tilde{v}\hat{e}_3,\mu}, \tag{4.19}$$

where \wedge is the cross product. The above expression is $\sim J\partial_{x_1}\varphi_{\tilde{v}\hat{e}_3,\mu}$. Analogously for $v = (0, v_2, v_3)$ we find

$$\partial_{v_2}\varphi_{v,\mu}|_{v=(0,0,\tilde{v})} = \frac{1}{\tilde{v}}\hat{e}_1 \cdot (x \wedge \nabla_x)\varphi_{\tilde{v}\hat{e}_3,\mu} \sim J\partial_{x_2}\varphi_{\tilde{v}\hat{e}_3,\mu}. \tag{4.20}$$

Recall from proposition 4.5 that $\partial_{\tilde{v}}\varphi_{v,\mu} \in H^1_{\hat{e}_3}$ and thus $\sim J\partial_{x_3}\varphi_{\tilde{v}\hat{e}_3,\mu}$. Therefore, we have shown the corollary for a given coordinate system, i.e. coordinates such that $\varphi_{v,\mu}$ is $\varphi_{\tilde{v}\hat{e}_3,\mu}$; rotation R , once again, from \hat{e}_3 to \hat{v} of this case gives the general result. \square

4.2. Proof of exponential decay of tangent vectors

In this subsection we prove proposition 4.3(iii), i.e. the pointwise exponential decay of the tangent vectors $\{\partial_{x_j}\varphi_{v,\mu}, \partial_{v_j}\varphi_{v,\mu}, J\varphi_{v,\mu}, \partial_\mu\varphi_{v,\mu}\}$. In [2, appendix C] we showed that $\varphi_{v,\mu}$ satisfies the bound

$$|\varphi_{v,\mu}| \leq c_1(\beta)e^{-\beta|x|}, \quad (4.21)$$

for $0 < \beta < \min(m, (\mu - \mu_l)(1 - v^2)^{-1/2})$, where μ_l is defined in proposition 4.3. All the remaining tangent vectors satisfy an equation of the type

$$\mathbf{u} = F(\mathbf{u}, \mathbf{w}), \quad (4.22)$$

with

$$F(\mathbf{u}, \mathbf{w}) := (H_v + \mu)^{-1}(W_1\mathbf{u} + W_2(\mathbf{u}) + \mathbf{w}) \quad (4.23)$$

and

$$H_v = \sqrt{-\Delta + m^2} - m - Jv \cdot \nabla, \quad W_1 := \frac{1}{|x|} * |\varphi_{v,\mu}|^2, \quad W_2(\mathbf{u}) := \left(\frac{2}{|x|} * (\mathbf{u} \cdot \varphi_{v,\mu})\right)\varphi_{v,\mu}, \quad (4.24)$$

for $\mu > \mu_l$ and some \mathbf{w} depending on the tangent vector, see (4.4)–(4.5).

We have the following result.

Lemma 4.9. *Let $m > 0$, $\mu > \mu_l$ and let \mathbf{u} be a solution to (4.22), for some \mathbf{w} with $|\mathbf{w}| \leq c_2e^{-\beta_2|x|}$ where c_2, β_2 are some positive constants. Then, there is $\theta > 0$ and a constant $C(\theta) > 0$ such that*

$$|\mathbf{u}| \leq C(\theta)e^{-\theta|x|}. \quad (4.25)$$

We now return to equations (4.4)–(4.5) to find pairs (\mathbf{u}, \mathbf{w}) that solve (4.22). These are $(\partial_{x_j}\varphi_{v,\mu}, 0)$, $(\partial_\mu\varphi_{v,\mu}, -\varphi_{v,\mu})$ and $(\partial_{v_j}\varphi_{v,\mu}, J\partial_{x_j}\varphi_{v,\mu})$. The first pair, inserted in lemma 4.9, ensures that $\partial_{x_j}\varphi_{v,\mu}$ is pointwise exponentially decaying. Thus, we know that the last two pairs also satisfy the assumptions of lemma 4.9 and hence both $\partial_\mu\varphi_{v,\mu}$ and $\partial_{v_j}\varphi_{v,\mu}$ are pointwise exponentially decaying. It remains to prove lemma 4.9.

Proof of lemma 4.9. The proof is based on [32] as presented in [33] and we extend the result to include the source terms. The integral kernel of $(H_v + \mu)^{-1}$, $G_{\mu,v}$, satisfies the bound [2, appendix C],

$$|G_{\mu,v}(x)| \leq c_3 \frac{e^{-\delta|x|}}{|x|^2}, \quad (4.26)$$

for some $\delta > 0$ depending on $m > 0$, $|v| < 1$ and $\mu > \mu_l$.

Our first goal is to use (4.22) to bound $|\mathbf{u}|$ as

$$|\mathbf{u}(x)| \leq h_\theta(x)M(x) + C_1e^{-\gamma|x|}, \quad (4.27)$$

where $\theta > 0$ remains to be chosen later, and

$$M(x) := \sup_{x'} |\mathbf{u}(x')|e^{-\theta|x-x'|}. \quad (4.28)$$

To this end, we need estimates on each term in (4.23) and we begin with the $W_2(\mathbf{u})$ term

$$W_2(\mathbf{u})(x') = \varphi_{v,\mu}(x') \int_{\mathbb{R}^3} \frac{1}{|x' - y|} \mathbf{u}(y) \cdot \varphi_{v,\mu}(y) dy. \quad (4.29)$$

The identity $\mathbf{u}(y) = \mathbf{u}(y)e^{-\theta|y-x|}e^{\theta|y-x|}$, the inequality $|y - x| \leq |y - x'| + |x' - x|$ and the upper bound (4.21) lead to

$$|W_2(\mathbf{u})(x')| \leq C'_2 M(x) e^{-\beta|x'|+\theta|x-x'|} \int_{\mathbb{R}^3} \frac{e^{\theta|x'-y|}}{|x'-y|} e^{-\beta|y|} dy. \tag{4.30}$$

We evaluate the integral, with $0 < \theta < \beta$, to find for some $C_2 = C_2(\theta) > 0$ that

$$|W_2(\mathbf{u})(x')| \leq C_2 \frac{e^{\theta|x-x'|}}{1+|x'|} M(x). \tag{4.31}$$

The estimate for W_1 follows similarly, by once again integrating an integral of the type that appears in (4.30):

$$|W_1(x')| \leq C_3 \frac{1}{1+|x'|}. \tag{4.32}$$

The ‘potentials’ W_1 and W_2 are hence bounded and decaying and we can choose h_θ to be

$$h_\theta(x) := C_4 \int_{\mathbb{R}^3} \frac{e^{-(\delta-\theta)|x-x'|}}{|x-x'|^2} \frac{1}{1+|x'|} dx', \tag{4.33}$$

where C_4 is composed of the constants c_3, C_2 and C_3 .

We use the integral kernel $G_{\mu,v}$ of $(H_v + \mu)^{-1}$ to express the last term of (4.23). By (4.26), we have

$$|G_{\mu,v} * \mathbf{w}| \leq c_3 \int_{\mathbb{R}^3} \frac{e^{-\delta|x'|}}{|x'|^2} |\mathbf{w}(x-x')| dx'. \tag{4.34}$$

The assumed, pointwise exponential decay of \mathbf{w} together with the inequality $|x - x'| \geq ||x| - |x'||$ yields

$$|G_{\mu,v} * \mathbf{w}| \leq C_1 e^{-\gamma|x|}, \tag{4.35}$$

where $\gamma = \min(\delta, \beta_2)$ and $C_1 > 0$ are suitable constants. We have thus established (4.27).

To proceed, we show that h_θ is bounded and that it decays pointwise as $|x| \rightarrow \infty$. The first of these properties follows from Young’s inequality, since for $\theta < \delta$, $e^{\theta|\cdot|} G_{\mu,v}(\cdot) \in L^1$ and $(1 + |\cdot|)^{-1} \in L^\infty$:

$$\sup_x |h_\theta| \leq C_4 \|e^{\theta|\cdot|} G_{\mu,v}(\cdot)\|_1 \sup_x |(1 + |x|)^{-1}| = C_5(\theta) < \infty. \tag{4.36}$$

To show the decay of h_θ as $|x| \rightarrow \infty$, let $\alpha := \delta - \theta, \alpha > 0$, we use (4.33) and split the region of integration into two parts $|x - x'| \leq \kappa, |x - x'| > \kappa$. In the outer region we use the uniform bound of $(1 + |x'|)^{-1} < 1$ to find

$$\int_{|x-x'|>\kappa} \frac{e^{-\alpha|x-x'|}}{|x-x'|^2} \frac{1}{1+|x'|} dx' \leq \frac{1}{\kappa^2} \int_{\mathbb{R}^3} e^{-\alpha|x'|} dx', \tag{4.37}$$

and in the inner region,

$$\int_{|x-x'|\leq\kappa} \frac{e^{-\alpha|x-x'|}}{|x-x'|^2} \frac{1}{1+|x'|} dx' \leq \int_{|y|\leq\kappa} \frac{dy}{|y|^2(1+|x-y|)} \leq \frac{4\pi\kappa}{1+||x|-\kappa|}. \tag{4.38}$$

The choice of $\kappa = |x|^{1/2}$ ensures that $h_\theta \leq C_6|x|^{-1/2}$ as $|x| \rightarrow \infty$ and $\theta \leq \delta$.

The following two identities will be used repeatedly in the next step of the proof, let $\theta > 0, \gamma > 0$:

$$\sup_y e^{-\theta|x-y|-\theta|y-x'|} = e^{-\theta|x-x'|}, \quad \sup_y e^{-\theta|y|-\gamma|y-x|} = e^{-\min(\theta,\gamma)|x|}. \tag{4.39}$$

The exponential decay of \mathbf{u} now follows from the properties of h_θ , through two inequalities. Since h_θ decay, for a fixed small $\theta < \delta$, there is a radius R , such that for $|x| > R$ we have that $h_\theta \leq C_6 R^{-1/2}$. In this exterior region, we use (4.27) together with (4.39) to obtain

$$\begin{aligned} \sup_{|x'| > R} |\mathbf{u}(x')| e^{-\theta|x-x'|} &\leq \sup_{|x'| > R} (h_\theta(x')M(x') + C_1 e^{-\gamma|x'|}) e^{-\theta|x-x'|} \\ &\leq C_6 R^{-1/2} M(x) + \sup_{|x'| > R} C_1 e^{-\gamma|x'| - \theta|x-x'|}. \end{aligned} \tag{4.40}$$

In the interior we have

$$\begin{aligned} \sup_{|x'| \leq R} |\mathbf{u}(x')| e^{-\theta|x-x'|} &\leq \sup_{|x'| \leq R} \left[h_\theta(x') \left(\sup_{|y| \leq R} |\mathbf{u}(y)| e^{-\theta|x'-y|} \right. \right. \\ &\quad \left. \left. + \sup_{|y| > R} |\mathbf{u}(y)| e^{-\theta|x'-y|} \right) + C_1 e^{-\gamma|x'|} \right] e^{-\theta|x-x'|}. \end{aligned} \tag{4.41}$$

Insert the result (4.40) into (4.41). The upper bound (4.36) ensures that $h_\theta < C_5$; the exterior term $\sup_{|y| \geq R} |\mathbf{u}(y)| e^{-\theta|x'-y|}$ is estimated by (4.40); for the interior term $\sup_{|y| \leq R} |\mathbf{u}(y)| e^{-\theta|x'-y|}$ we have by continuity and boundedness of \mathbf{u} that $|\mathbf{u}| \leq C'_7(R)$ and hence $\sup_{|y| \leq R} |\mathbf{u}(y)| e^{-\theta|x'-y|} \leq C_7(R, \theta) e^{-\theta|x'|}$ yields (4.41) to become

$$\begin{aligned} \sup_{|x'| \leq R} |\mathbf{u}(x')| e^{-\theta|x-x'|} &\leq C_1 \sup_{|x'| \leq R} e^{-\gamma|x'|} e^{-\theta|x-x'|} \\ &\quad + C_5 \sup_{|x'| \leq R} \left(C_7 e^{-\theta|x'|} + C_6 R^{-1/2} M(x') + C_1 \sup_{|y| > R} e^{-\gamma|y| - \theta|x-y|} \right) e^{-\theta|x-x'|}. \end{aligned} \tag{4.42}$$

Adding (4.40) to (4.42), rewriting and using (4.39) gives

$$M(x) \leq C_8 (R^{-1/2} M(x) + e^{-\min(\gamma, \theta)|x|}) + C_9(R, \theta) e^{-\theta|x|}, \tag{4.43}$$

for suitable constants $C_8 = C_8(\theta)$, C_9 . By choice of $R = R^*$ sufficiently large and $\theta > 0$ sufficiently small, we find

$$M(x) \leq C'(R^*, \theta) e^{-\theta|x|}. \tag{4.44}$$

This upper bound inserted into (4.27) together with (4.36) yields $|\mathbf{u}| \leq C(\theta) e^{-\theta|x|}$ and we have proved the lemma. \square

4.3. Proof of the stability condition

We now derive the ‘stability condition’ stated in proposition 4.3(iv) for unboosted ground states, $\varphi_{v=0, \mu}(x) = (\varphi_\mu, 0)$. As mentioned in remark 4.6 (also see [2, 13]), these functions can be assumed to be real-valued and spherically symmetric. In view of this, we introduce the subspace

$$H_{\text{rad}}^s(\mathbb{R}^3, \mathbb{R}) = \{ \psi \in H^s(\mathbb{R}^3; \mathbb{R}) : \psi \text{ is spherically symmetric} \}, \tag{4.45}$$

for $s \geq 0$.

Lemma 4.10. *Suppose that assumption 2.1 holds. Then, for almost every $0 < N < N_c$, there exists an unboosted ground state, $\varphi_* = \varphi_{v=0, \mu_*}$, with $\mathcal{N}((\varphi_*, 0)) = N$ and Lagrange multiplier, μ_* , satisfying the following properties. For every sufficiently small $\delta > 0$, there exists a C^∞ -map*

$$(\mu_* - \delta, \mu_* + \delta) \rightarrow H_{\text{rad}}^1(\mathbb{R}^3, \mathbb{R}), \quad \mu \mapsto \varphi_\mu, \tag{4.46}$$

where $(\varphi_\mu, 0)$ solves (2.19) with $v = 0$ and we have that $\varphi_{\mu_*} = \varphi_*$. In addition, there exists a non-empty interval $I \subset (\mu_* - \delta, \mu_* + \delta)$, such that

$$\frac{d}{d\mu} \mathcal{N}((\varphi_\mu, 0)) > 0 \tag{4.47}$$

holds for all $\mu \in I$.

Proof of lemma 4.10. By remark 4.6(b), we can assume that unboosted ground states $\varphi(x) = \varphi_{v=0,\mu}$ are spherical symmetric and real-valued. Let $E(N) := \inf\{\mathcal{E}_{0,0}(\psi) : \psi \in H^{\frac{1}{2}}, \mathcal{N}(\psi) = N\}$. It is known that the function $E : (0, N_c) \rightarrow \mathbb{R}$ is strictly concave [2, lemma 2.3]. This implies in particular the following properties.

- $E(N)$ is continuous on $(0, N_c)$.
- $E'_-(N)$ and $E'_+(N)$ (which denote the left and right derivatives, respectively) exist for all $N \in (0, N_c)$.
- $E'(N) = E'_-(N) = E'_+(N)$ for all $N \in (0, N_c) \setminus \Sigma$, where Σ is some countable set.

For convenience, we denote the set where $E'(N)$ exists by

$$\Sigma^c := (0, N_c) \setminus \Sigma. \tag{4.48}$$

Let us now pick $N_* \in \Sigma^c$ and a strictly decreasing sequence, (N_k) , in Σ^c such that

$$N_k \searrow N_*, \quad \text{as } k \rightarrow \infty. \tag{4.49}$$

By density $\Sigma^c \subset (0, N_c)$, this is always possible. Correspondingly, let $(\varphi_k) \subset H^{\frac{1}{2}}$ be a sequence of minimizers with $\mathcal{E}_{0,0}((\varphi_k, 0)) = E(N_k)$ and $\mathcal{N}((\varphi_k, 0)) = N_k$, which, by continuity of $E(N)$, implies that

$$\mathcal{E}_{0,0}((\varphi_k, 0)) \rightarrow E(N_*) \quad \text{and} \quad \mathcal{N}((\varphi_k, 0)) \rightarrow N_*, \quad \text{as } k \rightarrow \infty. \tag{4.50}$$

By arguments similar to those in the proof of [2, theorem 2] and the relative compactness property stated in [2, theorem 1], we see that (φ_k) , after passing to a subsequence, converges strongly in $H^{\frac{1}{2}}_{\text{rad}}$ to some minimizer φ_* with $\mathcal{N}((\varphi_*, 0)) = N_*$ and Lagrange multiplier $-\mu_*$. (Note that due to $v = 0$, we can restrict our attention to radial functions and translations do not have to be taken into account.)

Next, we observe that any φ_k satisfies the identity

$$E(N_k) - \frac{1}{4} \int_{\mathbb{R}^3} \left(\frac{1}{|x|} * |\varphi_{n_k}|^2 \right) |\varphi_{n_k}|^2 dx = -\mu_k N_k, \tag{4.51}$$

where $-\mu_k$ is the Lagrange multiplier for the minimizer φ_k . This identity follows from the multiplication of the Euler-Lagrange equation (2.19) with $(\varphi_k(x), 0)$ and integration. Now we claim that

$$E'(N_k) = -\mu_k \tag{4.52}$$

holds for all k . Note that $E'(N_k)$ exists due to $N_k \in \Sigma^c$ for all k . To prove (4.52), we observe that $\mathcal{E}_{0,0}(\sqrt{\tau}(\varphi_k, 0)) \geq E(\tau N_k)$ holds for all $\tau \geq 0$ with equality for $\tau = 1$. Hence it is straightforward to see that the right derivative, $E'_+(N_k)$, obeys the following estimate:

$$\begin{aligned} E'_+(N_k) &= \frac{1}{N_k} \lim_{N \searrow N_k} \frac{E(N) - E(N_k)}{N/N_k - 1} \leq \frac{1}{N_k} \lim_{\tau \searrow 1} \frac{\mathcal{E}_{0,0}(\sqrt{\tau}(\varphi_k, 0)) - \mathcal{E}_{0,0}((\varphi_k, 0))}{\tau - 1} \\ &= \frac{1}{N_k} \lim_{\tau \searrow 1} \frac{d}{d\tau} \mathcal{E}_{0,0}(\sqrt{\tau}(\varphi_k, 0)) = \frac{1}{2N_k \sqrt{\tau}} \Big|_{\tau=1} (\mathcal{E}'_{0,0}((\varphi_k, 0)), (\varphi_k, 0))_2 \end{aligned} \tag{4.53}$$

$$= \frac{-\mu_k}{N_k} \frac{1}{2} \int_{\mathbb{R}^3} |\varphi_k|^2 dx = -\mu_k, \tag{4.54}$$

using the Euler–Lagrange equation $\mathcal{E}'_{0,0}((\varphi_k, 0)) = -\mu_k(\varphi_k, 0)$. Similarly, we obtain $-\mu_k \leq E'_-(N_k)$. Since $E'(N_k)$ exists for $N_k \in \Sigma^c$, we have equality and we conclude that (4.52) holds.

Next, let us define the map

$$G(\psi, \mu) := (\sqrt{-\Delta + m^2} - m)\psi - \left(\frac{1}{|x|} * |\psi|^2\right)\psi + \mu\psi, \tag{4.55}$$

which is seen to be a C^∞ -map $G : H^1_{\text{rad}} \times \mathbb{R} \rightarrow L^2_{\text{rad}}$; see the proof of proposition 4.5 and remark 4.6(b). Moreover, we note that $G(\varphi_*, \mu_*) = 0$ holds and we have that $\partial_\psi G(\varphi_*, \mu_*)$ equals L_{11, μ_*} restricted on H^1_{rad} . But assumption 2.1 implies that L_{11, μ_*} restricted to H^1_{rad} has trivial kernel (since $\partial_{x_i} \phi_* \notin H^1_{\text{rad}}$). Thus, we can apply the implicit function theorem to find a unique C^∞ -map

$$(\mu_* - \delta, \mu_* + \delta) \longrightarrow U, \quad \mu \longmapsto \varphi_\mu, \tag{4.56}$$

for every sufficiently small $\delta > 0$, where $\varphi_{\mu_*} = \varphi_*$ and U is some open H^1_{rad} -neighbourhood around φ_* .

We now show that the strong convergence of φ_k to φ_* in $H^{\frac{1}{2}}_{\text{rad}}$ implies strong convergence in H^1_{rad} . This can be seen as follows. Each φ_k satisfies the equation

$$\varphi_k = R_{\mu_k} F(\varphi_k), \tag{4.57}$$

where $R_\mu := (H_0 + \mu)^{-1}$ with $H_0 := \sqrt{-\Delta + m^2} - m$ and $F(\varphi) := (|x|^{-1} * |\varphi|^2)\varphi$. Therefore we have

$$\begin{aligned} \|\varphi_k - \varphi_*\|_{H^1} &= \|R_{\mu_k} F(\varphi_k) - R_{\mu_*} F(\varphi_*)\|_{H^1} \leq \|(R_{\mu_k} - R_{\mu_*})(F(\varphi_k) + F(\varphi_*))\|_{H^1} \\ &\quad + \|(R_{\mu_k} + R_{\mu_*})(F(\varphi_k) - F(\varphi_*))\|_{H^1}. \end{aligned} \tag{4.58}$$

By (4.51), the fact that $\varphi_k \rightarrow \varphi_*$ in $H^{\frac{1}{2}}_{\text{rad}}$, and $N_k \searrow N_*$, we see that $\mu_k \searrow \mu_*$ (note that (4.52) holds and that $E'(N)$ is strictly decreasing on Σ^c). Using now the resolvent identity $R_{\mu_k} - R_{\mu_*} = (\mu_* - \mu_k)R_{\mu_k}R_{\mu_*}$, as well as $\|R_{\mu_k}\|_{L^2 \rightarrow H^1} \leq C/\mu_k$, we deduce that

$$\|\varphi_k - \varphi_*\|_{H^1} \leq C(|\mu_k - \mu_*| + \|\varphi_k - \varphi_*\|_{H^{\frac{1}{2}}}) \rightarrow 0 \quad \text{as } k \rightarrow \infty, \tag{4.59}$$

where we also used the local Lipschitz estimate

$$\|F(u) - F(v)\|_2 \leq \|F(u) - F(v)\|_{H^{\frac{1}{2}}} \leq C(\|u\|_{H^{\frac{1}{2}}}^2 + \|v\|_{H^{\frac{1}{2}}}^2)\|u - v\|_{H^{\frac{1}{2}}}, \tag{4.60}$$

see [3, section 3, lemma 1]. In estimate (4.59), $C = C(M, \mu_*)$ denotes a suitable constant with $M = \sup_k \|\varphi_k\|_{H^{\frac{1}{2}}}$.

By the strong convergence of φ_k to φ_* in H^1_{rad} shown above, we thus obtain that $\varphi_k \in U$ whenever $k \geq k_0$, where k_0 is sufficiently large. Moreover, since the left-hand side of (4.51) converges to its value at N_* , we conclude that μ_k converges to μ_* . In addition, by (4.52) and the strict concavity, we deduce that $\mu_k \searrow \mu_*$ (note $E'(N)$ has to be strictly decreasing on Σ^c). In summary, we find that $\varphi_{k_0} \in U$ and $\mu_{k_0} \in (\mu_* - \delta, \mu_* + \delta)$ for some k_0 and $\mu_{k_0} > \mu_*$. By the uniqueness of the map (4.56), we see that $\varphi_{k_0} = \varphi_{\mu_{k_0}}$, where φ_{k_0} belongs to the sequence (φ_k) and $\varphi_{\mu_{k_0}}$ is constructed via the map (4.56). Hence, we have that the C^∞ -function

$$f(\mu) := \mathcal{N}(\varphi_\mu) \tag{4.61}$$

satisfies $f(\mu_*) < f(\mu_{k_0})$. By the mean-value theorem, there exists some $\xi \in (\mu_* - \delta, \mu_* + \delta)$ such that $f'(\xi) > 0$. By continuity of f' , we conclude that $f'(\mu) > 0$ for all $\mu \in I$ with some open interval I containing ξ . This completes the proof of lemma 4.10. \square

4.4. Completion of the proof of proposition 4.3

In appendix A we prove proposition 4.3(v). Each part is shown for some small open neighbourhood in \mathbb{R}^2 around the point $(\mu_0, 0)$. We can now complete the proof of proposition 4.3.

Proof of proposition 4.3. Let $\mu_0 > 0$, with minimizer φ_{μ_0} , be a point where assumption 2.1 holds. In section 4.3 we showed that for almost all $N > 0$ with corresponding μ_* there is an open non-empty interval I around μ_* and a unique spherically symmetric, real function φ_μ such that $\partial_\mu \mathcal{N}(\varphi_\mu) \geq c$ and that c is independent of μ . We thus have singled out an ‘admissible’ φ_{μ_*} around which we construct our ground states.

In section 4.1 we constructed $\varphi_{v,\mu}$ and its symmetries around φ_{μ_*} for $(\tilde{v}, \mu) \in W$, where W is some open, non-empty neighbourhood in \mathbb{R}^2 and $v = \tilde{v}\hat{v}$ and $\hat{v} = v/|v|$. That $\mu_* > 0$ and $\mu_l(0) = 0$ ensure the existence of a, possibly smaller, open non-empty set, also denoted W with points so that $\mu > \mu_{|v|}$ is satisfied. Thus for all $(\tilde{v}, \mu) \in W$ we have shown proposition 4.3(i), (ii).

The proof of the exponential decay (iii) (section 4.2) does not further constrain the set W .

We now show proposition 4.3(iv), i.e. the stability condition for non-zero velocities. At $v = 0$ the stability condition holds, and since the above constructed $\varphi_{v,\mu}$ depends continuously on \tilde{v} and μ so does $\mathcal{N}(\varphi_{v,\mu})$ and there exists an open, non-empty, possibly smaller, region $W_1 \subset W$ such that $\partial_\mu \mathcal{N}(\varphi_{v,\mu}) > c/2$ for all $(\tilde{v}, \mu) \in W_1$.

The spectral properties of $L_{v,\mu}$, proposition 4.3(v) (see appendix A), are shown for $(\tilde{v}, \mu) \in W_2$, where W_2 is some open non-empty neighbourhood around $(0, \mu_*)$ such that $W_2 \subset W_1$.

Finally, we have this small open and non-empty set W_2 , where (i)–(ii) hold; we now choose $r_0 > 0$ and an open non-empty interval I_0 such that $W_3 := (-r_0, r_0) \times I_0$ and $W_3 \Subset W_2$. This concludes the proof of proposition 4.3.

5. The symplectic form reduced to the soliton manifold

The purpose of this section is to show that the symplectic form (2.9), $\omega(u, v) = -(u, Jv)_2$, reduced to a subset $M_1 = M(Z_1)$ of the soliton manifold $M_0 = M(Z_0)$, see (3.2) and (3.3), is non-degenerate, i.e. M_1 is a symplectic manifold. The manifold M_0 is eight-dimensional and we choose

$$\{z_{1,\zeta}, \dots, z_{8,\zeta}\} := \{\partial_{x_1}\varphi_\zeta, \partial_{x_2}\varphi_\zeta, \partial_{x_3}\varphi_\zeta, \partial_{v_1}\varphi_\zeta, \partial_{v_2}\varphi_\zeta, \partial_{v_3}\varphi_\zeta, \partial_{\vartheta}\varphi_\zeta, \partial_\mu\varphi_\zeta\} \tag{5.1}$$

as a basis on the tangent space $T_{\varphi_\zeta}M_0$ (see (3.6)).

We represent $\omega(\cdot, \cdot)$ on the tangent space $T_{\varphi_\zeta}M_0$ with respect to this basis with the matrix

$$(\Omega_{\varphi_\zeta})_{jk} := \omega(z_{j,\zeta}, z_{k,\zeta}). \tag{5.2}$$

If Ω_{φ_ζ} is invertible for all $\zeta \in Z_1$, then $\omega(\cdot, \cdot)$ is non-degenerate on M_1 and M_1 is symplectic. The invertibility of Ω_{φ_ζ} is used in the next section.

We have the result.

Proposition 5.1. *Let $Z(r, I)$ be as in (3.2) and let Ω_{φ_ζ} be defined as above. Under assumption 2.1, there are numbers $0 < r_1 < r_0$, $\kappa > 0$ and an open non-empty interval $I_1 \Subset I_0$ such that*

$$\det \Omega_{\varphi_\zeta} \geq \kappa > 0 \tag{5.3}$$

for all $\zeta \in Z(r_1, I_1)$. The constant κ depends only on r_1 and I_1 .

Using the definitions (3.2) and (3.3) we set $Z_1 := Z(r_1, I_1)$ and $M_1 := M(Z_1)$. We have now defined the first number and non-empty interval in the sequence mentioned in the main theorem. The size of $|\Omega_{\varphi_\zeta}^{-1}|$ may depend on r_1 and I_1 ; thus, the first requirement on ε is that of $|\Omega_{\varphi_\zeta}^{-1}| = \mathcal{O}(1)$. This is a natural requirement, as we will see in proposition 7.1. As a direct consequence of the invertability of Ω_{φ_ζ} we have the following corollary.

Corollary 5.2. *For all $0 \neq z \in T_{\varphi_\zeta} M_1$, there is at least one element, $\bar{z} \in T_{\varphi_\zeta} M_1$, such that $\omega(z, \bar{z}) \neq 0$.*

Proof of proposition 5.1. By the explicit form of $\omega(\cdot, \cdot)$ we have with $\zeta = (y, v, \vartheta, \mu)$

$$\Omega_{\varphi_\zeta} = \Omega_{\varphi_{v,\mu}} \quad \text{and} \quad (\Omega_{\varphi_{v,\mu}})_{jk} = -(\Omega_{\varphi_{v,\mu}})_{kj}. \tag{5.4}$$

Thus, it suffices to consider elements in the tangent space with $y = 0, \vartheta = 0$. By the anti-symmetry of $\Omega_{\varphi_{v,\mu}}$, it is sufficient to calculate the upper half triangle of the matrix. Without loss of generality we may choose coordinates so that v is parallel to the x_3 -axis, where $x = (x_1, x_2, x_3)$. That is, $v = |v|\hat{e}_3$ and for such v we use the notation $\varphi_{v,\mu}^{(3)}$ and $\Omega^{(3)}$.

The determinant of $\Omega_{\varphi_{v,\mu}}$ will be expressed in terms of $\tau, n_{,v}$ and $n_{,\mu}$, where

$$\tau_{jk} := \omega(\partial_{x_j} \varphi_{v,\mu}, \partial_{v_k} \varphi_{v,\mu}), \quad (n_{,v})_j := -\omega(\partial_{x_j} \varphi_{v,\mu}, \partial_\mu \varphi_{v,\mu}), \quad j, k = 1, 2, 3, \tag{5.5}$$

$$n_{,\mu}(\varphi_{v,\mu}^{(3)}) := \frac{1}{2} \partial_\mu \|\varphi_{v,\mu}\|_2^2. \tag{5.6}$$

Here $n(\mu, v) := \mathcal{N}(\varphi_{v,\mu})$. The relations (4.5) yield the identities

$$\tau_{jk} = (L_{v,\mu} \partial_{v_j} \varphi_{v,\mu}, \partial_{v_k} \varphi_{v,\mu})_2, \quad (n_{,v})_j = (\partial_{v_j} \varphi_{v,\mu}, \varphi_{v,\mu})_2, \quad j, k = 1, 2, 3. \tag{5.7}$$

The last equation yields $(n_{,v})_j = n_{,v_j}$. Once the coordinates $v = |v|\hat{e}_3$ are chosen, we obtain $\tau^{(3)}, n_{,v}^{(3)}$ and $n_{,\mu}^{(3)}$; the corresponding notation for $\varphi_{v,\mu}$ is $\varphi_{v,\mu}^{(3)}$.

Each element in Ω_{φ_ζ} is an integral of a product between a pair of tangent vectors. The reflection symmetry of the tangent vectors, shown under assumption 2.1 in proposition 4.3, is the key to this proposition. We have

$$\begin{aligned} \partial_{x_1} \varphi_{v,\mu}^{(3)} &\sim (oee, oeo), & \partial_{x_2} \varphi_{v,\mu}^{(3)} &\sim (eoe, eoo), & \partial_{x_3} \varphi_{v,\mu}^{(3)} &\sim (eoo, eee) \sim J\varphi_{v,\mu}^{(3)}, \\ \partial_{v_1} \varphi_{v,\mu}^{(3)} &\sim (oeo, oee), & \partial_{v_2} \varphi_{v,\mu}^{(3)} &\sim (eoo, eoe), & \partial_{v_3} \varphi_{v,\mu}^{(3)} &\sim (eee, eeo) \sim \partial_\mu \varphi_{v,\mu}^{(3)}. \end{aligned} \tag{5.8}$$

Let us calculate the cross term $\omega(\partial_{x_1} \varphi_{v,\mu}^{(3)}, \partial_{x_2} \varphi_{v,\mu}^{(3)})$. It is an integral over a product of functions with symmetries $J\partial_{x_1} \varphi_{v,\mu}^{(3)} \sim (oeo, oee)$ and $(eoe, eoo) \sim \partial_{x_2} \varphi_{v,\mu}^{(3)}$. Thus both components of $J\partial_{x_1} \varphi_{v,\mu}^{(3)}$ are odd in the first variable whereas $\partial_{x_2} \varphi_{v,\mu}^{(3)}$ is even; hence the integral over this product vanishes. Analogously, most of the other integrals vanish and by the repeated use of (5.8) and (5.5)–(5.7) we find the matrix

$$\Omega^{(3)} = \begin{pmatrix} 0 & 0 & 0 & \tau_{11}^{(3)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tau_{22}^{(3)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \tau_{33}^{(3)} & 0 & -n_{,v_3}^{(3)} \\ -\tau_{11}^{(3)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\tau_{22}^{(3)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\tau_{33}^{(3)} & 0 & 0 & 0 & n_{,v_3}^{(3)} & 0 \\ 0 & 0 & 0 & 0 & 0 & -n_{,v_3}^{(3)} & 0 & -n_{,\mu}^{(3)} \\ 0 & 0 & n_{,v_3}^{(3)} & 0 & 0 & 0 & n_{,\mu}^{(3)} & 0 \end{pmatrix}. \tag{5.9}$$

Its determinant is $\tilde{\kappa}(v, \mu) := \det \Omega^{(3)} = (\tau_{11}^{(3)} \tau_{22}^{(3)})^2 (\tau_{33}^{(3)} n_{,\mu}^{(3)} + (n_{,v_3}^{(3)})^2)^2$. By lemma 5.4 there is an $0 < \tilde{r}_1 \leq r_0 \leq 1$ such that $\tau_{jj}^{(3)} > 0$, and by part (iv) of proposition 4.3 $n_{,\mu}^{(3)} > 0$ for all $|v| \leq r_0$ and $\mu \in I_0$. Thus $\tilde{\kappa} > 0$. Now, let $r_1 < \tilde{r}_1$ and I_1 be an open non-empty subinterval of I_0 such that $\bar{I}_1 \subset I_0$, then on the closed set $[0, r_1] \times \bar{I}_1$, $\tilde{\kappa}$ attains its minimum $\kappa > 0$.

Corollary 5.3. *The matrices $\Omega_{\varphi_{v,\mu}}$ and $\Omega_{\varphi_{v,\mu}}^{-1}$ have the form*

$$\Omega_{\varphi_{v,\mu}} = \begin{pmatrix} 0 & \tau & 0 & -n_{,v} \\ -\tau & 0 & n_{,v} & 0 \\ 0 & -n_{,v}^T & 0 & -n_{,\mu} \\ n_{,v}^T & 0 & n_{,\mu} & 0 \end{pmatrix}, \quad \Omega_{\varphi_{v,\mu}}^{-1} = \begin{pmatrix} 0 & -g & 0 & q \\ g & 0 & -q & 0 \\ 0 & q^T & 0 & -\gamma \\ -q^T & 0 & \gamma & 0 \end{pmatrix}, \quad (5.10)$$

with $\tau, n_{,v}$ and $n_{,\mu}$ as in (5.5)–(5.6) and where

$$g = (\tau + n_{,\mu}^{-1} n_{,v} n_{,v}^T)^{-1}, \quad q = (\tau n_{,\mu} + n_{,v} n_{,v}^T)^{-1} n_{,v} \quad (5.11)$$

and

$$\gamma = n_{,\mu}^{-1} (-1 + n_{,v}^T (\tau n_{,\mu} + n_{,v} n_{,v}^T)^{-1} n_{,v}). \quad (5.12)$$

Proof. To obtain $\Omega_{\varphi_{v,\mu}}$, we observe that each block matrix, e.g. $\tau, n_{,v}, n_{,\mu}$, is related to the corresponding matrix block in equation (5.9) by a change in coordinates. Thus, matrix blocks that are in (5.9) identically zero remain so, and $\tau, n_{,v}$ and $n_{,\mu}$ remain as in the general form from (5.5)–(5.6). \square

Lemma 5.4. *Let $\tau_{jj}^{(3)}$ be as in (5.5). There is a number $0 < \tilde{r}_1 \leq r_0$ such that $\tau_{jj}^{(3)} > 0$ for $j = 1, 2, 3$.*

Proof. At $v = 0$, $\tau^{(3)}$ reduces to

$$\tau_{jj}|_{v=0} = (\beta_j, L_{22,\mu} \beta_j)_2, \quad (5.13)$$

where β_j is defined through $\beta_j := \partial_{v_j} \varphi_{v,\mu}|_{v=0}$ and $\beta_j = (0, \beta_j)$. The linear operator $L_{22,\mu} \geq 0$ has a non-degenerate zero eigenvalue, with the corresponding eigenfunction being φ_μ , see the proof of proposition A.1. But, since $\beta_j \sim \partial_{x_j} \varphi_\mu$ we have $\beta_j \perp \varphi_\mu$ and hence $\tau_{jj}|_{v=0} > 0$. By the continuity of $\tau_{jj}^{(3)}$ in v , there is a number $\tilde{r}_{1j} > 0$ such that for all $|v| < \tilde{r}_{1j}$ we have that $\tau_{jj}^{(3)} > 0$. Now let $\tilde{r}_1 = \min_j \tilde{r}_{1j}$.

6. Symplectically orthogonal decomposition

In this section we introduce the symplectically (or skew) orthogonal decomposition of a function ψ close to the soliton manifold. The decomposition has two components: one on the manifold, $\varphi_\zeta(\psi)$, and one in the symplectically orthogonal direction, ξ . We show that the decomposition uniquely defines the modulation parameter $\zeta = (y, v, \vartheta, \mu)$ and a perturbation ξ .

Recall from proposition 5.1 that the modulation parameter ζ is a point in the parameter space $Z_1 = Z(r_1, I_1)$ and

$$Z(r_1, I_1) := \mathbb{R}^3 \times B_{r_1}^3(0) \times [0, 2\pi) \times I_1. \quad (6.1)$$

All ground states described by the modulation parameters in Z_1 define the soliton manifold $M_1 := M(Z_1)$. Above, $B_r^n(0) \subset \mathbb{R}^n$ denotes an open ball of radius r and I_1 is an open interval

on \mathbb{R} . For subsets $Z_3 \subset Z_2 \subset Z_1$, to be introduced below, we define the tubular neighbourhood $U_\delta = U_\delta(Z_j)$ of $M(Z_j)$ by

$$U_\delta(Z_j) := \left\{ \psi \in \tilde{\Gamma} : \inf_{\zeta \in Z_j} \|\psi - \varphi_\zeta\|_{\tilde{\Gamma}} < \delta \right\}, \quad j = 2, 3. \tag{6.2}$$

Let b_R be an open ball around φ_ζ in the phase space $\tilde{\Gamma}$ with radius R , defined by

$$b_R(\varphi_\zeta) := \{ \psi \in \tilde{\Gamma} : \|\psi - \varphi_\zeta\|_{\tilde{\Gamma}} < R \}, \tag{6.3}$$

where $\tilde{\Gamma} := \{ \psi \in H^{\frac{1}{2}} : \|\psi\|_{\tilde{\Gamma}} < \infty \}$, see also (2.11).

We have the following result.

Proposition 6.1. *Suppose assumption 2.1 is satisfied and let Z_1 be defined as above. Given $\delta > 0$ sufficiently small, let r_2 and r_3 be such that $0 < r_3 < r_2 < r_1$ and let I_2, I_3 be open non-empty intervals with $\tilde{I}_{j+1} \subset I_j$, $j = 1, 2$, with corresponding parameter domains Z_2, Z_3 and soliton manifolds M_2, M_3 . Then for every $\psi \in U_\delta(Z_{k+1})$ and $k = 1, 2$ there is a unique $C^1(U_\delta(Z_{k+1}), Z_k)$ -map ζ such that*

(i) for each $\psi \in U_\delta(Z_{k+1})$, we have

$$\omega(\psi - \varphi_{\zeta(\psi)}, \mathbf{z}) = 0, \quad \forall \mathbf{z} \in T_{\varphi_{\zeta(\psi)}} M_k. \tag{6.4}$$

Furthermore, for each $\psi \in U_\delta(Z_{k+1})$ there exists a $\zeta^{(0)} \in Z_k$ such that $\varphi_{\zeta^{(0)}}$ is the orthogonal projection of ψ onto M_k , and

- (ii) $\|\varphi_{\zeta(\psi)} - \varphi_{\zeta^{(0)}}\|_{\tilde{\Gamma}} \leq C\delta$,
- (iii) $|\zeta^{(0)} - \zeta(\psi)| \leq C\delta$,

for some positive constant $C > 0$.

The above proposition defines a unique function $\zeta : U_{Z_{k+1}} \rightarrow Z_k$. Consequently, $\psi \mapsto (\zeta, \xi)$ with $\xi := \psi - \varphi_{\zeta(\psi)}$ defines a unique decomposition of $\psi \in U_\delta(Z_2)$.

Remark 6.2. Given r_j and I_j for $j = 1, 2, 3$, we have determined above a $\delta > 0$ such that $\zeta(\psi), \zeta^{(0)}(\psi) \in Z_{j-1}$, when $\psi \in U_\delta(Z_j)$. Thus we can now give the relation between the distances δ_M (δ_P), the minimal distance between the manifolds and δ introduced above (see section 11, figure 1). This relation is $\delta_M \leq c\delta$, $\delta_P \leq c\delta$. For some constant $c > 0$, which is partly determined by the size of C in parts (ii) and (iii).

Remark 6.3. If we choose in proposition 6.1 an even smaller distance, $\varepsilon < \delta$, and consider the tubular neighbourhood $U_\varepsilon(Z_3)$, then parts (ii) and (iii) hold with δ replaced by ε .

Proof. This theorem is a standard application of an implicit function theorem, applied first to the symplectic decomposition and then to the orthogonal decomposition. For analogous results see, e.g., [15, 18]. We start with the symplectic decomposition. Define the function $G : H^{\frac{1}{2}} \times Z_1 \rightarrow \mathbb{R}^8$ by

$$G_j(\psi, \zeta) := \omega(\psi - \varphi_\zeta, \mathbf{z}_{j,\zeta}), \quad j = 1, \dots, 8, \tag{6.5}$$

where $\mathbf{z}_{j,\zeta}$ is the j th function in (5.1) and thus an element in $T_{\varphi_\zeta} M_1$.

Note above that the theorem is done for elements in tubular neighbourhoods of M_2 respective to M_3 . For simplicity we prove the theorem only for M_2 ; the result for M_3 follows by repeating the procedure with the obvious replacements.

Let $(\varphi_{\zeta^{(c)}}, \zeta^{(c)})$ be a point in $(M(Z_1), Z_1)$ and let $b_0(\varphi_{\zeta^{(c)}})$ be a ball of radius 10δ in $\tilde{\Gamma}$. We use an implicit function theorem to solve the equation $G = 0$. We need to show that

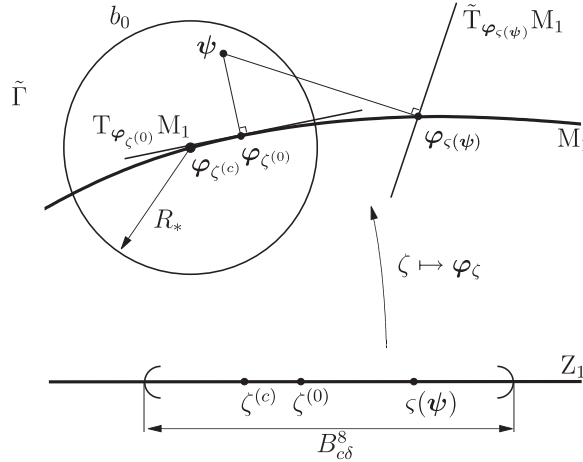


Figure 4. The above figure displays the sets Z_1 and M_1 , as well as $T_{\varphi_{\zeta^{(e)}}}M_1$ and the symplectically orthogonal plane $\tilde{T}_{\varphi_{\zeta(\psi)}}M_1$. Furthermore, we schematically sketch the orthogonal and symplectically orthogonal decomposition of ψ .

- (a) G is C^1 , (b) $G(\varphi_{\zeta^{(e)}}, \zeta^{(e)}) = 0$ and (c) $\partial_{\zeta} G(\varphi_{\zeta^{(e)}}, \zeta)|_{\zeta=\zeta^{(e)}}$ is invertible.
- (a) G is C^1 in ψ since it is linear in ψ . G is C^1 in ζ since both φ_{ζ} and $z_{j,\zeta}$ are C^1 in ζ , see proposition 4.3,
- (b) follows from the definition of G ,
- (c) calculate

$$\partial_{\zeta_k} G_j(\psi, \zeta) = -\omega(\partial_{\zeta_k} \varphi_{\zeta}, z_{j,\zeta}) + \omega(\psi - \varphi_{\zeta}, \partial_{\zeta_k} z_{j,\zeta}) \tag{6.6}$$

$$\text{at } (\psi, \zeta) = (\varphi_{\zeta^{(e)}}, \zeta^{(e)})$$

$$\partial_{\zeta_k} G_j(\psi, \zeta^{(e)})|_{\psi=\varphi_{\zeta^{(e)}}} = -\omega(\partial_{\zeta_k} \varphi_{\zeta}, z_{\zeta^{(e)},j}). \tag{6.7}$$

Recall the definition of $\Omega_{\varphi_{\zeta}}$ (5.2) to get

$$\partial_{\zeta_k} G_j(\psi, \zeta^{(e)})|_{\psi=\varphi_{\zeta^{(e)}}} = -\Omega_{\varphi_{\zeta^{(e)}}}. \tag{6.8}$$

Thus, by proposition 5.1, $\det \Omega_{\varphi_{\zeta^{(e)}}} > \kappa > 0$ and we have shown (c).

All assumptions in the implicit function theorem are thus satisfied and, therefore, there is a unique C^1 -function $\zeta : b_0 \rightarrow \mathbb{R}^8$ such that for all $\psi \in b_0(\varphi_{\zeta^{(e)}})$, $G(\psi, \zeta(\psi)) = 0$. An immediate consequence is that $\|\varphi_{\zeta(\psi)} - \varphi_{\zeta^{(e)}}\|_{\tilde{\Gamma}} \leq C\delta$ and $|\zeta(\psi) - \zeta^{(e)}| \leq C\delta$, for sufficiently small δ . The size of δ and the constant C might initially appear to depend on $\zeta^{(e)} \in Z_1$, but the argument below shows how to make C and δ uniform in $\zeta^{(e)}$. The functional G is invariant under translation and phase change; hence, both δ and C are independent of these two parameters. To get an uniform C and δ in μ and v , we shrink the initial domain such that $\zeta^{(e)} \in Z_2 = Z(r_2, I_2)$ for some $0 < r_2 < r_1$ and I_2 is an open non-empty interval with $\bar{I}_2 \subset I_1$. Since we are a small distance away from the boundary, one can choose constants in the proof of the implicit function theorem, to find that both C and δ can be chosen uniformly. Since $\zeta^{(e)}$ is arbitrary, and the radius of the ball b_0 is fixed and independent of position, we get the tubular neighbourhood of $M(Z_2)$ as the union of these balls. The price we pay for a uniform radius is that the range of ζ may remain in Z_1 ; we have thus got $\zeta : U_{\delta}(Z_2) \rightarrow Z_1$. See figure 4.

We now turn to the orthogonal decomposition. Recall that $\tilde{\Gamma}$ is a Hilbert space and define the function $f_{\psi}(\zeta) := \|\psi - \varphi_{\zeta}\|_{\tilde{\Gamma}}^2$. The minimizer $\zeta^{(0)}$ is the parameter that corresponds to

the soliton that is the orthogonal projection of ψ onto the soliton manifold. But first have a small intermediate result. Calculate

$$(f''_{\psi}(\zeta))_{jk} = (\psi - \varphi_{\zeta}, \partial_{\zeta_j} z_{k,\zeta})_{\bar{\Gamma}} + (z_{j,\zeta}, z_{k,\zeta})_{\bar{\Gamma}}, \tag{6.9}$$

where $z_{k,\zeta} := \partial_{\zeta_k} \varphi_{\zeta}$. The last term, $(f''_{\varphi_{\zeta}}(\zeta))_{jk} = (z_{j,\zeta}, z_{k,\zeta})_{\bar{\Gamma}}$ simplifies to $(z_{j,v,\mu}, z_{k,v,\mu})_{\bar{\Gamma}}$, since the phase and translation vanish in the integral. Analogously to proposition 5.1 we first have that there are r''_1 and I''_1 such that $0 < r''_1 < r_0$ and I''_1 is open and non-empty, with $I''_1 \Subset I_0$ such that $f''_{(\varphi_{v,\mu})}(\zeta) > 0$ for all $|v| \leq r''_1$ and $\mu \in I''_1$. This is shown by utilizing the reflection symmetries of $z_{k,v,\mu}$, see proposition 4.3(ii), together with the fact that both $|x|$ and $\sqrt{1 - \Delta}$ commute with rotation and reflection.

Let $Z''_1 := Z(r''_1, I''_1)$. Note that $f_{\psi} \in C^2(Z''_0, \mathbb{R})$, and that Z''_1 is a convex open set. Let $\psi \in b_0(\zeta^{(c)})$, with $\zeta^{(c)} \in Z''_1$ where once again b_0 is a ball in $\tilde{\Gamma}$ with radius 10δ . To show that f_{ψ} has a unique minimizer, it suffices to show that (i) there exists a $\zeta^{(0)} = \zeta^{(0)}(\psi)$ such that $f'_{\psi}(\zeta^{(0)}) = 0$ for all $\psi \in b_0$ and (ii) $f''_{\psi}(\zeta) > 0$, for all $\zeta \in B$. Define the map $h : \zeta \mapsto \varphi_{\zeta}$, then $B := h^{-1}(M_1 \cap b_0)$. To show (ii), first recall (6.9) and that $f''_{\varphi_{v,\mu}}(\zeta) > 0$. To extend the positivity of $f''_{\varphi_{v,\mu}}$ to positivity of f''_{ψ} , we see that the first term in (6.9) can be made small, when $\|\psi - \varphi_{\zeta}\|_{\bar{\Gamma}}$ is small. Thus, for sufficiently small δ we have $f''_{\psi} > 0$ and we have proved (ii). To show (i), we note that $f'_{\varphi_{\zeta}}(\zeta) = 0$, $f''_{\varphi_{\zeta}} > 0$ and $f'_{\psi} \in C^1$; thus, we can use the implicit function theorem to find a sufficiently small δ such that there exists a unique C^1 -map $\zeta^{(0)}(\psi) : b_0 \mapsto \mathbb{R}^8$ such that $f_{\psi}(\zeta^{(0)}) = 0$. This shows (i). Thus close to the manifold there exists a unique minimizer. In the procedure for showing this we have also defined the orthogonal decomposition and its corresponding parameter $\zeta^{(0)}$. As a byproduct of the implicit function theorem, we get

$$|\zeta^{(0)} - \zeta^{(c)}| \leq C\delta, \quad \|\varphi_{\zeta^{(0)}} - \varphi_{\zeta^{(c)}}\|_{\bar{\Gamma}} \leq C\delta. \tag{6.10}$$

Once again, we observe that δ and C appear to depend on $\zeta^{(c)}$, but entirely analogously to the above result for the symplectic decomposition, we can show that there is a smaller domain, also called Z_2 , where δ and C are independent of $\zeta^{(c)}$.

Using the triangle inequality we obtain (2) and (3) above and we have shown the proposition for a tubular neighbourhood around M_2 ; repeating the procedure, with obvious replacements, we obtain the result for $U_{\delta}(M_3)$. □

7. Dynamics in a moving frame

In this section we apply the unique decomposition provided by proposition 6.1 to a class of solutions of (1.1) and find the resulting equations for the decomposed parts. Another way to see this decomposition is that we make the change in variables $\psi \mapsto (\zeta, \xi)$ and derive the equation of motion for this set of variables. As mentioned above, ξ can be seen as a *perturbation* to a solitary wave parametrized by ζ .

For the decomposition of a solution ψ to equation (1.1) to exist, we require ψ to remain in the tubular neighbourhood $U_{\delta}(Z_2)$ for some interval of times. This is ensured by the requirement that the corresponding initial condition ψ_0 belongs to $U_{\delta}(Z_3)$. The decomposition is defined by the unique function $\varsigma(\psi)$, with $\varsigma = (y, v, \vartheta, \mu)$ that solves the equation $G(\psi, \zeta) = 0$, where $G(\psi, \zeta) := \omega(\psi - \varphi_{\zeta}, z_{\cdot,\zeta})$, and the relation

$$\psi(x, t) = e^{-\vartheta(t)J}(\varphi_{v(t),\mu(t)}(x - y(t)) + \xi(x - y(t), t)). \tag{7.1}$$

Thus, the existence of ζ ensures that $\omega(\xi, z) = 0$ for all z in $T_{\varphi_{v,\mu}} M_2$. Here

$$e^{-\vartheta J} = \begin{pmatrix} \cos \vartheta & -\sin \vartheta \\ \sin \vartheta & \cos \vartheta \end{pmatrix}. \tag{7.2}$$

The solution ψ depends on time, and consequently so does $\zeta(t) := \zeta(\psi(\cdot, t))$ and ξ .

By substituting the decomposition (7.1) into equation (2.15) (which is the Hamiltonian formulation of (1.1)) and upon applying ‘projections’ of the symplectic form we have the following result.

Proposition 7.1. *Let $U_\delta(\mathbb{Z}_3)$ be defined as above. Suppose that assumption 2.1 is satisfied. Let $\psi(\cdot, t)$ be a solution to (2.15) with initial condition $\psi_0 \in U_\delta(\mathbb{Z}_3)$ and let $\xi(\cdot, t)$ and $\zeta(t) = (y(t), v(t), \vartheta(t), \mu(t))$ be the decomposed parameters corresponding to $\psi(\cdot, t)$. Furthermore, let the external potential V satisfy (3.1) for some $\varepsilon > 0$. Then we have the following.*

(i) *The parameters $\zeta = (y, v, \vartheta, \mu)$ satisfy the modulation equations*

$$\alpha_j + \mathcal{N}(\varphi_{v,\mu}) \sum_{v=1}^3 (\Omega_{\varphi_{v,\mu}})^{-1}_{jv} \partial_{y_v} V(y) = Y_j, \tag{7.3}$$

where

$$\alpha := (v - \dot{y}, \dot{v}, \mu - \dot{\vartheta} - V(y), \dot{\mu}), \tag{7.4}$$

and the perturbation terms Y_j are given on the right-hand side of (7.23) and satisfy the estimate

$$|Y| \leq C(\|\xi\|_{H^{\frac{1}{2}}}^3 + \|\xi\|_{H^{\frac{1}{2}}}^2 + \varepsilon^2 + |\alpha| \|\xi\|_2). \tag{7.5}$$

Furthermore, we have $|\alpha| \leq C\varepsilon + |Y|$. Here $\Omega_{\varphi_{v,\mu}}$ is the matrix (5.10) and $n(\mu, v) = \mathcal{N}(\varphi_{v,\mu})$. The constant C depends only on r_1, r_2, I_1 and I_2 ; see propositions 5.1 and 6.1.

(ii) *Furthermore, the perturbation ξ satisfies the equation of motion*

$$\begin{aligned} \dot{\xi} = & J(L_{v,\mu}\xi + \mathcal{M}_{\varphi_{v,\mu}}(\xi) + (V_y - V(y))\xi + R_V\varphi_{v,\mu}) - ((v - \dot{y}) \cdot \nabla_x(\varphi_{v,\mu} + \xi) \\ & + \dot{v} \cdot \nabla_v\varphi_{v,\mu} + \dot{\mu}\partial_\mu\varphi_{v,\mu} + (\mu - \dot{\vartheta} - V(y))J(\varphi_{v,\mu} + \xi) \\ & - Jx \cdot \nabla_x V(y)\varphi_{v,\mu}), \end{aligned} \tag{7.6}$$

where $L_{v,\mu}$ is defined in (2.21),

$$-\mathcal{M}_{\varphi_{v,\mu}}(\xi) = \left(\frac{1}{|x|} * |\xi|^2\right)\varphi_{v,\mu} + \left(\frac{2}{|x|} * (\varphi_{v,\mu} \cdot \xi)\right)\xi + \left(\frac{1}{|x|} * |\xi|^2\right)\xi, \tag{7.7}$$

and

$$R_V(x) := V_y(x) - V(y) - x \cdot \nabla_y V(y), \tag{7.8}$$

where $V_y(x) = V(x + y)$.

Remark 7.2. With the explicit form of $\Omega_{\varphi_{v,\mu}}$ in corollary 5.3 we rewrite equations (7.3) and (7.4) as

$$\gamma \dot{v} + \nabla_y V(y) = \mathcal{N}(\varphi_{v,\mu})^{-1}(Y_4, Y_5, Y_6)^T \quad \dot{\mu} - n_{,\mu}^{-1} n_{,v}^T \gamma^{-1} \nabla_y V(y) = Y_8, \tag{7.9}$$

$$\dot{y} - v = -(Y_1, Y_2, Y_3) \quad \dot{\vartheta} - \mu + V(y) = -Y_7, \tag{7.10}$$

where $n := \mathcal{N}(\varphi_{v,\mu})$, $n_{,v_j} = \partial_{v_j} \mathcal{N}(\varphi_{v,\mu})$, $n_{,\mu} = \partial_\mu \mathcal{N}(\varphi_{v,\mu})$, $\tau_{jk} := (\partial_{v_j} \varphi_{v,\mu}, L_{v,\mu} \partial_{v_k} \varphi_{v,\mu})_2$ and $\gamma_{jk} := n^{-1}(\tau_{jk} + n_{,\mu}^{-1} n_{,v_j} n_{,v_k})$. Also see corollary 7.4.

Remark 7.3. The expression (7.6) is equivalent to (2.15) in the moving frame and with the decomposition (7.1) inserted. This equation does contain the information about equations (7.9)–(7.10). We can of course remove this information from (7.6) by a ‘projection’. But since we do not explicitly need this form of equation (7.6), we have refrained from writing out this expression.

Proof. By proposition 7.5 and $\psi_0 \in U_\delta(Z_3)$, the solution ψ to equation (2.15) satisfies $\psi \in U_\delta(Z_2)$ for some positive times, and the decomposition of ψ into (ζ, ξ) exists and is unique. For such times we express (2.15) in terms of ζ, ξ , with $\zeta \in Z_1$. First, we calculate the time derivative of (7.1):

$$d_t \psi = e^{-\vartheta J} (-\dot{y} \cdot (\nabla_x \varphi_{v,\mu}(x-y) + \nabla_x \xi(x-y, t)) + \dot{v} \cdot \nabla_v \varphi_{v,\mu}(x-y) + \dot{\mu} \partial_\mu \varphi_{v,\mu}(x-y) - \dot{\vartheta} J(\varphi_{v,\mu}(x-y) + \xi(x-y, t)) + \partial_t \xi(x-y, t)). \quad (7.11)$$

We denote the decomposition of ψ inserted into the right-hand side of (2.15) by A . That is,

$$A := J \mathcal{H}'_V(e^{-\vartheta J}(\varphi_{v,\mu}(x-y) + \xi(x-y, t))). \quad (7.12)$$

We expand \mathcal{H}'_V around $\varphi_{v,\mu}$; with the relations $\mathcal{E}'_{v,\mu}(\varphi_{v,\mu}) = \mathcal{H}'_{V=0}(\varphi_{v,\mu}) + \mu \varphi_{v,\mu} - v \cdot \nabla_x J \varphi_{v,\mu}$, $L_{v,\mu} := \mathcal{E}''_{v,\mu}(\varphi_{v,\mu})$ (for its explicit form see (2.21)) and $\mathcal{E}'_{v,\mu}(\varphi_\zeta) = 0$ we find

$$A = e^{-\vartheta J} J(L_{v,\mu} \xi - \mu(\varphi_{v,\mu} + \xi) + v \cdot \nabla_x J(\varphi_{v,\mu} + \xi) + \mathcal{M}_{\varphi_{v,\mu}}(\xi))(x-y, t) + (V(x) - V(y) + V(y))(\varphi_{v,\mu}(x-y) + \xi(x-y, t)), \quad (7.13)$$

where

$$\mathcal{M}_{\varphi_{v,\mu}}(\xi) := \mathcal{H}'_V(\varphi_{v,\mu} + \xi) - \mathcal{H}'_V(\varphi_{v,\mu}) - \mathcal{H}''_V(\varphi_{v,\mu})\xi. \quad (7.14)$$

We insert the explicit form of \mathcal{H}_V into the above expression for $\mathcal{M}_{\varphi_{v,\mu}}$, simplification gives the result (7.7).

The expressions (7.11) and (7.13) are on the right- and left-hand sides of (2.15). Both sides have a common phase which we cancel. Furthermore, both sides also have a common spatial translation $x \mapsto x - y(t)$, which we remove. That is, we consider the equation in a moving frame. Thus, we can rewrite (2.15) in the form

$$(v - \dot{y}) \cdot \nabla_x(\varphi_{v,\mu} + \xi) + \dot{v} \cdot \nabla_v \varphi_{v,\mu} + (\mu - \dot{\vartheta} - V(y))J(\varphi_{v,\mu} + \xi) + \dot{\mu} \partial_\mu \varphi_{v,\mu} - \nabla_y V(y) \cdot x J \varphi_{v,\mu} + \partial_t \xi = J(L_{v,\mu} \xi + \mathcal{M}_{\varphi_{v,\mu}}(\xi) + (V_y - V(y))\xi) + R_V \varphi_{v,\mu}, \quad (7.15)$$

after collecting terms of similar types. Here $V_y(x) := V(x+y)$, and $\varphi_{v,\mu}$ and ξ are evaluated at x and x, t , respectively. Furthermore, R_V is defined as

$$R_V(x) := V_y(x) - V(y) - x \cdot \nabla_y V(y). \quad (7.16)$$

Thus (7.15) is the desired equation (7.6) with the terms somewhat rearranged. We have showed part (ii) of the proposition. See also remark 7.3.

To show part (i), let $z_k \in T_{\varphi_{v,\mu}} M_1$ where $\{z_k\}$ are ordered as in (5.1) and apply the symplectic form $\omega(z_k, \cdot)$ to (7.15), then

$$\omega(z_k, (v - \dot{y}) \cdot \nabla_x(\varphi_{v,\mu} + \xi) + \dot{v} \cdot \nabla_v \varphi_{v,\mu} + (\mu - \dot{\vartheta} - V(y))J(\varphi_{v,\mu} + \xi) + \dot{\mu} \partial_\mu \varphi_{v,\mu} - \nabla_y V(y) \cdot x J \varphi_{v,\mu} + \partial_t \xi) = \omega(z_k, J L_{v,\mu} \xi + J \mathcal{M}_{\varphi_{v,\mu}}(\xi) + J(V_y - V(y))\xi + J R_V \varphi_{v,\mu}). \quad (7.17)$$

Denoting the right-hand side of this equation with B_k , we claim that the term $\omega(z_k, J L_{v,\mu} \xi)$ vanishes for all $z_k \in T_{\varphi_{v,\mu}} M_1$. To show this claim, first note the identity

$\omega(\mathbf{z}_k, JL_{v,\mu}\xi) = (L_{v,\mu}\mathbf{z}_k, \xi)_2$, where we used that $L_{v,\mu}$ is symmetric. Secondly, $L_{v,\mu}\mathbf{z}_k$ is either zero or $J\mathbf{z}_{k'}$ for some k' ; see equations (4.4) and (4.5). In the first case, we have showed the claim; in the latter case recall that ξ satisfies the decomposition conditions (6.4). Thus we have $\omega(\mathbf{z}_{k'}, \xi) = 0$ and we have shown the claim. The consequence is

$$B_k = \omega(\mathbf{z}_k, JM_{\varphi_{v,\mu}}(\xi) + J(V_y - V(y))\xi + JR_V\varphi_{v,\mu}). \tag{7.18}$$

To estimate the nonlinear term in B_k , we use the Hardy–Littlewood–Sobolev inequality [31] and a Sobolev embedding theorem. The remaining terms involve the potential and Taylor expansions of it; we bound these terms by using the fundamental theorem of calculus and equation (3.1). We find

$$B_k \leq C(\|\xi\|_{H^{\frac{1}{2}}}^3 + \|\xi\|_{H^{\frac{1}{2}}}^2 + \varepsilon^2). \tag{7.19}$$

We now return to (7.17); let the infinitesimal generators, K , and their coefficients, α , be defined by

$$K := (\nabla_x, \nabla_v, J, \partial_\mu), \quad \alpha := (v - \dot{y}, \dot{v}, \mu - \dot{\vartheta} - V(y), \dot{\mu}). \tag{7.20}$$

We keep the above notation $B = (B_1, \dots, B_8)$ to represent the right-hand side of (7.17), which by (7.19) is a perturbation for sufficiently small ξ and ε . With the observations that $\omega(\mathbf{z}_k, \nabla_x\xi) = -\omega(\nabla_x\mathbf{z}_k, \xi)$, $\omega(\mathbf{z}_k, J\xi) = -\omega(J\mathbf{z}_k, \xi)$, $\omega(\mathbf{z}_k, \xi) = 0$ and

$$0 = \partial_t\omega(\mathbf{z}_k, \xi) = \dot{v} \cdot \omega(\nabla_v\mathbf{z}_k, \xi) + \dot{\mu}\omega(\partial_\mu\mathbf{z}_k, \xi) + \omega(\mathbf{z}_k, \xi). \tag{7.21}$$

we rewrite (7.17) as

$$\sum_j ((\Omega_{\varphi_{v,\mu}})_{kj} - \omega(K_j\mathbf{z}_k, \xi))\alpha_j - \omega(\mathbf{z}_k, \nabla_y V(y) \cdot xJ\varphi_{v,\mu}) = B_k, \tag{7.22}$$

where $(\Omega_{\varphi_{v,\mu}})_{kj} = \omega(\mathbf{z}_k, K_j\varphi_{v,\mu})$ is as in corollary 5.3. Solving for the leading term in α we find

$$\alpha_j - \sum_k (\Omega_{\varphi_{v,\mu}})_{jk}^{-1}\omega(\mathbf{z}_k, \nabla_y V(y) \cdot xJ\varphi_{v,\mu}) = \sum_k (\Omega_{\varphi_{v,\mu}})_{jk}^{-1}(B_k + \sum_l \omega(K_l\mathbf{z}_k, \xi)\alpha_l). \tag{7.23}$$

Denote the right-hand side with Y_j , then the uniform lower bound on $\Omega_{\varphi_{v,\mu}}$, given by proposition 5.1, yields that

$$|Y| \leq C(\varepsilon^2 + \|\xi\|_2|\alpha| + \|\xi\|_{H^{\frac{1}{2}}}^2 + \|\xi\|_{H^{\frac{1}{2}}}^3) \tag{7.24}$$

and $|\alpha| \leq C\varepsilon + |Y|$. The constant C , in both cases, depends only on r_1, r_2, I_1 and I_2 . Since $\mathbf{z}_k = K_k\varphi_{v,\mu}$ we find that

$$\omega(\mathbf{z}_k, x \cdot \nabla_y V(y)) = -\delta_{kv}\nabla_{y_v} V(y)\mathcal{N}(\varphi_{v,\mu}), \tag{7.25}$$

for $v = 1, 2, 3$. Thus

$$\alpha_j + \mathcal{N}(\varphi_{v,\mu}) \sum_{v=1}^3 (\Omega_{\varphi_{v,\mu}}^{-1})_{jv}\nabla_{y_v} V(y) = Y_j, \tag{7.26}$$

and we have proved the proposition. □

We have derived a set of ordinary differential equations (7.3). The right-hand side remains small by the main theorem, provided the decomposition exists. Standard ODE theory shows that the solution to (7.3) is well defined as long as the decomposition is well defined. This agrees with what we expect from the global well-posedness of the solution ψ to (2.15).

We close this section with yet another form of (7.9).

Corollary 7.4. *With the change in variables $(\mu, v) \mapsto (\mathcal{N}, \mathcal{P})$, through $\mathcal{N} = \mathcal{N}(\varphi_{v,\mu})$ and $\mathcal{P} = \mathcal{P}(\varphi_{v,\mu})$, defined in (2.13) and (2.14), the equations for \dot{v} and $\dot{\mu}$ (equations (7.9)) take the form*

$$d_t \mathcal{P} + \mathcal{N} \nabla_y V(y) = (X_1, X_2, X_3)^T, \tag{7.27}$$

$$d_t \mathcal{N} = X_8, \tag{7.28}$$

where each of X_j , $j = 1, 2, 3, 8$, are related to Y_j above by $X_j = \sum_k (\Omega_{\varphi_{v,\mu}})_{jk} Y_k$.

Proof. Insert $z_k = (\nabla_x \varphi_{v,\mu}, J \varphi_{v,\mu})$ into (7.17) and simplify to obtain

$$d_t \mathcal{P}(\varphi_{v,\mu}) + \mathcal{N}(\varphi_{v,\mu}) \nabla_y V(y) = (X_1, X_2, X_3)^T \tag{7.29}$$

$$d_t \mathcal{N}(\varphi_{v,\mu}) = X_8, \tag{7.30}$$

where

$$X_j = \omega(K_j \varphi_{v,\mu}, J(\mathcal{M}_{\varphi_{v,\mu}}(\xi) + (V_y - V(y))\xi + R_V \varphi_{v,\mu})) + \sum_{k=1}^8 \alpha_k \omega(K_k K_j \varphi_{v,\mu}, \xi), \tag{7.31}$$

in which K is defined by (7.20). The change in variables $(\mu, v) \mapsto (\mathcal{N}, \mathcal{P})$ gives (7.27) and (7.28). To see the relation between X and Y , see equation (7.23). \square

Proposition 7.5. *If $\psi_0 \in \tilde{\Gamma}$, then the solution of (1.1) satisfies*

$$\psi \in C^0([0, T]; \tilde{\Gamma}),$$

where $T \in (0, \infty]$ is the maximal time of existence. Furthermore, we have that $T = \infty$ holds whenever $\mathcal{N}(\psi) < N_c$, for some universal constant $N_c > 2/\pi$.

Proof. For $\tilde{\Gamma} \subset H^{\frac{1}{2}}$ replaced by $H^{\frac{1}{2}}$, the claim follows from the well-posedness results for (1.1) proven in [3] (where also more general V s are treated).

It remains to show that $t \mapsto |x|^{1/2} \psi(t)$ is a continuous map from $[0, T)$ into L^2 . First, we note that $\psi_0 \in \tilde{\Gamma}$ implies that $\psi(t) \in \tilde{\Gamma}$ for all $0 \leq t < T$. This claim follows in particular from a direct adaption of [5][lemma A], yielding the formula

$$(\psi(t), |x| \psi(t))_2 = (\psi(0), |x| \psi(0))_2 + \int_0^t (\psi(s), J[|x|, \sqrt{-\Delta + m^2}] \psi(s))_2 ds. \tag{7.32}$$

Here the commutator $[|x|, \sqrt{-\Delta + m^2}]$ is a bounded operator on L^2 ; also see [5] for this.

Moreover, equation (7.32) shows in particular that $t \mapsto \| |x|^{1/2} \psi(t) \|_2$ is continuous. Assume now that $t_n \rightarrow t_* \in [0, T)$ is a sequence of times. Then $u_n := |x|^{1/2} \psi(t_n)$ is a bounded sequence in L^2 . By possibly passing to a subsequence, we have that u_n converges weakly to some u_* with $\liminf_{n \rightarrow \infty} \|u_n\|_2 \geq \|u_*\|_2$. But since $t \mapsto \| |x|^{1/2} \psi(t) \|_2$ is continuous, we have that $\lim_{n \rightarrow \infty} \|u_n\|_2 = \|u_*\|_2$ holds. Thus we conclude that u_n actually converges strongly to u_* in L^2 , showing that $t \mapsto |x|^{1/2} \psi(t)$ is a continuous map from $[0, T)$ into L^2 . \square

8. Weighted dynamics

Let ψ be a solution to (2.15) with initial condition $\psi_0 \in U_\varepsilon(Z_3)$. Then, under assumption 2.1, there is, for some positive time, a unique decomposition of ψ into ζ, ξ (proposition 6.1). Furthermore, if we add that V satisfies (3.1) for some $\varepsilon > 0$, we find the equations of motion for ξ in proposition 7.1. Let $\mathcal{Q}_{\delta_0}(t) := (\xi(\cdot, t)|_x|e^{-\delta_0|x|}, \xi(\cdot, t))_2$. In this section we use the equations of motions for ξ to show that the weighted expectation value $\mathcal{Q}(t) = \mathcal{Q}_{\delta_0=0}(t)$ is well defined and small. That is, we consider a time derivative of $\mathcal{Q}_{\delta_0}(t)$, where the regularization of x to $xe^{-\delta_0|x|}$ ensures that $\mathcal{Q}_{\delta_0}(t)$ is C^1 in time.

We have the following result.

Proposition 8.1. *Let $U_\varepsilon(Z_3)$ be defined as above. Let assumption 2.1 be satisfied, and let ψ be a solution to (2.15) with initial condition $\psi_0 \in U_\varepsilon(Z_3)$. Denote its decomposition by (ζ, ξ) . Let the external potential V satisfy (3.1), for some small parameter $\varepsilon > 0$. Then, for times such that the decomposition is unique there is a constant $0 < c < \infty$ which depends only on $r_j, I_j, j = 1, 2, 3$ such that*

$$\begin{aligned} \sup_{s \leq t} \mathcal{Q}_{\delta_0}(s) &\leq \mathcal{Q}_{\delta_0}(0) + ct \sup_{s \leq t} (\mathcal{Q}_{\delta_0}(s) (\|\xi(s)\|_{H^{\frac{1}{2}}} + \|\xi(s)\|_{H^{\frac{1}{2}}}^2) + (\varepsilon + |\alpha(s)|) \|\xi(s)\|_2 \\ &\quad + \|\xi(s)\|_{H^{\frac{1}{2}}}^2 + \|\xi(s)\|_{H^{\frac{1}{2}}}^3), \end{aligned} \tag{8.1}$$

where α is defined in (7.4).

Let $\mathcal{Q}(t) := (\xi(\cdot, t)|_x|, \xi(\cdot, t))_2$. From the above result and the assumption on the initial condition in main theorem 3.1 we have corollary 8.2.

Corollary 8.2. *Assume, in addition to the assumptions of proposition 8.1, that $\|\psi_0 - \varphi_{\zeta_0}\|_{\tilde{\Gamma}} \leq \varepsilon < \delta$. Then there is a constant $0 < C < \infty$ depending only on I_j and r_j for $j = 1, 2, 3$ such that*

$$\sup_{s \leq t} \mathcal{Q}(s) \leq C\varepsilon + \sup_{s \leq t} \|\xi(s)\|_{H^{\frac{1}{2}}}, \tag{8.2}$$

for positive times t such that $\psi \in U_\delta(Z_1)$ and such that time, t , satisfies the inequality

$$t \leq \frac{1}{2c\varepsilon + \sup_{s \leq t} (|\alpha(s)| + \|\xi(\cdot, s)\|_{H^{\frac{1}{2}}} + \|\xi(\cdot, s)\|_{H^{\frac{1}{2}}}^2)}, \tag{8.3}$$

with constant c as in proposition 8.1.

Proof of corollary 8.2. By assumption $\|\psi_0 - \varphi_{\zeta_0}\|_{\tilde{\Gamma}} \leq \varepsilon$, thus by the definition of $\|\cdot\|_{\tilde{\Gamma}}$, $\|(\varepsilon|x|^{1/2})(\psi_0 - \varphi_{\zeta_0})\|_2 \leq \varepsilon$. Hence

$$\|\psi_0 - \varphi_{\zeta(\psi_0)}\|_{\tilde{\Gamma}} \leq \varepsilon + \|\varphi_{\zeta(\psi_0)} - \varphi_{\zeta_0}\|_{\tilde{\Gamma}}. \tag{8.4}$$

Part (ii) in proposition 6.1 yields that $\|\varphi_{\zeta_0} - \varphi_{\zeta(\psi_0)}\|_{\tilde{\Gamma}} \leq C\varepsilon$. Thus $\mathcal{Q}_{\delta_0}(0) \leq \mathcal{Q}(0) \leq C\varepsilon$.

As we consider times, the decomposition time, such that $\psi \in U_\delta(Z_2)$ (and hence $\zeta(t) \in Z_1$), we have that the result in proposition 8.1 holds. Let t be such that it is smaller than the minimum of the decomposition time and the times such that (8.3) holds. For such times, estimate (8.1) simplifies to

$$\sup_{s \leq t} \mathcal{Q}_{\delta_0}(s) \leq C\varepsilon + \frac{1}{2} \|\xi\|_{H^{\frac{1}{2}}}. \tag{8.5}$$

The right-hand side is independent of δ_0 ; we can thus take the limit to find the result (8.2). □

Proof of proposition 8.1. Note that

$$\mathcal{Q}_{\delta_0}(t) = \mathcal{Q}_{\delta_0}(0) + \int_0^t d_s \mathcal{Q}_{\delta_0}(s) ds. \quad (8.6)$$

Since \mathcal{Q}_{δ_0} is positive for all times, we find

$$\mathcal{Q}_{\delta_0}(t) \leq \mathcal{Q}_{\delta_0}(0) + t \sup_{s \leq t} |d_s \mathcal{Q}_{\delta_0}(s)|. \quad (8.7)$$

The right-hand side does not change when interchanging t and s and taking $\sup_{s \leq t}$. Thus we find

$$\sup_{s \leq t} \mathcal{Q}_{\delta_0}(s) \leq \mathcal{Q}_{\delta_0}(0) + t \sup_{s \leq t} |d_s \mathcal{Q}_{\delta_0}(s)|. \quad (8.8)$$

To bound this we need to estimate $d_s \mathcal{Q}_{\delta_0}(s)$. As mentioned in the introduction to this section we have assumed $\psi_0 \in U_\varepsilon(\mathbb{Z}_3)$; thus for some times (to be determined) $\psi \in U_\delta(\mathbb{Z}_2)$. For such ψ , there is, under assumption 2.1 and by proposition 6.1, a unique decomposition of $\psi \mapsto (\xi, \zeta)$. Proposition 7.1 yields the equation of motion for ξ in (7.6), that is,

$$\begin{aligned} \partial_t \xi = & J(L_{v,\mu} \xi + \mathcal{M}_{\varphi_{v,\mu}}(\xi) + (V_y - V(y))(\xi + \varphi_{v,\mu})) + (\dot{y} - v) \cdot \nabla(\varphi_{v,\mu} + \xi) \\ & - \dot{v} \cdot \nabla_v \varphi_{v,\mu} - \dot{\mu} \partial_\mu \varphi_{v,\mu} - (\mu - \dot{v} - V(y))J(\varphi_{v,\mu} + \xi), \end{aligned} \quad (8.9)$$

where $L_{v,\mu}$ is the 4×4 matrix operator introduced in (2.21). We repeat the explicit form of $L_{v,\mu}$ for clarity:

$$L_{11} \xi_1 := \sqrt{-\Delta + m^2} \xi_1 + (-m + \mu) \xi_1 - \frac{1}{|x|} * |\varphi_{v,\mu}|^2 \xi_1 - \left(\frac{2}{|x|} * (\xi_1 \varphi_{v,\mu}^{(1)}) \right) \varphi_{v,\mu}^{(1)}, \quad (8.10)$$

$$L_{12} \xi_2 := v \cdot \nabla \xi_2 - \left(\frac{2}{|x|} * (\xi_2 \varphi_{v,\mu}^{(2)}) \right) \varphi_{v,\mu}^{(1)}, \quad (8.11)$$

$$L_{21} \xi_1 := -v \cdot \nabla \xi_1 - \left(\frac{2}{|x|} * (\xi_1 \varphi_{v,\mu}^{(1)}) \right) \varphi_{v,\mu}^{(2)}, \quad (8.12)$$

$$L_{22} \xi_2 := \sqrt{-\Delta + m^2} \xi_2 + (-m + \mu) \xi_2 - \left(\frac{1}{|x|} * |\varphi_{v,\mu}|^2 \right) \xi_2 - \left(\frac{2}{|x|} * (\xi_2 \varphi_{v,\mu}^{(2)}) \right) \varphi_{v,\mu}^{(2)}, \quad (8.13)$$

where we have used the notation $\varphi_{v,\mu} = (\varphi_{v,\mu}^{(1)}, \varphi_{v,\mu}^{(2)})^T$. We regularize $|x|$ by

$$f_{\delta_0}(x) := |x| e^{-\delta_0 |x|}. \quad (8.14)$$

This is a bounded function, and $2\mathcal{Q}_{\delta_0} := (\xi, f_{\delta_0} \xi)_2$ is well defined, since $\xi \in L^2$ and $f_{\delta_0} \in L^\infty$.

The time derivative of \mathcal{Q}_{δ_0} can be expressed as

$$2d_t \mathcal{Q}_{\delta_0} = (\partial_t \xi, f_{\delta_0} \xi)_2 + (\xi, f_{\delta_0} \partial_t \xi)_2 = 2(\partial_t \xi, f_{\delta_0} \xi)_2. \quad (8.15)$$

Inserting the above equation for ξ , we find

$$\begin{aligned} d_t \mathcal{Q}_{\delta_0} = & (J(L_{v,\mu} \xi + \mathcal{M}_{\varphi_{v,\mu}}(\xi) + (V_y - V(y))(\xi + \varphi_{v,\mu})), f_{\delta_0} \xi)_2 \\ & + (-\dot{v} \cdot \nabla_v \varphi_{v,\mu} - \dot{\mu} \partial_\mu \varphi_{v,\mu} - (\mu - \dot{v} - V(y))J(\varphi_{v,\mu} + \xi), f_{\delta_0} \xi)_2 \\ & + ((\dot{y} - v) \cdot \nabla(\varphi_{v,\mu} + \xi), f_{\delta_0} \xi)_2. \end{aligned} \quad (8.16)$$

To simplify this expression, we note that $(Ja, r(x)a)_2 = 0$ for all bounded scalar functions $r(x)$. Similarly $2(\nabla a, ra)_2 = -(a, a \nabla r)_2$, for any scalar function $r(x)$. Thus

$$\begin{aligned} d_t \mathcal{Q}_{\delta_0} = & (J(L_{v,\mu} \xi + \mathcal{M}_{\varphi_{v,\mu}}(\xi)), f_{\delta_0} \xi)_2 + (J(V_y - V(y))\varphi_{v,\mu}, f_{\delta_0} \xi)_2 \\ & + (\dot{y} - v) \cdot ((\nabla \varphi_{v,\mu}, f_{\delta_0} \xi)_2 - \frac{1}{2}(\xi, \xi \nabla f_{\delta_0})_2) - (\mu - \dot{v} - V(y))(J\varphi_{v,\mu}, f_{\delta_0} \xi)_2 \\ & - (\dot{v} \cdot \nabla_v \varphi_{v,\mu} + \dot{\mu} \partial_\mu \varphi_{v,\mu}, f_{\delta_0} \xi)_2. \end{aligned} \quad (8.17)$$

To estimate $d_t \mathcal{Q}_{\delta_0}$, we begin with recalling the definition of α in equation (7.4); thus all the terms $\dot{y} - v, \dot{v}$, etc are bounded by $|\alpha|$. Furthermore, we note that $|\nabla f_{\delta_0}| \leq 1$. Thus

$$\begin{aligned}
 |d_t \mathcal{Q}_{\delta_0}| &\leq |(JL_{v,\mu} \xi, f_{\delta_0} \xi)_2| + |(\mathcal{M}_{\varphi_{v,\mu}}(\xi), f_{\delta_0} \xi)_2| + C\varepsilon \| |x| f_{\delta_0} \varphi_{v,\mu} \|_2 \| \xi \|_2 \\
 &\quad + |\alpha| (\| f_{\delta_0} \nabla \varphi_{v,\mu} \|_2 + \| f_{\delta_0} J \varphi_{v,\mu} \|_2 + \| f_{\delta_0} \partial_\mu \varphi_{v,\mu} \|_2 \\
 &\quad + \| f_{\delta_0} \nabla_v \varphi_{v,\mu} \|_2) \| \xi \|_2 + \frac{1}{2} \| \xi \|_2^2.
 \end{aligned} \tag{8.18}$$

By proposition 4.3, we know that all terms of the form $\| f_{\delta_0} z_j \|_2$ satisfy $\| f_{\delta_0} z_j \|_2 \leq \| |x| z_j \|_2 \leq C$, where z_j is of the form $K \varphi_{v,\mu}$ and $K \in \{ |x|, \nabla_x, J, \partial_\mu, \nabla_v \}$, and C being independent of δ_0 . Hence

$$|d_t \mathcal{Q}_{\delta_0}| \leq |(JL_{v,\mu} \xi, f_{\delta_0} \xi)_2| + |(JM_{\varphi_{v,\mu}}(\xi), f_{\delta_0} \xi)_2| + C(\varepsilon + |\alpha|) \| \xi \|_2 + \frac{1}{2} \| \xi \|_2^2. \tag{8.19}$$

To estimate the term $(L_{v,\mu} \xi, f_{\delta_0} \xi)_2$, we write down this expression in detail

$$\begin{aligned}
 (JL_{v,\mu} \xi, f_{\delta_0} \xi)_2 &= \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \xi, f_{\delta_0} \xi \right)_2 = \left(\begin{pmatrix} L_{21} & L_{22} \\ -L_{11} & -L_{12} \end{pmatrix} \xi, f_{\delta_0} \xi \right)_2 \\
 &= (L_{21} \xi^{(1)} + L_{22} \xi^{(2)}, f_{\delta_0} \xi^{(1)})_2 - (L_{11} \xi^{(1)} + L_{12} \xi^{(2)}, f_{\delta_0} \xi^{(2)})_2.
 \end{aligned} \tag{8.20}$$

Inserting the explicit expressions for the operator $L_{v,\mu}$ yields

$$\begin{aligned}
 (L_{12} \xi^{(2)}, f_{\delta_0} \xi^{(2)})_2 &= (v \cdot \nabla \xi^{(2)}, f_{\delta_0} \xi^{(2)})_2 - \left(\frac{2}{|x|} * (\varphi_{v,\mu}^{(2)} \xi^{(2)}) \varphi_{v,\mu}^{(1)}, f_{\delta_0} \xi^{(2)} \right)_2 \\
 &= -\frac{v}{2} \cdot (\xi^{(2)}, \xi^{(2)} \nabla f_{\delta_0})_2 - \left(\frac{2}{|x|} * (\varphi_{v,\mu}^{(2)} \xi^{(2)}), f_{\delta_0} \varphi_{v,\mu}^{(1)} \xi^{(2)} \right)_2.
 \end{aligned} \tag{8.21}$$

Once again we observe that $\sup_x |\nabla f_{\delta_0}| \leq 1$ holds, independent of δ_0 . Using the Hardy–Littlewood–Sobolev inequality we thus find

$$\begin{aligned}
 |(L_{12} \xi^{(2)}, f_{\delta_0} \xi^{(2)})_2| &\leq 2|v| \| \xi^{(2)} \|_2^2 + C \| \varphi_{v,\mu}^{(2)} \xi^{(2)} \|_{12/5} \| f_{\delta_0} \varphi_{v,\mu}^{(1)} \xi^{(2)} \|_{12/5} \\
 &\leq 2|v| \| \xi^{(2)} \|_2^2 + C \| \xi^{(2)} \|_{12/5}^2 (\| \varphi_{v,\mu}^{(2)} \|_\infty^{12/5} \| f_{\delta_0} \varphi_{v,\mu}^{(1)} \|_\infty^{12/5})^{5/12}.
 \end{aligned} \tag{8.22}$$

Observe that $f_{\delta_0} \leq |x|$, $\sup |x| \varphi_{v,\mu} < C$ and that $\| \xi \|_{12/5} \leq C \| \xi \|_{H^{\frac{1}{2}}}$. We find that

$$|(L_{12} \xi^{(2)}, f_{\delta_0} \xi^{(2)})_2| \leq 2|v| \| \xi^{(2)} \|_2^2 + C \| \xi^{(2)} \|_{H^{\frac{1}{2}}}^2. \tag{8.23}$$

For the L_{21} -term we have

$$\begin{aligned}
 (L_{21} \xi^{(1)}, f_{\delta_0} \xi^{(1)})_2 &= (-v \cdot \nabla \xi^{(1)}, f_{\delta_0} \xi^{(1)})_2 - \left(\frac{2}{|x|} * (\varphi_{v,\mu}^{(1)} \xi^{(1)}) \varphi_{v,\mu}^{(2)}, f_{\delta_0} \xi^{(1)} \right)_2 \\
 &= \frac{v}{2} \cdot (\xi^{(1)}, \xi^{(1)} \nabla f_{\delta_0})_2 - \left(\frac{2}{|x|} * (\varphi_{v,\mu}^{(1)} \xi^{(1)}), f_{\delta_0} \varphi_{v,\mu}^{(2)} \xi^{(1)} \right)_2.
 \end{aligned} \tag{8.24}$$

Similarly to the estimate of L_{12} , we obtain that

$$|(L_{21} \xi^{(1)}, f_{\delta_0} \xi^{(1)})_2| \leq 2|v| \| \xi^{(1)} \|_2^2 + C \| \xi^{(1)} \|_{H^{\frac{1}{2}}}^2. \tag{8.25}$$

The last two terms yield

$$\begin{aligned}
 (L_{22}\xi^{(2)}, f_{\delta_0}\xi^{(1)})_2 - (L_{11}\xi^{(1)}, f_{\delta_0}\xi^{(2)})_2 &= \left(\sqrt{-\Delta + m^2}\xi^{(2)} + (-m + \mu)\xi^{(2)} \right. \\
 &\quad - \left. \left(\frac{1}{|x|} * |\varphi_{v,\mu}|^2 \right) \xi^{(2)} - \left(\frac{2}{|x|} * (\xi^{(2)}\varphi_{v,\mu}^{(2)}) \right) \varphi_{v,\mu}^{(2)}, f_{\delta_0}\xi^{(1)} \right)_2 \\
 &\quad - \left(\sqrt{-\Delta + m^2}\xi^{(1)} + (-m + \mu)\xi^{(1)} - \left(\frac{1}{|x|} * |\varphi_{v,\mu}|^2 \right) \xi^{(1)} \right. \\
 &\quad - \left. \left(\frac{2}{|x|} * (\xi^{(1)}\varphi_{v,\mu}^{(1)}) \right) \varphi_{v,\mu}^{(1)}, f_{\delta_0}\xi^{(2)} \right)_2 = ([f_{\delta_0}, \sqrt{-\Delta + m^2}]\xi^{(2)}, \xi^{(1)})_2 \\
 &\quad - \left(\left(\frac{2}{|x|} * (\xi^{(2)}\varphi_{v,\mu}^{(2)}) \right) \varphi_{v,\mu}^{(2)}, f_{\delta_0}\xi^{(1)} \right)_2 + \left(\left(\frac{2}{|x|} * (\xi^{(1)}\varphi_{v,\mu}^{(1)}) \right) \varphi_{v,\mu}^{(1)}, f_{\delta_0}\xi^{(2)} \right)_2,
 \end{aligned} \tag{8.26}$$

where $[A, B] = AB - BA$. The last two terms are both bounded by $C\|\xi\|_{H^{\frac{1}{2}}}^2$, analogous to the estimate for L_{12} . For the first term we use [2, lemma A.3], see also Stein [34], that shows

$$|([f_{\delta_0}, \sqrt{-\Delta + m^2}]\xi^{(2)}, \xi^{(1)})_2| \leq C\|\xi\|_2^2, \tag{8.27}$$

with C independent of δ_0 . Thus we find, using that $|v| \leq 1$,

$$|(JL_{v,\mu}\xi, f_{\delta_0}\xi)_2| \leq C\|\xi\|_{H^{\frac{1}{2}}}^2. \tag{8.28}$$

The last term to estimate is $(\mathcal{M}_{\varphi_{v,\mu}}(\xi), f_{\delta_0}\xi)_2$. To this end, we recall from proposition 7.1 that

$$-\mathcal{M}_{\varphi_{v,\mu}}(\xi) = \frac{1}{|x|} * |\xi|^2\varphi_{v,\mu} + \frac{2}{|x|} * (\varphi_{v,\mu} \cdot \xi)\xi + \frac{1}{|x|} * |\xi|^2\xi. \tag{8.29}$$

In this case we cannot use the Hardy–Littlewood–Sobolev estimate. But instead we can use the Kato [35, section V.5.4, equation (5.33)] inequality:

$$\int_{\mathbb{R}^3} \frac{1}{|y|} |\xi(y) \cdot \mathbf{u}(y)| \, dy \leq C\|\mathbf{u}\|_{H^{\frac{1}{2}}}\|\xi\|_{H^{\frac{1}{2}}}, \tag{8.30}$$

see, e.g., [8]. We estimate $(\mathcal{M}_{\varphi_{v,\mu}}(\xi), f_{\delta_0}\xi)_2$ as follows:

$$\begin{aligned}
 |(\mathcal{M}_{\varphi_{v,\mu}}(\xi), f_{\delta_0}\xi)_2| &\leq \|f_{\delta_0}|\xi|^2\|_1 \sup_x \int_{\mathbb{R}^3} \left(\frac{1}{|y|} (|\xi(x+y)|^2 + |\xi(y+x) \cdot \varphi_{v,\mu}(y+x)|) \right) dy \\
 &\quad + \|f_{\delta_0}\varphi_{v,\mu} \cdot \xi\|_1 \sup_x \int_{\mathbb{R}^3} \frac{1}{|y|} |\xi(y+x)|^2 dy.
 \end{aligned} \tag{8.31}$$

Using (8.30) we find

$$|(\mathcal{M}_{\varphi_{v,\mu}}(\xi), f_{\delta_0}\xi)_2| \leq \|f_{\delta_0}|\xi|^2\|_1 (\|\xi\|_{H^{\frac{1}{2}}}^2 + \|\xi\|_{H^{\frac{1}{2}}}\|\varphi_{v,\mu}\|_{H^{\frac{1}{2}}}) + \|f_{\delta_0}\varphi_{v,\mu}\xi\|_1 \|\xi\|_{H^{\frac{1}{2}}}^2. \tag{8.32}$$

Note that $\|f_{\delta_0}|\xi|^2\|_1 = \mathcal{Q}_{\delta_0}(t)$ and $\|f_{\delta_0}\varphi_{v,\mu} \cdot \xi\|_1 \leq C\|\xi\|_2$, where C is independent of δ_0 . Thus

$$|(\mathcal{M}_{\varphi_{v,\mu}}(\xi), f_{\delta_0}\xi)_2| \leq c\|\xi\|_{H^{\frac{1}{2}}}^3 + c\|\xi\|_{H^{\frac{1}{2}}}(1 + \|\xi\|_{H^{\frac{1}{2}}})\mathcal{Q}_{\delta_0}(t), \tag{8.33}$$

where c is independent of δ_0 .

Inserting the results in (8.28) and (8.33) into (8.19) yields

$$|d_t \mathcal{Q}_{\delta_0}| \leq C\|\xi\|_{H^{\frac{1}{2}}}^2 + c\|\xi\|_{H^{\frac{1}{2}}}^3 + c\|\xi\|_{H^{\frac{1}{2}}}(1 + \|\xi\|_{H^{\frac{1}{2}}})\mathcal{Q}_{\delta_0} + C(\varepsilon + |\alpha|)\|\xi\|_2. \tag{8.34}$$

This concludes the proof of the proposition.

9. Estimates of the Lyapunov functional from below

In this section, we define a Lyapunov functional $\mathcal{S} = \mathcal{S}(t)$ as

$$\mathcal{S} := \mathcal{U}_{\mu,v,y}(\psi) - \mathcal{U}_{\mu,v,y}(\varphi_\zeta). \tag{9.1}$$

Here $\mathcal{U}_{\mu,v,y}$ is defined by

$$\mathcal{U}_{\mu,v,y}(\psi) := (\mu - V(y))\mathcal{N}(\psi) - \frac{1}{2}v \cdot (\nabla J\psi, \psi)_2 + \mathcal{H}_V(\psi), \tag{9.2}$$

where $\mathcal{N}(\psi) := \frac{1}{2}\|\psi\|_2^2$ and

$$\mathcal{H}_V(\psi) := \frac{1}{2}(\psi, (\sqrt{-\Delta + m^2} - m)\psi)_2 + \frac{1}{2}(V\psi, \psi)_2 - \left(\frac{1}{4|x|} * |\psi|^2, |\psi|^2\right)_2. \tag{9.3}$$

The function $\mathcal{U}_{\mu,v,y}$ is a linear combination of conserved and almost conserved quantities, \mathcal{H}_V , \mathcal{N} and \mathcal{P} .

The above parameters $\zeta = (y, v, \vartheta, \mu)$ are chosen such that if ψ is decomposed then $\zeta = \zeta(\psi)$ (see proposition 6.1, for the construction of ζ). In this section we show that this Lyapunov functional is coercive up to small corrections. This will be used to bound the perturbations ξ from above.

We recall the notation and a result shown in previous sections: if $\psi \in U_\delta(\mathbb{Z}_2)$ then, under assumption 2.1, there exists a unique decomposition of $\psi \rightarrow (\zeta = (y, v, \vartheta, \mu), \xi)$ by $\zeta \in \mathbb{Z}_1$ (see proposition 6.1).

We have the following result.

Proposition 9.1. *Let $\psi \in U_\delta(\mathbb{Z}_2)$ and let assumption 2.1 be satisfied. Denote the unique decomposition of ψ by (ζ, ξ) . Let the external potential V satisfy (3.1) for some number $\varepsilon > 0$ and let $\rho = \rho(r_1, I_1) > 0$, be defined as in proposition B.1. Then*

$$\mathcal{S} \geq \frac{7}{8}\rho\|\xi\|_{H^{\frac{1}{2}}}^2 - C\varepsilon\mathcal{Q} - C\varepsilon^2 - C\|\xi\|_{H^{\frac{1}{2}}}^4, \tag{9.4}$$

with $\mathcal{Q} := (\xi, |x|\xi)_2$.

Remark 9.2. The major limitation of ε appears here. The lower bound, ρ , depends on the distance from zero to the start of the essential spectrum and is hence of size ℓ_{sol}^{-1} , whereas the upper bound (next section) is given in terms of gradients of the potential, initial distances both parametrized by ε together with the so far unknown size of the perturbation ξ . Thus the requirement that $\varepsilon = \ell_{\text{sol}}/\ell_{\text{pot}} \ll 1$ arises here.

Proof. Using the decomposition (proposition 6.1) of ψ into ζ, ξ , with $(y, v, \vartheta, \mu) = \zeta := \zeta(\psi)$ we can write ψ as $U_\delta(\mathbb{Z}_2) \ni \psi(x, \cdot) = e^{-J\vartheta}(\varphi_{v,\mu}(x - y) + \xi(x - y, \cdot))$. Inserting this into \mathcal{S} gives

$$\begin{aligned} \mathcal{S} &= \mathcal{U}_{\mu,v,y}(\psi) - \mathcal{U}_{\mu,v,y}(\varphi_\zeta) = (\mathcal{U}'_{\mu,v,y}(\varphi_{v,\mu}(\cdot - y)), \xi(\cdot - y))_2 \\ &\quad + \frac{1}{2}(\mathcal{U}''_{\mu,v,y}(\varphi_{v,\mu}(\cdot - y))\xi(\cdot - y), \xi(\cdot - y))_2 + \mathcal{R}_{\mu,v}(\xi, \varphi_{v,\mu}) \\ &= A + B + \mathcal{R}_{\mu,v}(\xi, \varphi_{v,\mu}). \end{aligned} \tag{9.5}$$

Here we define A as

$$A := (\mathcal{U}'_{\mu,v,y}(\varphi_{v,\mu}(\cdot - y)), \xi(\cdot - y))_2 \tag{9.6}$$

and B as

$$B := \frac{1}{2}(\mathcal{U}''_{\mu,v,y}(\varphi_{v,\mu}(\cdot - y))\xi(\cdot - y), \xi(\cdot - y))_2. \tag{9.7}$$

The remainder, $\mathcal{R}_{\mu,v}(\xi, \varphi_{v,\mu})$, is defined as

$$\begin{aligned} \mathcal{R}_{\mu,v}(\xi, \varphi_{v,\mu}) := & \frac{1}{4} \left(\frac{1}{|x|} * (|\varphi_{v,\mu} + \xi|^2), |\varphi_{v,\mu} + \xi|^2 \right)_2 - \frac{1}{4} \left(\frac{1}{|x|} * (|\varphi_{v,\mu}|^2), |\varphi_{v,\mu}|^2 \right)_2 \\ & - \left(\frac{1}{|x|} * (|\varphi_{v,\mu}|^2), \varphi_{v,\mu} \cdot \xi \right)_2 - \frac{1}{2} \left(\frac{1}{|x|} * (|\varphi_{v,\mu}|^2), |\xi|^2 \right)_2 \\ & - \left(\frac{1}{|x|} * (\varphi_{v,\mu} \cdot \xi), \varphi_{v,\mu} \cdot \xi \right)_2. \end{aligned} \tag{9.8}$$

$\mathcal{R}_{\mu,v}$ can also be defined directly from (9.5) as the remainder of the given Taylor expansion of $\mathcal{U}_{\mu,v,y}(\psi)$ around $\varphi_{v,\mu}$ to second order. Thus the remaining term contains only the Taylor expansion of the nonlinear term in $\mathcal{U}_{\mu,v,y}$ which is what is written out above in detail. By expansion of the polynomials, $\mathcal{R}_{\mu,v}$ simplifies to

$$\mathcal{R}_{\mu,v}(\xi, \varphi_{v,\mu}) = - \left(\frac{1}{|x|} * |\xi|^2, \varphi_{v,\mu} \cdot \xi \right)_2 - \frac{1}{4} \left(\frac{1}{|x|} * |\xi|^2, |\xi|^2 \right)_2. \tag{9.9}$$

We now proceed to estimate the terms A , B and $\mathcal{R}_{\mu,v}$. We begin with $\mathcal{R}_{\mu,v}$. The Hardy–Littlewood–Sobolev inequality yields

$$|\mathcal{R}_{\mu,v}(\xi, \varphi_{v,\mu})| \leq c(\|\xi\|_{12/5}^4 + \|\xi\|_{12/5}^3). \tag{9.10}$$

From the Sobolev inequality we have $\|f\|_{12/5} \leq C\|f\|_{H^{1/4}} \leq C\|f\|_{H^{1/2}}$. Hence,

$$|\mathcal{R}_{\mu,v}(\xi, \varphi_{v,\mu})| \leq c'(\|\xi\|_{H^{1/2}}^4 + \|\xi\|_{H^{1/2}}^3). \tag{9.11}$$

Cauchy’s inequality in the form $2ab \leq \eta a^2 + \eta^{-1}b^2$ finally yields that

$$|\mathcal{R}_{\mu,v}(\xi, \varphi_{v,\mu})| \leq C\|\xi\|_{H^{1/2}}^4 + \frac{\rho}{16}\|\xi\|_{H^{1/2}}^2. \tag{9.12}$$

To estimate A as defined in (9.6), we relate it to $\mathcal{E}'_{v,\mu}$. The functional $\mathcal{E}_{v,\mu}$ is defined in (2.17) as

$$\mathcal{E}_{v,\mu}(\psi) := \mu\mathcal{N}(\psi) - \frac{1}{2}v \cdot (\nabla J\psi, \psi)_2 + \mathcal{H}_{V=0}(\psi). \tag{9.13}$$

All ground states $\varphi_{v,\mu}$ satisfy (2.19), that is

$$\mathcal{E}'_{v,\mu}(\varphi_{v,\mu}) = 0. \tag{9.14}$$

We write out the terms in A explicitly and identify $\mathcal{E}'_{v,\mu}(\varphi_{v,\mu})$. This gives

$$\begin{aligned} A = & (\mathcal{U}'_{\mu,v,y}(\varphi_{v,\mu}(\cdot - y)), \xi(\cdot - y))_2 = (\mu - V(y))(\varphi_{v,\mu}, \xi)_2 - v \cdot (\nabla J\varphi_{v,\mu}, \xi)_2 \\ & + (\mathcal{H}'_{V=0}(\varphi_{v,\mu}), \xi)_2 + (V_y\varphi_{v,\mu}, \xi)_2 = (\mathcal{E}'_{v,\mu}(\varphi_{v,\mu}), \xi)_2 + (V_y - V(y))\varphi_{v,\mu}, \xi)_2. \end{aligned} \tag{9.15}$$

Using that $\mathcal{E}'_{v,\mu}(\varphi_{v,\mu}) = 0$ and $(\varphi_{v,\mu}, \xi)_2 = 0$, together with that V satisfies (3.1) for some small ε , we find that

$$|A| = |(V_y - V(y))\varphi_{v,\mu}, \xi)_2| \leq c_1\varepsilon\|\xi\|_2. \tag{9.16}$$

Next we estimate B defined in (9.7). We also rewrite B in terms of $L_{v,\mu} := \mathcal{E}''_{v,\mu}(\varphi_{v,\mu})$. A calculation, similar to the one above, shows the relation

$$(\xi(\cdot - y), \mathcal{U}''_{\mu,v,y}(\varphi_{v,\mu}(\cdot - y))\xi(\cdot - y))_2 = (\xi, L_{v,\mu}\xi)_2 + (\xi, (V_y - V(y))\xi)_2. \tag{9.17}$$

Denote

$$B_1 := (\xi, (V_y - V(y))\xi)_2. \tag{9.18}$$

On the space where w is symplectically orthogonal to $T_{\varphi_{v,\mu}}M$, we have

$$(\xi, L_{v,\mu}\xi)_2 \geq \rho \|\xi\|_{H^{\frac{1}{2}}}^2, \tag{9.19}$$

which is shown in [appendix B](#). Note that ρ here and in [proposition B.1](#) depends only on r_1 and I_1 .

To bound B_1 , we expand V_y around y to obtain

$$|B_1| \leq C\varepsilon \| |x|^{1/2} \xi \|_2^2. \tag{9.20}$$

Hence B obeys the lower bound

$$B \geq \rho \|\xi\|_{H^{\frac{1}{2}}}^2 - C\varepsilon \| |x|^{1/2} \xi \|_2^2. \tag{9.21}$$

To complete the proof, we use estimates [\(9.12\)](#), [\(9.16\)](#) and [\(9.21\)](#) to obtain

$$\mathcal{S} \geq \rho \|\xi\|_{H^{\frac{1}{2}}}^2 - C\varepsilon \| |x|^{1/2} \xi \|_2^2 - c_1\varepsilon \|\xi\|_2 - C \|\xi\|_{H^{\frac{1}{2}}}^4 - \frac{\rho}{16} \|\xi\|_{H^{\frac{1}{2}}}^2. \tag{9.22}$$

By using $c_1\varepsilon \|\xi\|_2 \leq 4c_1^2\varepsilon^2/\rho + (\rho/16)\|\xi\|_2^2$ in the above equation we conclude the proof. \square

10. Estimates of the Lyapunov functional from above

In this section we show that the above-defined Lyapunov functional \mathcal{S} is almost conserved, to cubic order in terms of small quantities. First, we recall that mass and energy are conserved and that the momentum satisfies the Ehrenfest identity, i.e.

$$d_t \mathcal{N} = 0, \quad d_t (J \nabla \psi, \psi)_2 = -(\psi, (\nabla V) \psi)_2, \quad d_t \mathcal{H}_V = 0. \tag{10.1}$$

The conservation laws are proved in [\[3\]](#) and for Ehrenfest’s lemma see the comment after [\(2.16\)](#). Once again, recall that the Lyapunov–Schmidt functional is defined as

$$\mathcal{S} := \mathcal{U}_{\mu,v,y}(\psi) - \mathcal{U}_{\mu,v,y}(\varphi_\zeta), \tag{10.2}$$

with $\zeta = (y, v, \vartheta, \mu) = \zeta(\psi)$, provided that ζ exists (see [proposition 6.1](#)), and where

$$\mathcal{U}_{\mu,v,y}(\psi) := (\mu - V(y))\mathcal{N}(\psi) - \frac{1}{2}v \cdot (\nabla J \psi, \psi)_2 + \mathcal{H}_V(\psi). \tag{10.3}$$

We can now state the following result.

Proposition 10.1. *Let assumption [2.1](#) be satisfied and let ψ be a solution to [\(2.15\)](#), with initial condition $\psi_0 \in U_\varepsilon(\mathbb{Z}_3)$, $\varepsilon \leq \delta$. The decomposition of ψ , which exists for some times, is denoted by $(\zeta = (y, v, \vartheta, \mu), \xi)$. Let $\alpha := (\dot{y} - v, \dot{v}, \mu - \vartheta - V(y), \dot{\mu})$, see [equation \(7.4\)](#), and let the external potential V satisfy [\(3.1\)](#) for some small parameter $\varepsilon > 0$. Finally, let $\mathcal{S}(t)$ be defined as above. Then*

$$|\mathcal{S}(t)| \leq |\mathcal{S}(0)| + Ct \sup_{s \leq t} ((\varepsilon + |\alpha(s)|)(\varepsilon^2 + \|\xi(\cdot, s)\|_{H^{\frac{1}{2}}}^2) + \|\xi(\cdot, s)\|_{H^{\frac{1}{2}}}^3 (1 + \|\xi(\cdot, s)\|_{H^{\frac{1}{2}}}^2)), \tag{10.4}$$

where C depends only on r_j and I_j , $j = 1, 2, 3$.

Using [proposition 6.1](#), we have the following corollary.

Corollary 10.2. *In addition to the assumptions in [proposition 10.1](#), we assume that $\|\psi_0 - \varphi_{\zeta_0}\|_{\tilde{\Gamma}} \leq \varepsilon < \delta$, for ε sufficiently small. Then*

$$|\mathcal{S}(t)| \leq C\varepsilon^2 + Ct \sup_{s \leq t} ((\varepsilon + |\alpha(s)|)(\varepsilon^2 + \|\xi(\cdot, s)\|_{H^{\frac{1}{2}}}^2) + \|\xi(\cdot, s)\|_{H^{\frac{1}{2}}}^3 (1 + \|\xi(\cdot, s)\|_{H^{\frac{1}{2}}}^2)). \tag{10.5}$$

Proof. Expanding $\mathcal{S}(0) = \mathcal{U}_{\mu,v,y}(\psi) - \mathcal{U}_{\mu,v,y}(\varphi_\zeta)|_{t=0}$ around the soliton yields

$$|\mathcal{S}(0)| \leq |((V_y - V(y))\varphi_{v,\mu}, \xi)_2| + |(L_{v_0,\mu_0}\xi, \xi)_2| + |((V_y - V(y))\xi, \xi)_2| + |\mathcal{R}_{\mu,v}(\xi, \varphi_{v_0,\mu_0})|_{t=0}. \tag{10.6}$$

Here $\mathcal{R}_{\mu,v}$ is defined in (9.9). Using estimate (9.12) yields

$$|\mathcal{S}(0)| \leq C(\varepsilon^2 + (1 + \varepsilon)\|\xi\|_{H^{\frac{1}{2}}}^2 + \|\xi\|_{H^{\frac{1}{2}}}^4)|_{t=0}. \tag{10.7}$$

To estimate $\|\xi\|_{H^{\frac{1}{2}}}|_{t=0}$, we recall from proposition 6.1(ii) with $\psi_0 \in U_\varepsilon(Z_3)$ that

$$\|\xi\|_{H^{\frac{1}{2}}}|_{t=0} = \|\psi_0 - \varphi_{\mathcal{S}(\psi_0)}\|_{H^{\frac{1}{2}}} \leq \|\psi_0 - \varphi_{\zeta_0}\|_{H^{\frac{1}{2}}} + \|\varphi_{\zeta_0} - \varphi_{\mathcal{S}(\psi_0)}\|_{H^{\frac{1}{2}}} \leq C\varepsilon. \tag{10.8}$$

Here C depends only on I_j and r_j , $j = 1, 2, 3$. Since $\varepsilon^4 \leq C\varepsilon^2$, we find that

$$|\mathcal{S}(0)| \leq C\varepsilon^2, \tag{10.9}$$

whenever ε is sufficiently small. □

Proof of proposition 10.1. The proof of this proposition is a straightforward calculation. Using that

$$\mathcal{S}(t) - \mathcal{S}(0) = \int_0^t d_s \mathcal{S}(s) ds, \tag{10.10}$$

we find

$$|\mathcal{S}(t)| \leq |\mathcal{S}(0)| + t \sup_{s \leq t} |d_s \mathcal{S}(s)|. \tag{10.11}$$

Thus the desired result corresponds to controlling $|d_t \mathcal{S}|$ in terms which are of third order or higher in ε , $|\alpha|$ and $\|\xi\|_{H^{\frac{1}{2}}}$.

By proposition 7.5 and $\psi_0 \in U_\delta(Z_3)$ there is some positive time such that the solution ψ to (2.15) satisfies $\psi \in U_\delta(Z_2)$ to (2.15) and that it has a unique decomposition into ζ, ξ .

We now calculate the time derivative of $\mathcal{U}_{\mu,v,y}(\psi)$ for a solution ψ to (2.15). By (10.1), we find

$$d_t \mathcal{U}_{\mu,v,y}(\psi) = \mathcal{N}(\psi) d_t(\mu - V(y)) - \frac{1}{2} \dot{v} \cdot (J \nabla \psi, \psi)_2 + \frac{1}{2} v \cdot (\psi, \nabla V \psi)_2. \tag{10.12}$$

Here μ, v and y are taken from the decomposition of ψ .

Next, we rewrite it in terms of the $\mathcal{E}_{v,\mu}$ functional for the (boosted) solitary waves:

$$\begin{aligned} \mathcal{U}_{\mu,v,y}(\varphi_{v,\mu}(\cdot - y)) &= \mu \mathcal{N}(\varphi_{v,\mu}) - \frac{1}{2} v \cdot (J \nabla \varphi_{v,\mu}, \varphi_{v,\mu})_2 + \mathcal{H}_{V=0}(\varphi_{v,\mu}) \\ &\quad + (\varphi_{v,\mu}(V_y - V(y)), \varphi_{v,\mu})_2 = \mathcal{E}_{v,\mu}(\varphi_{v,\mu}) + (R_V \varphi_{v,\mu}, \varphi_{v,\mu})_2, \end{aligned} \tag{10.13}$$

where we used the symmetry properties of $\varphi_{v,\mu}$ to conclude that $|\varphi_{v,\mu}|^2$ is even in all directions and hence $(\varphi_{v,\mu} x_j, \varphi_{v,\mu})_2 = 0$, for $j = 1, 2, 3$. Thus

$$d_t \mathcal{U}_{\mu,v,y}(\varphi_{v,\mu}(\cdot - y)) = \dot{\mu} \mathcal{N}(\varphi_{v,\mu}) - \frac{1}{2} \dot{v} \cdot (J \nabla \varphi_{v,\mu}, \varphi_{v,\mu})_2 + d_t (R_V \varphi_{v,\mu}, \varphi_{v,\mu})_2, \tag{10.14}$$

where we used that $\mathcal{E}'_{v,\mu}(\varphi_{v,\mu}) = 0$.

Subtracting (10.14) from (10.12), using the orthogonality relations $(\varphi_{v,\mu}, \xi)_2 = 0$ and $(J \nabla \varphi_{v,\mu}, \xi)_2 = 0$, gives

$$\begin{aligned} d_t \mathcal{S} &= \frac{1}{2} \|\xi\|_2^2 \dot{\mu} - \frac{1}{2} \dot{v} \cdot (J \nabla \xi, \xi)_2 + \frac{1}{2} v \cdot (\psi, \nabla V \psi)_2 - \dot{y} \cdot \nabla V(y) \mathcal{N}(\psi) \\ &\quad - d_t (R_V \varphi_{v,\mu}, \varphi_{v,\mu})_2. \end{aligned} \tag{10.15}$$

The first two terms are of cubical order, quadratic in $\|\xi\|_{H^{\frac{1}{2}}}$ and linear in α ; recall the definition of α in (7.4). The last term is also of third order or higher. Indeed, let

$$A_2 := d_t (R_V \varphi_{v,\mu}, \varphi_{v,\mu})_2. \tag{10.16}$$

Then

$$|A_2| = |\dot{y} \cdot (\varphi_{v,\mu}, \nabla_y R_V, \varphi_{v,\mu})_2 + (R_V \varphi_{v,\mu}, \dot{\mu} \partial_\mu \varphi_{v,\mu} + \dot{v} \cdot \nabla_v \varphi_{v,\mu})_2|. \tag{10.17}$$

To bound A_2 , we recall that $|v| \leq 1$ and that $\dot{y}_j = v_j - Y_j(\zeta, \xi)$, $j = 1, 2, 3$ (see (7.9)) together with equation (7.5). This gives the estimate for Y_j ; see proposition 7.1. We thus find

$$|A_2| \leq C\varepsilon^2(|\alpha| + \varepsilon|Y|). \tag{10.18}$$

The middle two terms of (10.15) are also of at least cubic order. Indeed, let

$$A_1 := \frac{1}{2}v \cdot (\psi, \nabla V \psi)_2 - \dot{y} \cdot \nabla V(y)\mathcal{N}(\psi). \tag{10.19}$$

Decomposing ψ gives

$$\begin{aligned} A_1 &= \frac{1}{2}v \cdot ((\varphi_{v,\mu}, \nabla V_y \varphi_{v,\mu})_2 + 2(\xi, \nabla V_y \varphi_{v,\mu})_2 + (\xi, \nabla V_y \xi)_2) \\ &\quad - \frac{1}{2}\dot{y} \cdot \nabla V(y)(\|\xi\|_2^2 + \|\varphi_{v,\mu}\|_2^2). \end{aligned} \tag{10.20}$$

Using the orthogonality relation $(\xi, \varphi_{v,\mu})_2 = 0$ and $(\varphi_{v,\mu}, x_j \varphi_{v,\mu})_2 = 0$, we find

$$\begin{aligned} A_1 &= \frac{1}{2}v \cdot ((\varphi_{v,\mu}, \nabla_y R_V \varphi_{v,\mu})_2 + (\xi, (\nabla V_y - \nabla V(y))\xi)_2) \\ &\quad + \frac{1}{2}(v - \dot{y}) \cdot \nabla V(y)(\|\xi\|_2^2 + \|\varphi_{v,\mu}\|_2^2) + v \cdot (\xi, (\nabla V_y - \nabla V(y))\varphi_{v,\mu})_2. \end{aligned} \tag{10.21}$$

Hence,

$$|A_1| \leq C\varepsilon(\varepsilon^2 + \|\xi\|_2^2 + |Y|(1 + \|\xi\|_{H^{\frac{1}{2}}}^2)), \tag{10.22}$$

with Y defined as in proposition 7.1. From this, we infer

$$|d_t \mathcal{S}| \leq C|\alpha| \|\xi\|_{H^{\frac{1}{2}}}^2 + |A_1| + |A_2|. \tag{10.23}$$

Inserting the above estimates for A_1 and A_2 gives

$$|d_t \mathcal{S}| \leq C(\varepsilon + |\alpha|)(\|\xi\|_{H^{\frac{1}{2}}}^2 + \varepsilon^2) + C\varepsilon|Y|(1 + \varepsilon^2 + \|\xi\|_{H^{\frac{1}{2}}}^2) \tag{10.24}$$

Note that $\varepsilon \leq C$; inserting the bound of $|Y|$, given in (7.5), we simplify the above result to obtain

$$|d_t \mathcal{S}| \leq C(\varepsilon + |\alpha|)(\|\xi\|_{H^{\frac{1}{2}}}^2 + \varepsilon^2) + C\|\xi\|_{H^{\frac{1}{2}}}^3 (1 + \|\xi\|_{H^{\frac{1}{2}}}^2) \tag{10.25}$$

for some constant $0 < C < \infty$ depending only on r_j and I_j , $j = 1, 2, 3$. □

11. Proof of theorem 3.1

In this section we use the lower and upper bounds on the Lyapunov functional together with the modulation equations to bound $\|\xi\|_{\tilde{\Gamma}}$ and $|\alpha|$.

Proof of theorem 3.1. Theorem 3.1 assumes that assumption 2.1 is satisfied and that the external potential satisfies (3.1) for some $\varepsilon > 0$. Furthermore, we require that the initial condition ψ_0 satisfies the inequality

$$\|\psi_0 - \varphi_{\zeta_0}\|_{\tilde{\Gamma}} \leq \varepsilon, \tag{11.1}$$

for some $\zeta_0 \in Z_3$. By proposition 7.5 $\psi(\cdot, t) \in U_\delta(Z_2)$, for some δ and up to some time T_δ . Here U_δ is constructed in proposition 6.1. Thus all the assumptions for corollary 10.2 are satisfied, and we obtain

$$|\mathcal{S}(t)| \leq C\varepsilon^2 + Ct \sup_{s \leq t} f(s), \tag{11.2}$$

where

$$f(s) = (\varepsilon + |\alpha(s)|)(\varepsilon^2 + \|\xi(\cdot, s)\|_{H^{\frac{1}{2}}}^2) + \|\xi(\cdot, s)\|_{H^{\frac{1}{2}}}^3 (1 + \|\xi(\cdot, s)\|_{H^{\frac{1}{2}}}^2). \tag{11.3}$$

For times $t \leq T_\delta$, we can invoke proposition 9.1 and corollary 10.2 to find

$$\frac{7}{8}\rho \|\xi(\cdot, t)\|_{H^{\frac{1}{2}}}^2 \leq C\varepsilon^2 + C \sup_{s \leq t} (tf(s) + \varepsilon \| |x|^{1/2} \xi(\cdot, s) \|_2^2 + \|\xi(\cdot, s)\|_{H^{\frac{1}{2}}}^4). \tag{11.4}$$

Thus, for all such times $t \leq T_\delta$ we have the above inequality, and since the right-hand side is independent under $t \rightarrow s'$, $\sup_{s' \leq t}$, for $t \leq T_\delta$ we can also apply this to the left-hand side. This gives us

$$\frac{7}{8}\rho \sup_{s \leq t} \|\xi(\cdot, s)\|_{H^{\frac{1}{2}}}^2 \leq C\varepsilon^2 + C \sup_{s \leq t} (tf(s) + \varepsilon \| |x|^{1/2} \xi(\cdot, s) \|_2^2 + \|\xi(\cdot, s)\|_{H^{\frac{1}{2}}}^4). \tag{11.5}$$

Consider the inequality

$$t \leq \frac{\rho}{8C} \frac{1}{\varepsilon + \sup_{s \leq t} (|\alpha(s)| + \|\xi(\cdot, s)\|_{H^{\frac{1}{2}}})}. \tag{11.6}$$

This inequality implicitly defines a maximal time, T_2 , dependent on ε , the size of $|\alpha|$ and $\|\xi\|_{H^{\frac{1}{2}}}$, such that when $t \leq T_2$ the inequality holds. We now choose the minimal time of T_2 and T_δ . Since this minimal time is necessarily smaller than the right-hand side of the inequality (11.6), we can use this inequality to rewrite (11.5) as

$$\begin{aligned} \frac{7}{8}\rho \sup_{s \leq t} \|\xi(\cdot, s)\|_{H^{\frac{1}{2}}}^2 &\leq C\varepsilon^2 + \sup_{s \leq t} \left(\frac{\rho}{8} (\varepsilon^2 + 2\|\xi(\cdot, s)\|_{H^{\frac{1}{2}}}^2 + \|\xi(\cdot, s)\|_{H^{\frac{1}{2}}}^4) \right. \\ &\quad \left. + C(\varepsilon \| |x|^{1/2} \xi(\cdot, s) \|_2^2 + \|\xi(\cdot, s)\|_{H^{\frac{1}{2}}}^4) \right). \end{aligned} \tag{11.7}$$

Let T_1 be the maximal time such that for $t \leq T_1$ equation (8.3) in corollary 8.2 holds. By choosing the minimal of the three times T_δ , T_1 and T_2 we can apply the result in corollary 8.2. That is, we use

$$\sup_{s \leq t} \| |x|^{1/2} \xi(\cdot, s) \|_2^2 \leq C\varepsilon + \sup_{s \leq t} \|\xi(\cdot, s)\|_{H^{\frac{1}{2}}} \tag{11.8}$$

in (11.7). We find, for this minimal time,

$$\frac{1}{2}\rho \sup_{s \leq t} \|\xi(\cdot, s)\|_{H^{\frac{1}{2}}}^2 \leq C(\varepsilon^2 + \sup_{s \leq t} \|\xi(\cdot, s)\|_{H^{\frac{1}{2}}}^4). \tag{11.9}$$

Recalling that the initial condition is small enough we simplify the inequality to find

$$\|\xi(\cdot, t)\|_{H^{\frac{1}{2}}}^2 \leq \sup_{s \leq t} \|\xi(\cdot, s)\|_{H^{\frac{1}{2}}}^2 \leq C\varepsilon^2. \tag{11.10}$$

We now use definition (2.11) to find from (11.10) and (11.8) that

$$\|\xi\|_{\bar{F}} \leq C\varepsilon. \tag{11.11}$$

We insert the result (11.10) into the modulation equations (7.3); we find that $|\alpha|$, as defined in proposition 7.1, satisfies the inequality

$$\sup_{t \leq s} |\alpha(s)| \leq C' \sup_{s \leq t} (|\alpha(s)|\varepsilon + 2\varepsilon). \tag{11.12}$$

Choosing ε sufficiently small, i.e. $C'\varepsilon \leq 1/2$, leads to

$$\sup_{t \leq s} |\alpha(s)| \leq c'\varepsilon. \tag{11.13}$$

This inserted into (7.5) gives that $|Y_j| \leq C\varepsilon^2$ for all j and hence the finite-dimensional modulation equations are bounded by $C\varepsilon^2$. We insert the above upper bounds on $\|\xi\|_2, |\alpha|$, into the inequalities (8.3) and (11.6) that determine the times T_1 and T_2 ; both inequalities simplify to

$$t \leq \frac{c}{\varepsilon}. \tag{11.14}$$

By possibly reducing the constant c we find that $c/\varepsilon < T_\delta$, and we have proved the theorem. \square

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Appendix A. Proof of proposition 4.3(v)

Here we prove proposition 4.3(v). In [2, appendix C] we showed that the essential spectrum starts at $\mu - \mu_l(v)$, where $\mu_l(v) = (1 - \sqrt{1 - v^2})m$. For the remaining claims we have proposition A.1.

Proposition A.1. *Suppose assumption 2.1 is satisfied for the frequency μ_0 . There is a neighbourhood, $W \subset \mathbb{R}^2$, around $(0, \mu_0)$ such that*

$$\dim \text{Ker}(L_{v,\mu}) = 4, \tag{A.1}$$

for all $(|v|, \mu) \in W$. Furthermore, $L_{v,\mu}$ has exactly one negative eigenvalue, and around zero there is a gap to the next spectral point.

Proof. To prove the proposition, we begin with the point $(v, \mu) = (0, \mu_0)$; here $\varphi_\mu = (\varphi_\mu, 0)$ and $L_{v,\mu}$ reduces to $L_{\mu_0} = \text{diag}(L_{11,\mu_0}, L_{22,\mu_0})$. That assumption 2.1 implies $\dim \text{Ker}(L_{\mu_0}) = 4$ has already been shown in the proof of proposition 4.5.

For general velocities, $v \neq 0$, and frequencies, let $K := \text{Ker}(L_{v,\mu})$ and let $k = \dim K$. Equations (4.4) show that

$$J\varphi_{v,\mu}, \partial_j \varphi_{v,\mu} \in K, \tag{A.2}$$

and consequently $k \geq 4$; to show that $k = 4$, we use Kato’s perturbation of the spectrum: define the operator $A := L_{v,\mu} - L_{\mu_0} = -v \cdot \nabla J + \mathcal{E}''_{0,\mu}(\varphi_{v,\mu}) - \mathcal{E}''_{0,\mu_0}(\varphi_{\mu_0}, 0)$; it is L_{μ_0} -bounded,

$$\|A\mathbf{u}\|_2 \leq c_{|v|,\mu-\mu_0} \|\mathbf{u}\|_2 + C_{|v|,\mu-\mu_0} \|L_{\mu_0}\mathbf{u}\|_2, \tag{A.3}$$

where both constants approach zero as both $|v|$ and $|\mu - \mu_0|$ approach zero, which follows from the fact $\|\varphi_{v,\mu} - \varphi_{\mu_0}\|_{H^1} \leq (|v| + |\mu - \mu_0|)C$, for small enough $|v|$ and $|\mu - \mu_0|$.

Denote the spectral distance in $\sigma(L_{\mu_0})$ from zero to the nearest spectral point d and consider the inequality:

$$c_{|v|,\mu-\mu_0} + C_{v,\mu-\mu_0} d \leq d/2. \tag{A.4}$$

For neighbourhoods W such that the above inequality is satisfied for all $(|v|, \mu) \in W$, [35, section V.4.3] states that within the circle with centre zero and radius $d/2$ there are exactly four (repeated) eigenvalues of $L_{v,\mu}$ (since L_{μ_0} has a degeneracy four zero eigenvalue). Thus $k = \dim K \leq 4$, consequently $k = 4$. Furthermore, $\sigma(L_{v,\mu})$ has a spectral gap of at least $d/2$ from zero to the next spectral point. The circle thus separates the spectrum into three parts.

The function φ_{μ_0} is a minimizer with one constraint; thus its corresponding Hessian L_{μ_0} can have at most one negative eigenvalue, see, e.g., [23]. But by

$$(\partial_\mu \varphi_\mu, L_\mu \partial_\mu \varphi_\mu)_2 = -\mathcal{N}'(\varphi_\mu) < 0, \quad (\text{A.5})$$

it has at least one negative eigenvalue. Thus it has exactly one negative eigenvalue. The above separation of the spectrum together with the fact that the eigenvalues (of self-adjoint operators) are constrained to the real axis ensures that $L_{v,\mu}$ has exactly one negative eigenvalue. \square

Appendix B. Positivity of $(L_{v,\mu} \varphi_v, \varphi_v)_2$

In this section we show that $(\xi, L_{v,\mu} \xi)_2 \geq \rho \|\xi\|_2^2$ for $\omega(\xi, z) = 0$ for all $z \in T_{\varphi_\zeta} M_1$. From proposition 4.3 we know that $L_{v,\mu}$ has one negative eigenvalue. We have the following result.

Proposition B.1. *Under assumption 2.1 and with $|v| < r_1 < 1$ and $\mu \in I_1$ there is a $\rho > 0$ dependent only on I_1 and r_1 defined in proposition 5.1 such that if $\omega(\xi, z) = 0$ for all $z \in T_{\varphi_\zeta} M_1$, then*

$$(\xi, L_{v,\mu} \xi)_2 \geq \rho \|\xi\|_{H^{\frac{1}{2}}}^2. \quad (\text{B.1})$$

We follow the proof of proposition D.1 in [23] with necessary modifications to the pseudo-relativistic Hartree equation. But we repeat the proof here for completeness. We break the proposition into three steps.

Lemma B.2 (Step 1). *Let $X_1 := \{\xi \in H^{\frac{1}{2}} : \|\xi\|_2 = 1, (\xi, \varphi_{v,\mu})_2 = 0\}$ and $|v| \leq r_1, \mu \in I_1$. Then*

$$\inf_{\xi \in X_1} (\xi, L_{v,\mu} \xi)_2 = 0. \quad (\text{B.2})$$

Proof. Let $a := \inf_{\xi \in X_1} (\xi, L_{v,\mu} \xi)_2$. Clearly $v \leq a \leq 0$, where $v < 0$ is the negative eigenvalue of $L_{v,\mu}$. That $a \leq 0$ is clear as $\xi = J\varphi_{v,\mu} / \|\varphi_{v,\mu}\|_2 \in X_1$ yields $(\xi, L_{v,\mu} \xi)_2 = 0$. Moreover $a \neq v$. Indeed if $a = v$, then the (local) minimizer, ϕ , of (B.2) would be an eigenfunction of $L_{v,\mu}$ corresponding to the smallest eigenvalue v and $\phi \in X_1$ and $\varphi_{v,\mu} \perp \phi$. Now, since $\varphi_{v,\mu} \perp \text{Ker}(L_{v,\mu})$ and since v is the only negative eigenvalue, we conclude that $\varphi_{v,\mu}$ is in the spectral subspace of $L_{v,\mu}$ corresponding to the interval $[\delta, \infty)$ for some $\delta > 0$. Therefore $L_{v,\mu}^{-1} \varphi_{v,\mu}$ is well defined and $(\varphi_{v,\mu}, L_{v,\mu}^{-1} \varphi_{v,\mu})_2 > 0$. On the other hand the equation $L_{v,\mu} \partial_\mu \varphi_{v,\mu} = -\varphi_{v,\mu}$ implies that

$$(\varphi_{v,\mu}, L_{v,\mu}^{-1} \varphi_{v,\mu})_2 = -\mathcal{N}'(\varphi_{v,\mu}) < 0, \quad (\text{B.3})$$

which contradicts $(\varphi_{v,\mu}, L_{v,\mu}^{-1} \varphi_{v,\mu})_2 > 0$. Hence $a = v$ is impossible.

To show that $a = 0$ we use the Euler–Lagrange equations corresponding to (B.2):

$$L_{v,\mu} \xi = a\xi + b\varphi_{v,\mu}, \quad (\text{B.4})$$

where a and b are Lagrange multipliers corresponding to $\|\xi\|_2 = 1$ and $(\varphi_{v,\mu}, \xi)_2 = 0$, respectively. Assume $v < a < 0$. If $b = 0$, then a would be a negative eigenvalue in $(v, 0)$, which contradicts that v is the only negative eigenvalue. Thus $b \neq 0$. Given $v < a < 0$, we can solve the Euler–Lagrange equation as

$$\xi = b(L_{v,\mu} - a)^{-1} \varphi_{v,\mu}. \quad (\text{B.5})$$

The inner product of the above equation with $\varphi_{v,\mu}$, together with the orthogonality relation $(\varphi_{v,\mu}, \xi)_2 = 0$ and $b \neq 0$, gives

$$0 = (\varphi_{v,\mu}, (L_{v,\mu} + |a|)^{-1} \varphi_{v,\mu})_2 =: q(|a|). \quad (\text{B.6})$$

$q(\lambda)$ is analytic in $\lambda \in (0, |v|)$ and hence differentiable. Moreover it is monotonically decreasing, since

$$q'(\lambda) = -(\varphi_{v,\mu}, (L_{v,\mu} + \lambda)^{-2} \varphi_{v,\mu})_2 = -\|(L_{v,\mu} + \lambda)^{-1} \varphi_{v,\mu}\|_2^2 < 0. \tag{B.7}$$

Furthermore, by (B.3) $q(0) = (\varphi_{v,\mu}, L_{v,\mu}^{-1} \varphi_{v,\mu})_2 < 0$. Thus $q(|a|) \neq 0$, for $a \in (v, 0)$, which contradicts (B.6). Hence $a = 0$.

Lemma B.3 (Step 2). *Let $X := \{\xi \in H^{\frac{1}{2}} : \|\xi\|_2 = 1, \omega(\xi, z) = 0, \forall z \in T_{\varphi_\zeta} M_1\}$. Then*

$$\inf_{\xi \in X} (\xi, L_{v,\mu} \xi)_2 > 0. \tag{B.8}$$

Proof. The Euler–Lagrange equation corresponding to (B.8) is

$$L_{v,\mu} \xi = a \xi + \sum_k \gamma_k J z_k, \tag{B.9}$$

where $\{z_k\}$ is a basis for $T_{\varphi_\zeta} M_1$. Here a and $\{\gamma_k\}$ are the Lagrange multipliers corresponding to the constraints $\|\xi\|_2 = 1$ and $\omega(\xi, z_k) = 0 \forall k$, respectively. Note that $a = (\xi, L \xi)_2$ and that $X \subset X_1$; hence $a \geq 0$. Assume that $a = 0$ and that $\gamma_j \neq 0$ for some j . Then, by corollary 5.2, there exists a $z = \sum_{j,l} \gamma_j (\Omega_{\varphi_{v,\mu}}^{-1})_{jl} z_l \in T_{\varphi_\zeta} M_1$ such that

$$(z, L \xi)_2 = \sum_{j,k,l} \gamma_j (\Omega_{\varphi_{v,\mu}}^{-1})_{jl} (\Omega_{\varphi_{v,\mu}})_{lk} \gamma_k = \sum_j |\gamma_j|^2 > 0, \tag{B.10}$$

which contradicts $(z, L_{v,\mu} \xi)_2 = (L_{v,\mu} z, \xi)_2 = 0$. Here we have used that $z = \sum_j b_j z_j$ and z_j is either a zero-eigenfunction or an associated zero-mode for $L_{v,\mu}$. Thus either $a > 0$ or $a = 0$ and $\gamma_j = 0$. Consider the latter case. In this case

$$L_{v,\mu} \xi = 0, \tag{B.11}$$

which implies that $\xi \in \text{Ker}(L_{v,\mu})$. Since $\text{Ker}(L_{v,\mu}) \subset T_{\varphi_\zeta} M_1$, the relation $\omega(\xi, z) = 0$ for all $z \in T_{\varphi_\zeta} M_1$ contradicts the non-degeneracy of $\Omega_{\varphi_{v,\mu}}$ on M_1 (see corollary 5.2). Thus $a > 0$. \square

End of Proof (Step 3). Equation (B.8) implies that there exists a $\rho' > 0$ such that

$$(\xi, L_{v,\mu} \xi)_2 \geq \rho' \|\xi\|_2^2, \tag{B.12}$$

for some $\rho' = \rho'(\mu, v)$ and all $\xi \in X$. To improve the coercivity from L^2 to $H^{\frac{1}{2}}$, we let $0 < \delta < 1$ and estimate $(\xi, L_{v,\mu} \xi)_2$ using (B.12) as

$$(1 - \delta) \rho' \|\xi\|_2^2 + \delta (\xi, L_{v,\mu} \xi)_2 \leq (\xi, L_{v,\mu} \xi)_2. \tag{B.13}$$

Upon using the explicit form of $L_{v,\mu}$ we find that

$$(\xi, L_{v,\mu} \xi)_2 \geq (\xi, \sqrt{-\Delta + m^2} \xi)_2 - v \cdot (J \nabla \xi, \xi)_2 - C_\mu \|\xi\|_2^2, \tag{B.14}$$

where

$$C_\mu \leq |m - \mu| + C(3, 1) \|\varphi_{v,\mu}\|_3^2 + \sup_x \left| \frac{1}{|x|} * |\varphi_{v,\mu}|^2 \right|. \tag{B.15}$$

Here we have used Kato’s inequality (see (8.30)) and the Hardy–Littlewood–Sobolev inequality with sharp constant $C(3, 1)$ see, e.g., in [31, theorem 4.3]. The two estimates (B.13) and (B.14) with $\delta := \rho'(1 + \rho' + C_\mu)^{-1}$ imply

$$(\xi, L_{v,\mu} \xi)_2 \geq \rho \|\xi\|_{H^{\frac{1}{2}}}^2, \tag{B.16}$$

where $0 < \rho = \inf_{\mu \in I_1, v < r_1} (1 - |v|) \rho'(1 + \rho' + C_\mu)^{-1}$. Thus for $\mu \in I_1, |v| < r_1$, we find that ρ depends only on I_1 and r_1 . This concludes the proof of proposition B.1. \square

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