CONSTRUCTION OF THE REAL NUMBERS

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1. Motivation.

It will not come as a big surprise to anyone when I say that we need the real numbers in mathematics. We need to be able to talk precisely about real numbers and their properties, and to rigorously prove theorems whose statements and/or proofs involve the system of real numbers. We must be able to do this on an axiomatic basis.

One solution to this is to simply introduce a set of axioms for the real number system. Whether you are aware of this or not, this is what you have already been subjected to in previous courses.

There are two reasons why this is not acceptable in the long run. The first reason is that it is a question of aesthetics: if we can introduce the real numbers without accepting any further axioms than the axioms of set theory then that seems like a more satisfactory situation.

The second reason is far more important though. On a deeper level it is in fact connected with the first reason but goes in the direction of answering the following question: "How do we know that the axiomatic basis of mathematics does not allow us to prove contradictory statements?" I.e., how can we know that it is not possible to prove both a theorem and the negation of that theorem on the basis of our accepted axiomatic system?

The point is that it is usually extremely hard to answer such questions. For instance, one can prove that it is *not possible* for the system of axioms of set theory – i.e., the modern basis of all mathematics – to prove its own "consistency", that is, that there are no hidden contradictions, – unless, of course, the system does in fact contain such contradictions (for in that case one would be able to prove *any* statement on the basis of the axioms). (Do not worry too much about this though; in the unlikely event that a contradiction should turn up you can be sure that bridges will not suddenly start to collapse, or that space ships will begin to miss their destinations because of that. If a contradiction turned up we would simply have to reconsider the situation and construct a new axiomatic system that does for us what we want of it).

However, we can take a few precautions: We can be as conservative as we can when it comes to introducing new axioms in mathematics. Thus, by constructing the real numbers on the basis of the axioms that we already have we can be certain that there will not result any *new* contradictions for some deep or hidden reason.

So, how do we construct the real numbers?

The fundamental idea about the real numbers is that – whatever they are – they should be objects that can be approximated by rational numbers. For instance, if we think of real numbers as having – possibly infinite – decimal expansions then this is just one way of thinking about real numbers as "limits" of rational numbers:

For a(n) (infinite) decimal expansion is nothing but a sequence of *finite* decimal expansions where we add more and more digits in the expansion. And a finite decimal expansion is a rational number (Exercise 1).

But there is nothing special about decimal expansions: We can consider any sequences of rational numbers. The important question then is: Which sequences of rational numbers should be thought of as "approximating real numbers"? Notice that, from a formal point of view, the question is meaningless: For since we have not yet constructed the real numbers we can not attach any precise meaning to the phrase "approximating real numbers" ...

But that the question is formally meaningless does *not* mean that it is mathematically pointless or trivial. Rather, the point is that we are searching for a *good definition*, i.e., a definition that will ultimately lead to a satisfactory theory.

To make a long story short, it turns out that the notion of a "Cauchy sequence of rational numbers" – see the precise definition in the next section – not only captures much of the intuition we may have about "approximating sequences of rational numbers", but also leads to a satisfactory theory. How people came up with this definition in the first place is a question in the history of mathematics that we will not go further into here.

When we have defined what Cauchy sequences of rational numbers are, the great idea is then this: we want to think about the Cauchy sequences as sequences of approximations to real numbers but we still do not have the real numbers. How do we get them? Answer: we simply *identify* them with these sequences ...

This causes a small problem to consider: our intuition tells us that there are many different sequences approximating a real number. For instance, it seems reasonable to think of the sequences

$$(1+1, 1+\frac{1}{2}, 1+\frac{1}{4}, \ldots, 1+\frac{1}{2^n}, \ldots) \quad \text{and} \quad (1+1, 1+\frac{1}{2}, 1+\frac{1}{3}, \ldots, 1+\frac{1}{n}, \ldots)$$

as both being sequences approximating 1. But when should we generally consider two sequences $(a_1, a_2, \ldots)$ and $(b_1, b_2, \ldots)$ as approximations of the "same real number"? Intuition tells us that this should be so precisely if the sequence $(a_1 - b_1, a_2 - b_2, \ldots)$ approximates the rational number 0; and if this is the case, we should consider the sequences $(a_1, a_2, \ldots)$ and $(b_1, b_2, \ldots)$ as being – or rather *representing* – the same real number.

The mathematically exact way of doing this is to introduce a certain equivalence relation between Cauchy sequences of rational numbers. The real numbers are then *defined* as the corresponding equivalence classes.

2. Definition of the real numbers.

On the basis of the motivational remarks above we now proceed to actually construct the system of real numbers.

2.1. **Fundamental definitions and basic properties.**

Definition 1. *Let $\alpha = (a_1, a_2, \ldots, a_n, \ldots)$ be a sequence of rational numbers, i.e., $a_n \in \mathbb{Q}$, for all $n \in \mathbb{N}$.*

We say that α is a **Cauchy sequence** *of rational numbers if for every positive rational number ϵ there is (depending on ϵ) an $N \in \mathbb{N}$ such that:*

$$|a_m - a_n| < \epsilon \quad \textit{whenever} \quad m, n \geq N.$$

I.e., α is called a Cauchy sequence if

$$\forall \epsilon \in \mathbb{Q}_+ \exists N \in \mathbb{N} \forall m,n \in \mathbb{N} \ : \ m,n \geq N \Rightarrow |a_m - a_n| < \epsilon.$$

We denote the set of Cauchy sequences of rational numbers by $\mathcal{C}$:

$$\mathcal{C} := \{\alpha \text{ sequence of rational numbers} \mid \alpha \text{ is a Cauchy sequence}\}.$$

If $\alpha = (a_1, a_2, \ldots, a_n, \ldots)$ is a sequence of rational numbers and $a \in \mathbb{Q}$ we say that α **converges to** *a if for every positive rational number ϵ there is (depending on ϵ) an $N \in \mathbb{N}$ such that:*

$$|a - a_n| < \epsilon \quad \text{whenever} \ \ n \geq N.$$

I.e., we say that α converges to $a \in \mathbb{Q}$ if

$$\forall \epsilon \in \mathbb{Q}_+ \exists N \in \mathbb{N} \forall n \in \mathbb{N} \ : \ n \geq N \Rightarrow |a - a_n| < \epsilon.$$

If α converges to a we also write $\alpha \to a$, or $a_n \to a$ for $n \to \infty$, or we say that a **is the limit of** *α.*

We call the sequence α **convergent in** *$\mathbb{Q}$ if it converges to some $a \in \mathbb{Q}$.*

The sequence α is called a **null-sequence** *if it converges to the rational number 0. We denote the set of null-sequences by $\mathcal{N}$:*

$$\mathcal{N} := \{\alpha \text{ sequence of rational numbers} \mid \alpha \text{ is a null-sequence}\}.$$

Proposition 1. *Let $\alpha = (a_1, a_2, \ldots, a_n, \ldots)$ be a sequence of rational numbers.*

If α is convergent in $\mathbb{Q}$ then α is a Cauchy sequence.

In particular, every null-sequence is a Cauchy sequence:

$$\mathcal{N} \subseteq \mathcal{C}.$$

Proof. Suppose that α is convergent in $\mathbb{Q}$. To show that α is a Cauchy sequence, let the positive rational number ϵ be given.

Since α converges in $\mathbb{Q}$ there is some $a \in \mathbb{Q}$ such that α converges to a. Since $\epsilon/2$ is a positive rational number there is an $N \in \mathbb{N}$ such that:

$$|a - a_n| < \epsilon/2 \quad \text{whenever} \ \ n \geq N.$$

Let $m, n \in \mathbb{N}$ be such that $m, n \geq N$. We then find:

$$|a_m - a_n| = |(a_m - a) + (a - a_n)| \leq |(a_m - a)| + |(a - a_n)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Since ϵ was arbitrary we have shown that α is a Cauchy sequence. □

Definition 2. *Let $a \in \mathbb{Q}$. The sequence $(a, a, \ldots, a, \ldots)$ is called the* **constant sequence** *with term a. It is clear from the definition that any such constant sequence is a Cauchy sequence which in fact converges in $\mathbb{Q}$ to $a \in \mathbb{Q}$.*

It is a little harder to show the existence of Cauchy sequences that are *not* convergent in $\mathbb{Q}$. We will construct an example in exercise 3.

Next we proceed to show that Cauchy sequences can be added and multiplied in a natural way. We first need a little lemma.

Lemma 1. *Suppose that $\alpha = (a_1, a_2, \ldots, a_n, \ldots)$ is a Cauchy sequence of rational numbers.*

Then there is (depending on α) a positive rational number c such that:

$$|a_n| \leq c \quad \text{for all} \ \ n \in \mathbb{N}.$$

(One says that the sequence α is **bounded***).*

If α is not a null-sequence there is (depending on α) a positive rational number d and an $M \in \mathbb{N}$ such that:

$$|a_n| \geq d \quad \textit{whenever} \quad n \geq M.$$

Proof. Since α is a Cauchy sequence there is an $N \in \mathbb{N}$ such that $|a_m - a_n| < 1$ whenever $m, n \geq N$. Let c_0 be the maximum of the finitely many rational numbers $|a_1|, \ldots, |a_N|$, and put $c := c_0 + 1$. Then c is a positive rational number.

Let $n \in \mathbb{N}$. If $n \leq N$ we have $|a_n| \leq c_0 < c$. And if $n \geq N$ we find:

$$|a_n| = |(a_n - a_N) + a_N| \leq |a_n - a_N| + |a_N| \leq 1 + c_0 = c.$$

Suppose then that α is a Cauchy sequence which is *not* a null-sequence. This means that α does not converge to 0. By definition, this means that:

$$\exists \epsilon \in \mathbb{Q}_+ \forall N \in \mathbb{N} \exists n \in \mathbb{N} \; : \; n \geq N \text{ and } |0 - a_n| \geq \epsilon,$$

i.e., there is a positive rational number ϵ so that whenever $N \in \mathbb{N}$ we can always find an $n \geq N$ such that $|a_n| \geq \epsilon$. Fix one such positive rational number ϵ.

Now, since α is a Cauchy sequence, and since $\epsilon/2$ is a positive rational number there is an $N \in \mathbb{N}$ such that

$$|a_m - a_n| < \epsilon/2 \quad \text{whenever} \quad m, n \geq N.$$

By the above there is an $n_0 \geq N$ such that $|a_{n_0}| \geq \epsilon$. But then we have for any $n \geq N$ that:

$$|a_n| = |a_{n_0} - (a_{n_0} - a_n)| \geq |a_{n_0}| - |a_{n_0} - a_n| \geq \epsilon - \epsilon/2 = \epsilon/2.$$

So we can take $d = \epsilon/2$ and $M = N$. □

Proposition 2. *Suppose that $\alpha = (a_1, a_2, \ldots, a_n, \ldots)$ and $\beta = (b_1, b_2, \ldots, b_n, \ldots)$ are Cauchy sequences of rational numbers. Define the sequences $\alpha + \beta$ and $\alpha \cdot \beta$ (also often simply written as $\alpha\beta$) of rational numbers as follows:*

$$\alpha + \beta := (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n, \ldots),$$

and

$$\alpha \cdot \beta := (a_1 b_1, a_2 b_2, \ldots, a_n b_n, \ldots).$$

Then $\alpha + \beta$ and $\alpha \cdot \beta$ are both Cauchy sequences of rational numbers.

Proof. Let a positive rational number ϵ be given.

Since α and β are Cauchy sequences there exist according to Lemma 1 positive rational numbers c_1 and c_2 such that $|a_n| \leq c_1$ and $|b_n| \leq c_2$ for all $n \in \mathbb{N}$. Let c be the largest of the 3 numbers 1, c_1, and c_2. Then we certainly have $|a_n| \leq c$ and $|b_n| \leq c$ for all $n \in \mathbb{N}$.

The number $\frac{\epsilon}{2c}$ is a positive rational number. Again since α and β are Cauchy sequences there exist by definition $N_1, N_2 \in \mathbb{N}$ such that $|a_m - a_n| < \frac{\epsilon}{2c}$ if $m, n \geq N_1$, and such that $|b_m - b_n| < \frac{\epsilon}{2c}$ if $m, n \geq N_2$.

Now let N be the largest of the two numbers N_1 and N_2. Then, whenever $m, n \geq N$ we deduce:

$$\begin{aligned} |(a_m + b_m) - (a_n + b_n)| &= |(a_m - a_n) + (b_m - b_n)| \\ &\leq |a_m - a_n| + |b_m - b_n| < \frac{\epsilon}{2c} + \frac{\epsilon}{2c} = \frac{\epsilon}{c} \leq \epsilon, \end{aligned}$$

since $c \geq 1$, and furthermore:

$$\begin{aligned} |a_m b_m - a_n b_n| &= |a_m(b_m - b_n) + b_n(a_m - a_n)| \leq |a_m||b_m - b_n| + |b_n||a_m - a_n| \\ &< c \cdot \frac{\epsilon}{2c} + c \cdot \frac{\epsilon}{2c} = \epsilon. \end{aligned}$$

□

If $\alpha = (a_1, a_2, \dots, a_n, \dots) \in \mathcal{C}$ and $q \in \mathbb{Q}$ we can define:

$$q \cdot \alpha := (qa_1, qa_2, \dots, qa_n, \dots)$$

which we may also simply denote by $q\alpha$. But one notices that $q\alpha$ is in fact nothing but the product of the constant sequence with term q with the sequence α. Thus, by Proposition 2 we have $q\alpha \in \mathcal{C}$.

Also, if $\alpha = (a_1, a_2, \dots, a_n, \dots)$ and $\beta = (b_1, b_2, \dots, b_n, \dots)$ are elements of $\mathcal{C}$ we can define their difference $\alpha - \beta$:

$$\alpha - \beta := (a_1 - b_1, a_2 - b_2, \dots, a_n - b_n, \dots),$$

but we notice that this is just the sequence $\alpha + (-1) \cdot \beta$. Again by Proposition 2 we deduce that $\alpha - \beta \in \mathcal{C}$.

The following proposition is completely straightforward to prove.

Proposition 3. *Let $\alpha, \beta, \gamma \in \mathcal{C}$. Then:*

$$\alpha + \beta = \beta + \alpha, \;\; \alpha\beta = \beta\alpha,$$

$$\alpha \pm (\beta \pm \gamma) = (\alpha \pm \beta) \pm \gamma, \;\; \alpha(\beta\gamma) = (\alpha\beta)\gamma,$$

and

$$\alpha(\beta \pm \gamma) = \alpha\beta \pm \alpha\gamma.$$

In particular, if $q \in \mathbb{Q}$ we have:

$$q(\alpha\beta) = (q\alpha)\beta = \alpha(q\beta), \;\; q(\alpha + \beta) = q\alpha + q\beta.$$

Proposition 4. *Suppose that $\alpha = (a_1, a_2, \dots, a_n, \dots)$ and $\beta = (b_1, b_2, \dots, b_n, \dots)$ are sequences of rational numbers that converge in $\mathbb{Q}$ to a and b, respectively. Let q be an arbitrary rational number.*

Then the sequences $\alpha + \beta$, $q\alpha$, and $\alpha \cdot \beta$ are also convergent in $\mathbb{Q}$, with limits $a + b$, qa, and $a \cdot b$, respectively.

Corollary 1. *Suppose that $\alpha = (a_1, a_2, \dots, a_n, \dots)$ and $\beta = (b_1, b_2, \dots, b_n, \dots)$ are null-sequences of rational numbers. Suppose also that q is a rational number.*

Then the sequences $\alpha + \beta$, $q\alpha$, and $\alpha \cdot \beta$ are also null-sequences.

Proof. This follows immediately from the definition of "null-sequence" and Proposition 4. □

The fact that $q\alpha$ is a null-sequence if α is can be generalized a bit:

Proposition 5. *Suppose that $\alpha = (a_1, a_2, \dots, a_n, \dots)$ and $\beta = (b_1, b_2, \dots, b_n, \dots)$ are Cauchy sequences of rational numbers and that β is a null-sequence.*

Then $\alpha\beta$ is also a null-sequence.

Proof. Let ϵ be any positive rational number.

Since α is a Cauchy sequence there is according to Lemma 1 a positive rational number c such that $|a_n| \leq c$ for all $n \in \mathbb{N}$.

Then ϵ/c is also a positive rational number. Since β is a null-sequence there is $N \in \mathbb{N}$ such that $|b_n| < \epsilon/c$ whenever $n \geq N$. But then:

$$|a_n b_n| = |a_n||b_n| < c \cdot \epsilon/c = \epsilon$$

for all $n \geq N$.

We conclude that $\alpha\beta$ is a null-sequence. □

2.2. **Definition of the set of real numbers.**

Definition 3. *Define a relation $\sim$ between elements of $\mathcal{C}$, i.e., between Cauchy sequences of rational numbers, as follows: if $\alpha, \beta \in \mathcal{C}$ we write $\alpha \sim \beta$ if the Cauchy sequence $\alpha - \beta$ is a null-sequence. In other words:*

$$\alpha \sim \beta \overset{def}{\Leftrightarrow} \alpha - \beta \in \mathcal{N}.$$

Proposition 6. *The relation $\sim$ is an equivalence relation in $\mathcal{C}$.*

Proof. Proof that $\sim$ is reflexive: Let $\alpha = (a_1, a_2, \ldots, a_n, \ldots) \in \mathcal{C}$. Then:

$$\alpha - \alpha = (0, 0, \ldots)$$

is the constant sequence with term 0. This is obviously a null-sequence, so we have $\alpha - \alpha \in \mathcal{N}$, i.e., $\alpha \sim \alpha$ by definition.

Proof that $\sim$ is symmetric: Suppose that $\alpha, \beta \in \mathcal{C}$ and that $\alpha \sim \beta$. Then $\alpha - \beta$ is a null-sequence. Then by Proposition 3 we have

$$\beta - \alpha = (-1) \cdot (\alpha - \beta)$$

and so by Corollary 1, $\beta - \alpha$ is again a null-sequence. That is, $\beta \sim \alpha$.

Proof that $\sim$ is transitive: Let $\alpha, \beta, \gamma \in \mathcal{C}$, and suppose that $\alpha \sim \beta$ and $\beta \sim \gamma$. That is, the sequences $\alpha - \beta$ and $\beta - \gamma$ are both null-sequences. Now, by Proposition 3 we have:

$$\alpha - \gamma = (\alpha - \beta) + (\beta - \gamma),$$

and so by Corollary 1 the sequence $\alpha - \gamma$ is again a null-sequence. That is, we have $\alpha \sim \gamma$. □

Definition 4. *The set of real numbers $\mathbb{R}$ is defined as the set of equivalence classes of elements of $\mathcal{C}$ with respect to the equivalence relation $\sim$:*

$$\mathbb{R} := \mathcal{C}/\sim.$$

If α is a Cauchy sequence of rational numbers we will write $\bar{\alpha}$ for the equivalence class containing α in $\mathcal{C}/\sim$. Thus, for $\alpha \in \mathcal{C}$ we have an element $\bar{\alpha} \in \mathbb{R}$.

We would now like to define sums and products of elements in $\mathbb{R}$. Since we have already defined sums and products of elements in $\mathcal{C}$ it seems rather clear how to proceed: if $x, y \in \mathbb{R}$ there are elements $\alpha, \beta \in \mathcal{C}$ such that $x = \bar{\alpha}$ and $y = \bar{\beta}$; it seems reasonable to define:

$$x + y := \overline{\alpha + \beta},$$

that is, we are proposing to define $+$ on equivalence classes in $\mathcal{C}/\sim$ thus:

$$\bar{\alpha} + \bar{\beta} := \overline{\alpha + \beta}.$$

Be sure that you understand fully that this is *not* a statement but a proposed definition.

Before we can formally make this definition we must be certain that it makes sense. What is the problem? The problem is that the above "definition" relied on a choice: We chose the representatives α and β of the classes x and y in $\mathbb{R} = \mathcal{C}/\sim$. We must convince ourselves that the definition of $x+y$ does not depend on these choices; that is, we must show that the class $\overline{\alpha+\beta}$ does not depend on the choices of representatives α and β for the classes x and y. In mathematical jargon we must show that our proposed $+$ on real numbers is "well-defined" (i.e., does not depend on any choices made). This is the purpose of the next proposition.

Proposition 7. *Let $x, y \in \mathbb{R}$. Suppose that $\alpha_1, \alpha_2 \in \mathcal{C}$ are both representatives of the class x, and that $\beta_1, \beta_2 \in \mathcal{C}$ are both representatives of the class y. Then:*

$$\overline{\alpha_1+\beta_1} = \overline{\alpha_2+\beta_2}$$

and

$$\overline{\alpha_1\beta_1} = \overline{\alpha_2\beta_2}.$$

Proof. The statement $\overline{\alpha_1+\beta_1} = \overline{\alpha_2+\beta_2}$ means that the two Cauchy sequences $\alpha_1+\beta_1$ and $\alpha_2+\beta_2$ are in the same class with respect to $\sim$, that is, that

$$\alpha_1+\beta_1 \sim \alpha_2+\beta_2.$$

By definition, this is the statement that the Cauchy sequence $(\alpha_1+\beta_1)-(\alpha_2+\beta_2)$ is a null-sequence. Now:

$$(\alpha_1+\beta_1)-(\alpha_2+\beta_2) = (\alpha_1-\alpha_2)+(\beta_1-\beta_2)$$

by proposition 3.

But $\alpha_1-\alpha_2$ is a null-sequence since α_1 and α_2 are both representatives for the class x. Similarly, the sequence is $\beta_1-\beta_2$ is a null-sequence. Now Corollary 1 implies that $(\alpha_1+\beta_1)-(\alpha_2+\beta_2)$ is a null-sequence, as required.

We also have:

$$\alpha_1\beta_1-\alpha_2\beta_2 = \alpha_1(\beta_1-\beta_2)+\beta_2(\alpha_1-\alpha_2).$$

Combining Proposition 5 with Corollary 1 we conclude that $\alpha_1\beta_1-\alpha_2\beta_2$ is a null-sequence. That is:

$$\overline{\alpha_1\beta_1} = \overline{\alpha_2\beta_2}.$$

□

We can now formally introduce sums and products of elements in $\mathbb{R}$:

Definition 5. *Let x and y be real numbers. We define the sum $x+y$ and the product xy as follows:*

Let α and β be Cauchy sequences of rational numbers such that:

$$x = \bar{\alpha},\ y = \bar{\beta}.$$

Then we define:

$$x+y := \overline{\alpha+\beta} \quad \text{and} \quad xy := \overline{\alpha\cdot\beta}.$$

By Proposition 7 this is well-defined, i.e., these definitions do not depend on the choices of representatives α and β of x and y, respectively.

Proposition 8. *Consider the map $\phi\colon \mathbb{Q} \to \mathbb{R}$ given by:*

$$\phi(a) := \overline{(a, a, \ldots, a \ldots)},$$

i.e., by mapping a rational number a to the class in $\mathbb{R} = \mathcal{C}/\sim$ containing the constant sequence with term a.

The map ϕ is injective and has the properties:

$$\phi(a+b) = \phi(a) + \phi(b) \quad \textit{and} \quad \phi(ab) = \phi(a)\phi(b).$$

Proof. Suppose that a and b are rational numbers such that $\phi(a) = \phi(b)$. By definition of ϕ this means that:

$$\overline{(a, a, \ldots, a \ldots)} = \overline{(b, b, \ldots, b \ldots)}$$

i.e., that $(a, a, \ldots, a \ldots) \sim (b, b, \ldots, b \ldots)$. By definition, this means that the sequence

$$(a, a, \ldots, a \ldots) - (b, b, \ldots, b \ldots) = (a-b, a-b, \ldots, a-b \ldots)$$

is a null-sequence. So, the rational number $|a-b|$ is smaller than any positive rational number (*why?*). Since $|a-b| \geq 0$ this can only happen if $|a-b| = 0$. So we must have $a - b = 0$, i.e., $a = b$. We have proved that ϕ is injective.

The proof that ϕ has the other stated properties is left to the exercises. □

Notice in the Proposition that we have two different "plusses" in the game: In the equality $\phi(a+b) = \phi(a) + \phi(b)$ the $+$ in $a+b$ means addition of the rational numbers a and b whereas the $+$ in $\phi(a) + \phi(b)$ is the addition of real numbers that was defined in Definition 5. Similarly, in the equality $\phi(ab) = \phi(a)\phi(b)$ there are two different (and implicit) multiplication signs $\cdot$.

Proposition 8 means that we can view $\mathbb{R}$ as containing a copy of the rational numbers, – namely the set $\phi(\mathbb{Q})$. Abusing notation and denoting this subset $\phi(\mathbb{Q})$ again by the symbol $\mathbb{Q}$ we can then say that $\mathbb{Q}$ is a subset of $\mathbb{R}$. So, if a is a rational number we will view it as an element $a \in \mathbb{R}$. That is, we write a but actually mean $\phi(a)$.

Does this not introduce a potential ambiguity? Namely, if a and b are rational numbers the expression $a+b$ can be interpreted in two different ways: Either we can say that $a+b$ is the "original" sum of a and b as rational numbers. Or, we can say that $a+b$ is the sum of a and b viewed as elements of $\mathbb{R}$. But the proposition states that if we view the sum as an element of $\mathbb{R}$ then it is the same whether we view it in the one or the other way. Thus, the ambiguity is harmless. Similarly, there is in principle an ambiguity in the expression ab, but again by the proposition it does not matter much.

Using notation in a slightly ambiguous way like in the above occurs frequently in mathematics. The reason is not that mathematicians love ambiguity but rather that they like notation as convenient and transparent as possible. Thus, in the above we would like to be able to write $a \in \mathbb{R}$ though we really should be writing $\phi(a) \in \mathbb{R}$.

Ambiguities such as the ones we have just discussed are perfectly all right to use *as long* as one is aware of them, and is able – at any time – to explain the situation without any ambiguities at all.

Now we can prove the first theorem on the properties of the set of real numbers $\mathbb{R}$.

Theorem 1. *The set $\mathbb{R}$ of real numbers equipped with the two binary operations $+$ and $\cdot$ defined in Definition 5 is a field.*

Proof. There is a lot of small things to verify, and then a larger one, namely that elements $\neq 0$ are invertible w.r.t. multiplication.

But let us first notice the following: In the field axioms there occur certain elements 0 and 1 of $\mathbb{R}$. We must clarify what we mean by these symbols. But this is easy: We let them mean exactly what they seem to mean, namely the rational numbers 0 and 1, – interpreted, however, as elements of $\mathbb{R}$ as in the discussion following Proposition 8. In other words, the element $0 \in \mathbb{R}$ is actually the class:

$$0 := \overline{(0, 0, \ldots, 0, \ldots)}$$

containing the constant sequence with term 0. Similarly, 1 as an element of $\mathbb{R}$ is the element:

$$1 := \overline{(1, 1, \ldots, 1, \ldots)}.$$

We prove now the least trivial of the properties which is the existence of multiplicative inverses to non-zero elements of $\mathbb{R}$. That is, we must prove that if $x \in \mathbb{R}$ with $x \neq 0$ then there exists $y \in \mathbb{R}$ such that:

$$xy = 1.$$

So let $x \in \mathbb{R}$ be arbitrary but with $x \neq 0$. Let $\alpha = (a_1, a_2, \ldots, a_n, \ldots)$ be a Cauchy sequence representing x, i.e., such that $x = \bar{\alpha}$. Since $x \neq 0$ we have:

$$\overline{(a_1, a_2, \ldots, a_n, \ldots)} = \bar{\alpha} \neq 0 = \overline{(0, 0, \ldots, 0, \ldots)};$$

by definition, this means that the sequence

$$(a_1, a_2, \ldots, a_n, \ldots) - (0, 0, \ldots, 0, \ldots) = (a_1, a_2, \ldots, a_n, \ldots)$$

is not a null-sequence. By Lemma 1 there is then a positive rational number d and an $M \in \mathbb{N}$ such that:

$$|a_n| \geq d \quad \text{whenever} \quad n \geq M.$$

Since d is positive we must then have $a_n \neq 0$ for $n \geq M$. Since a_n is a rational number we can meaningfully speak of the rational number $\frac{1}{a_n}$ whenever $a_n \neq 0$, – and hence in particular for $n \geq M$. We have:

$$(*) \qquad \frac{1}{|a_n|} \leq \frac{1}{d} \quad \text{whenever} \quad n \geq M.$$

Now let us consider the following sequence β of rational numbers:

$$\beta := (\underbrace{0, \ldots, 0}_{M}, \frac{1}{a_{M+1}}, \frac{1}{a_{M+2}}, \ldots).$$

We first claim that β is a Cauchy sequence. To see this let ϵ be an arbitrary positive rational number. Then $d^2\epsilon$ is also a positive rational number. Since $(a_1, a_2, \ldots, a_n, \ldots)$ is a Cauchy sequence there is $N \in \mathbb{N}$ such that:

$$(**) \qquad |a_m - a_n| < d^2\epsilon \quad \text{whenever} \quad m, n \geq N.$$

Let now L be the largest of the two numbers M and N. If $m, n \geq L$ we then find:

$$\left|\frac{1}{a_m} - \frac{1}{a_n}\right| = \left|\frac{a_m - a_n}{a_m a_n}\right| = \frac{|a_m - a_n|}{|a_m||a_n|} < d^2\epsilon \cdot \frac{1}{d^2} = \epsilon,$$

because of $(*)$ and $(**)$. Hence β is a Cauchy sequence.

So we can consider the class $y := \bar{\beta}$ in $\mathbb{R} = \mathcal{C}/\sim$. We claim that $xy = 1$. To prove this, we have to show that $\bar{\alpha}\bar{\beta} = 1 := \overline{(1, 1, \ldots, 1, \ldots)}$ in $\mathbb{R}$, i.e., that

$$\alpha\beta \sim (1, 1, \ldots, 1, \ldots).$$

We compute:

$$\begin{aligned}\alpha\beta &= (a_1, a_2, \ldots, a_n, \ldots) \cdot (\underbrace{0, \ldots, 0}_{M}, \frac{1}{a_{M+1}}, \frac{1}{a_{M+2}}, \ldots) \\ &= (\underbrace{0, \ldots, 0}_{M}, 1, 1, \ldots),\end{aligned}$$

so that

$$\begin{aligned}\alpha\beta - (1, 1, \ldots, 1, \ldots) &= (\underbrace{0, \ldots, 0}_{M}, 1, 1, \ldots) - (1, 1, \ldots, 1, \ldots) \\ &= (\underbrace{-1, \ldots, -1}_{M}, 0, 0, \ldots);\end{aligned}$$

since this is clearly a null-sequence, the claim follows. □

3. The order in $\mathbb{R}$ and further properties.

3.1. **Order and absolute value.** We wish to be able to measure sizes of real numbers. The key to doing that is to introduce an order relation. The key to define an order relation $<$ is to define a subset $\mathbb{R}_+$ with the following properties:

- For any $x \in \mathbb{R}$, exactly one of the following holds: $x \in \mathbb{R}_+$ or $x = 0$ or $-x \in \mathbb{R}_+$,
- for $x, y \in \mathbb{R}_+$ we have $x + y, xy \in \mathbb{R}_+$.

If we can define such a subset $\mathbb{R}_+ \subseteq \mathbb{R}$, the order relation $<$ is then defined:

$$x < y \quad \text{if and only if} \quad y - x \in \mathbb{R}_+,$$

and one also defines, as usual, the relation $x \leq y$ to mean that either $x = y$ or $x < y$.

Definition 6. *Let $x \in \mathbb{R}$ and let $\alpha = (a_1, a_2, \ldots, a_n, \ldots)$ be a Cauchy sequence of rational numbers that represents x in $\mathbb{R}$.*

We say that x is **positive** *if there exists a positive rational number d and an $N \in \mathbb{N}$ such that:*

$$a_n \geq d \quad \textit{whenever} \quad n \geq N.$$

The set of positive real numbers is denoted by $\mathbb{R}_+$.

Once again, we have a potential problem with the definition: We must show that the definition does not depend on the choice of the representative α. That is, we must show that a given x could not simultaneously be shown to be positive by choosing one representative, and be shown to be not positive by choosing another representative. The following lemma rules out exactly this hypothetical situation and thus makes the definition that we have just made possible.

Lemma 2. *Let $\alpha = (a_1, a_2, \ldots, a_n, \ldots)$ and $\beta = (b_1, b_2, \ldots, b_n, \ldots)$ be Cauchy sequences of rational numbers that both represent a given $x \in \mathbb{R}$. Suppose that there exists a positive rational number d and an $N \in \mathbb{N}$ such that:*

$$a_n \geq d \quad \textit{whenever} \quad n \geq N.$$

Then there exists a positive rational number d', and a natural number N' such that:

$$b_n \geq d' \quad \textit{whenever} \quad n \geq N'.$$

Proof. Since α and β represent the same real number x, we have that $\alpha - \beta$ is a null-sequence. Now, the number $d/2$ is a positive rational number. Since $\alpha - \beta$ is a null-sequence there exists an $M \in \mathbb{N}$ such that:

$$|a_n - b_n| < d/2 \quad \text{whenever} \quad n \geq M.$$

In particular, we have $a_n - b_n < d/2$ and so $b_n > a_n - d/2$ for $n \geq M$. If also $n \geq N$ we deduce $b_n > d - d/2 = d/2$.

So we can take $d' := d/2$ and let N' be the largest of the two numbers M and N. □

Theorem 2. *Suppose that $x, y \in \mathbb{R}_+$. Then $x + y, xy \in \mathbb{R}_+$.*

If x is any real number then exactly one of the following holds: (i) $x \in \mathbb{R}_+$, (ii) $x = 0$, (iii) $-x \in \mathbb{R}_+$.

Proof. Let us prove the second statement. So let $x \in \mathbb{R}$ be arbitrary. Let $\alpha = (a_1, a_2, \ldots, a_n, \ldots)$ be a Cauchy sequence of rational numbers that represents x in $\mathbb{R}$.

Suppose that $x \neq 0$. This means that α is not a null-sequence. By Lemma 1 there is then a positive rational number d and an $M \in \mathbb{N}$ such that:

$$|a_n| \geq d \quad \text{whenever} \quad n \geq M,$$

i.e., for any $n \geq M$ we have either $a_n \geq d$ or $a_n \leq -d$.

Now, since the number d is a positive rational number and since α is a Cauchy sequence there is $N \in \mathbb{N}$ such that

$$(\ddagger) \qquad |a_m - a_n| < d \quad \text{whenever} \quad m, n \geq N.$$

Let K be the largest of the two numbers M and N. Then we have either $a_K \geq d$ or $a_K \leq -d$.

Suppose that $a_K \geq d$ and let $m \geq K$. Then either $a_m \geq d$ or $a_m \leq -d$; but if $a_m \leq -d$ we would obtain $a_m - a_K \leq -2d$ and thus $|a_m - a_K| \geq 2d$ which contradicts (‡). So, we conclude that $a_m \geq d$ for all $m \geq K$. By definition this means that $x \in \mathbb{R}_+$.

If on the other hand we have $a_K \leq -d$ then we have $-a_K \geq d$. Now, the sequence $-\alpha = (-a_1, -a_2, \ldots, -a_n, \ldots)$ is a representative for $-x$ in $\mathbb{R}$; repeating the previous argument with $-x$ instead of x we can then conclude that $-x \in \mathbb{R}_+$.

So we have now shown that at least 1 of the 3 possibilities *(i), (ii), (iii)* materializes for any $x \in \mathbb{R}$. We must still show that the 3 possibilities are mutually exclusive. So for instance we must show that the real number 0 is not positive.

The verifications are easy. □

As described in the beginning of this section we can now introduce the order $\leq$ in $\mathbb{R}$: If $x, y \in \mathbb{R}$ we write $x < y$ if $y - x \in \mathbb{R}_+$. We also write $x \leq y$ if $y - x \in \mathbb{R}_+ \cup \{0\}$. And as usual we write $x \geq y$ if $y \leq x$, and correspondingly with $>$.

We then easily prove the following proposition:

Proposition 9. *The relation $<$ is what is called a total order on the set $\mathbb{R}$ of real numbers. This means that whenever $x, y \in \mathbb{R}$, exactly one of the following holds: $x < y$ or $x = y$ or $y < x$.*

We can also introduce the absolute value $|x|$ of a real number x:

Definition 7. *$|x|$ is defined to be x if $x \in \mathbb{R}_+ \cup \{0\}$, that is, if $x \geq 0$. Otherwise, $|x|$ is defined to be $-x$.*

One easily proves that this absolute value has the properties that we are used to.

But we have to face a potential ambiguity with our definition of $\leq$ for real numbers: suppose that a and b are *rational* numbers. Suppose that $a \leq b$ *as rational numbers*. Is it then still true that $a \leq b$ if we now view a and b as real numbers via the map ϕ of Proposition 8? The answer is yes and this is stated formally in the next proposition.

Proposition 10. *Consider the injective map $\phi\colon \mathbb{Q} \to \mathbb{R}$ of Proposition 8.*

For $a, b \in \mathbb{Q}$ we have:

$$a \leq b \Rightarrow \phi(a) \leq \phi(b),$$

and

$$|\phi(a)| = \phi(|a|).$$

We will now prove a theorem which says intuitively that the set $\mathbb{Q}$ of rational numbers is "dense" in $\mathbb{R}$, or, alternatively, that any real number can be approximated arbitrarily close by rational numbers.

Theorem 3. *(1). (The Archimedean property of $\mathbb{R}$). Let ϵ be a positive real number and let x be any real number. Then there is a natural number k such that:*

$$k \cdot \epsilon > x.$$

(2). Let ϵ be any positive real number. Then there is a positive rational number q such that:

$$q < \epsilon.$$

(3). Let x be a real number, and let ϵ be any positive real number.

Then there is a rational number q such that:

$$|x - q| < \epsilon.$$

Proof. Proof of (1): Let $(e_1, e_2, \ldots, e_n, \ldots)$ and $(a_1, a_2, \ldots, a_n, \ldots)$ be Cauchy sequences of rational numbers representing ϵ and x in $\mathbb{R}$, respectively. By Lemma 1 there is a rational number c such that $|a_n| \leq c$ for all $n \in \mathbb{N}$. Thus in particular, $a_n \leq c$ for all $n \in \mathbb{N}$.

Since ϵ is positive we know by definition that there exists a positive rational number d and an $N \in \mathbb{N}$ such that:

$$e_n \geq d \quad \text{whenever} \quad n \geq N.$$

Consider the rational number $\frac{c+1}{d}$. There is a natural number k such that $k > \frac{c+1}{d}$. With such a k we have:

$$k \cdot e_n > \frac{c+1}{d} \cdot d = c + 1 \geq a_n + 1$$

for all $n \geq N$. This shows that the terms of the Cauchy sequence

$$(k \cdot e_1 - a_1, k \cdot e_2 - a_2, \ldots, k \cdot e_n - a_n, \ldots)$$

are ≥ 1 for $n \geq N$. By definition this means that the sequence represents a positive real number. But it clearly represents the real number $k \cdot \epsilon - x$. So, $k \cdot \epsilon - x$ is a positive real number, i.e., we have $x < k \cdot \epsilon$.

Proof of (2): The number $1 \in \mathbb{R}$ is positive. The number ϵ is positive and hence in particular not 0. So we can consider its inverse ϵ^{-1}.

By *(1)* we know that there is $k \in \mathbb{N}$ such that $k \cdot 1 > \epsilon^{-1}$. But then $1/k < \epsilon$ and $1/k$ is a positive rational number.

Proof of (3): If $x = 0$ we can choose $q = 0$.

Also, suppose that we have proved the statement for x positive. If then x is such that $-x$ is positive, and if q is a rational number such that $|-x-q| < \epsilon$ then we have $|x - (-q)| < \epsilon$.

So, by Theorem 2 it is sufficient to prove the theorem for positive x.

Assume then that x is positive. If $x < \epsilon$ we can take $q = 0$, so we may, and will, further assume that $x \geq \epsilon$. According to *(2)* there is a positive rational number d such that $d < \epsilon$. According to *(1)* there is $k \in \mathbb{N}$ such that $k \cdot d > x$. Let $k \in \mathbb{N}$ be smallest with this property. Since $d < \epsilon \leq x$ we must have $k \geq 2$. Since k was chosen smallest possible we must then have $(k-1) \cdot d \leq x$. But then:

$$0 \leq x - (k-1) \cdot d < k \cdot d - (k-1) \cdot d = d < \epsilon.$$

So, we can take $q = (k-1) \cdot d$. □

3.2. **Sequences in $\mathbb{R}$ and completeness.** Now we are ready to study sequences of real numbers. We introduce the notions of Cauchy sequences of real numbers, of convergent sequences, and of limits of sequences in complete analogy with Definition 1:

Definition 8. *Let $(x_1, x_2, \ldots, x_n, \ldots)$ be a sequence of real numbers, i.e., $x_n \in \mathbb{R}$, for all $n \in \mathbb{N}$.*

We say that the sequence is a **Cauchy sequence** *if for every positive real number ϵ there is (depending on ϵ) an $N \in \mathbb{N}$ such that:*

$$|x_m - x_n| < \epsilon \quad \textit{whenever} \;\; m, n \geq N.$$

The sequence $(x_1, x_2, \ldots, x_n, \ldots)$ is said to **converge** *to an $a \in \mathbb{R}$ if for every positive real number ϵ there is (depending on ϵ) an $N \in \mathbb{N}$ such that:*

$$|a - x_n| < \epsilon \quad \textit{whenever} \;\; n \geq N.$$

If the sequence converges to a we also write $x_n \to a$ for $n \to \infty$, or we say that a **is the limit of the sequence** *$(x_1, x_2, \ldots, x_n, \ldots)$*

We call the sequence **convergent** *if it converges to some $a \in \mathbb{R}$.*

The following small lemma is useful. The proof is very easy.

Lemma 3. *Let $(x_1, x_2, \ldots, x_n, \ldots)$ be a sequence of real numbers.*

The sequence is a Cauchy sequence if and only if for every positive rational number ϵ there is (depending on ϵ) an $N \in \mathbb{N}$ such that:

$$|x_m - x_n| < \epsilon \quad \textit{whenever} \;\; m, n \geq N.$$

The sequence converges to the real number a if and only if for every positive rational number ϵ there is (depending on ϵ) an $N \in \mathbb{N}$ such that:

$$|a - x_n| < \epsilon \quad \textit{whenever} \;\; n \geq N.$$

Now we can say that the real numbers is the set of limits of Cauchy sequences of rational numbers:

Theorem 4. *Every real number is the limit of a Cauchy sequence of rational numbers.*

More precisely, if the real number x is represented by the Cauchy sequence $\alpha = (a_1, a_2, \ldots, a_n, \ldots)$ of rational numbers then α converges to x in $\mathbb{R}$.

Proof. Let ϵ be any positive rational number. Then $\epsilon/2$ is also a positive rational number. Since α is a Cauchy sequence there is an $N \in \mathbb{N}$ such that

$$|a_m - a_n| < \epsilon/2 \quad \text{whenever} \quad m, n \geq N.$$

That is,

$$-\epsilon/2 \leq a_m - a_n \leq \epsilon/2 \quad \text{whenever} \quad m, n \geq N,$$

that we can also write as:

$$(\sharp) \qquad \epsilon/2 \leq a_m - a_n + \epsilon \quad \text{and} \quad \epsilon/2 \leq -a_m + a_n + \epsilon, \quad \text{whenever} \quad m, n \geq N.$$

Now let n be any natural number with $n \geq N$ and consider the sequence

$$(a_1 - a_n + \epsilon, a_2 - a_n + \epsilon, \ldots, \epsilon, a_{n+1} - a_n + \epsilon, \ldots) = \alpha - (a_n - \epsilon, a_n - \epsilon, \ldots, a_n - \epsilon, \ldots)$$

which is a Cauchy sequence of rational numbers. Hence it represents a real number. By ($\sharp$) and Definition 6 it represents a positive real number. On the other hand, we see that it represents the real number $x - (a_n - \epsilon)$. So this number is positive, that is, $x - a_n + \epsilon > 0$. We proved this solely under the assumption that $n \geq N$. So we can conclude that

$$-\epsilon < x - a_n \quad \text{for all} \quad n \geq N.$$

Arguing similarly with the inequality $\epsilon/2 \leq -a_m + a_n + \epsilon$ which holds for all $m, n \geq N$ we deduce that

$$x - a_n < \epsilon \quad \text{for all} \quad n \geq N.$$

Combining the two inequalities we have that

$$(\sharp\sharp) \qquad |x - a_n| < \epsilon \quad \text{for all} \quad n \geq N.$$

We have shown: Given any positive rational number ϵ there is $N \in \mathbb{N}$ such that ($\sharp\sharp$) holds. By Lemma 3 we conclude that the sequence $(a_1, a_2, \ldots, a_n, \ldots)$ converges in $\mathbb{R}$ to the real number x. □

The relation between Cauchy sequences and convergent sequences is much simpler in $\mathbb{R}$ than in $\mathbb{Q}$:

Theorem 5. *A sequence $(x_1, x_2, \ldots, x_n, \ldots)$ of real numbers is convergent if and only if it is a Cauchy sequence.*

Proof. That the sequence is Cauchy if it is convergent is proved in exactly the same manner as in the case of sequences of rational numbers that converge in $\mathbb{Q}$.

Suppose that $(x_1, x_2, \ldots, x_n, \ldots)$ is a Cauchy sequence. We must show that it converges to some number $y \in \mathbb{R}$.

Let n be any natural number. Then $1/n$ is a positive real number. So, according to Theorem 3, *(3)*, there exists a rational number q such that $|x_n - q| < 1/n$. We choose for each $n \in \mathbb{N}$ such a rational number q_n. I.e., we choose q_n such that:

$$(\flat) \qquad |x_n - q_n| < 1/n \quad \text{for every} \quad n \in \mathbb{N}.$$

Consider now the sequence $(q_1, q_2, \ldots, q_n, \ldots)$ of rational numbers.

Suppose that ϵ is any positive real number. Then $\epsilon/2$ is also a positive real number. Since $(x_1, x_2, \ldots, x_n, \ldots)$ is a Cauchy sequence there exists an $N_1 \in \mathbb{N}$ such that:

$$|x_m - x_n| < \epsilon/2 \quad \text{whenever} \quad m, n \geq N_1.$$

Since $1 \in \mathbb{R}$ is positive there is according to Theorem 3, *(1)*, an $N_2 \in \mathbb{N}$ such that $N_2 = N_2 \cdot 1 > 4 \cdot \epsilon^{-1}$. Then, if we denote by N the largest of the two numbers N_1 and N_2 we obtain for $m, n \geq N$:

$$\begin{aligned} |q_m - q_n| &= |q_m - x_m + x_m - x_n + x_n - q_n| \\ &\leq |x_m - q_m| + |x_m - x_n| + |x_n - q_n| \\ &\leq \frac{1}{m} + \frac{\epsilon}{2} + \frac{1}{n} \leq \frac{2}{N} + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

In particular, we now know that $(q_1, q_2, \ldots, q_n, \ldots)$ is a Cauchy sequence of rational numbers. Let y denote the class of this sequence in $\mathbb{R}$, i.e.:

$$y := \overline{(q_1, q_2, \ldots, q_n, \ldots)}.$$

We claim that the sequence $(x_1, x_2, \ldots, x_n, \ldots)$ converges to y in $\mathbb{R}$.

For let ϵ be an arbitrary positive real number.

We know by Theorem 4 that the sequence $(q_1, q_2, \ldots, q_n, \ldots)$ converges in $\mathbb{R}$ to the number y. Consequently, there is $M \in \mathbb{N}$ such that:

$$(\flat\flat) \qquad |y - q_n| \leq \epsilon/2 \quad \text{whenever} \quad n \geq M.$$

Again by Theorem 3, *(1)*, there is $K \in \mathbb{N}$ such that $K = K \cdot 1 > 2\epsilon^{-1}$, i.e.,

$$(\flat\flat\flat) \qquad \frac{1}{K} < \epsilon/2.$$

Now, if $n \in \mathbb{N}$ is larger than each of the two numbers M and K, we obtain by combining $(\flat)$, $(\flat\flat)$, and $(\flat\flat\flat)$ that:

$$|x_n - y| = |x_n - q_n + q_n - y| \leq |x_n - q_n| + |q_n - y| \leq \frac{1}{n} + \epsilon/2 \leq \frac{1}{K} + \epsilon/2 < \epsilon,$$

and the claim follows. □

3.3. **The least upper bound property.** We will now prove that the set of real numbers has the "least upper bound property". In elementary texts on analysis, this property is often simply introduced as part of the axioms for the real numbers. But for us, now that we have constructed the system of real numbers, this is not an axiom but actually a theorem.

But let us first consider the precise definitions.

Definition 9. *Let $A \subseteq \mathbb{R}$ be a non-empty subset of the real numbers.*

We say that A is **bounded from above** *if there exist $y \in \mathbb{R}$ such that $x \leq y$ for all $x \in A$.*

Any y with this property is called an **upper bound of** A.

If $A \subseteq \mathbb{R}$ is non-empty and bounded from above we say that a number $y \in \mathbb{R}$ is **a least upper bound of** A *if the following conditions hold:*

- *y is an upper bound of A,*
- *If z is any upper bound of A then $z \geq y$.*

If A is a non-empty subset of $\mathbb{R}$ which has a least upper bound y then y is the only least upper bound of A. In other words, the least upper bound is uniquely determined if it exists. This is easy to see and is left as an exercise.

In this case, we also call y the **supremum** of A and write $y = \mathrm{Sup}A$.

The "least upper bound property" of the system of real numbers is the content of the following theorem.

Theorem 6. *Every non-empty subset of $\mathbb{R}$ which is bounded from above has a least upper bound.*

Before the proof we prove a useful lemma, the so-called "squeeze lemma":

Lemma 4. *(The squeeze lemma). Suppose that we have sequences of real numbers $\alpha = (x_1, x_2, \ldots, x_n, \ldots)$ and $\beta = (y_1, y_2, \ldots, y_n, \ldots)$ such that:*

$$x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots,$$

and

$$y_1 \geq y_2 \geq \ldots \geq y_n \geq \ldots,$$

and such that the sequence $(y_1 - x_1, y_2 - x_2, \ldots, y_n - x_n, \ldots)$ converges to 0.

Then the sequences α and β both converge and have the same limit.

Proof. We first claim that

$$x_m \leq y_n \quad \text{for all} \quad m, n \in \mathbb{N}.$$

For suppose not. Then there would exist a pair of natural numbers k, l such that $y_l < x_k$. But then we would find $y_n \leq y_l < x_k \leq x_n$ whenever n was larger then both of k and l. We would thus have

$$x_n - y_n \geq x_k - y_n \geq x_k - y_l$$

for all such n; since $x_k - y_l$ was a positive number this contradicts the fact that the sequence $(y_1 - x_1, y_2 - x_2, \ldots, y_n - x_n, \ldots)$ converges to 0.

Let now ϵ be an arbitrary positive real number. Since the sequence:

$$(y_1 - x_1, y_2 - x_2, \ldots, y_n - x_n, \ldots)$$

converges to 0 there is an $N \in \mathbb{N}$ such that $|y_n - x_n| < \epsilon$ for all $n \geq N$. In particular, we have $y_n - x_n < \epsilon$ for $n \geq N$.

Suppose now that $m, n \in \mathbb{N}$ with $m, n \geq N$. Using what we showed above we then get that $0 \leq x_m - x_n \leq y_n - x_n < \epsilon$ if $m \geq n$; and if $n \geq m$ we get $0 \leq x_n - x_m \leq y_m - x_m < \epsilon$. I.e., we have:

$$|x_m - x_n| < \epsilon \quad \text{whenever} \quad m, n \geq N.$$

We have proved that α is a Cauchy sequence of real numbers. By Theorem 5 we have then that this sequence converges in $\mathbb{R}$. Call its limit a.

In an analogous manner we deduce that the sequence β is convergent in $\mathbb{R}$. Call its limit b.

Since $(y_1 - x_1, y_2 - x_2, \ldots, y_n - x_n, \ldots)$ converges to 0 and since

$$\alpha + (y_1 - x_1, y_2 - x_2, \ldots, y_n - x_n, \ldots) = \beta$$

in the sense of Exercise 4, the result of that exercise shows that we have:

$$a = a + 0 = b.$$

□

Proof of Theorem 6. Let $A \subseteq \mathbb{R}$ be a non-empty subset that is bounded from above.

Since A is not empty there is some number $x \in A$. Since A is bounded from above there is some real number y such that $z \leq y$ for all $z \in A$.

In particular, we have $x \leq y$. Put:

$$\eta := y - x.$$

We will define by recursion two sequences of real numbers

$$\alpha = (x_1, x_2, \dots, x_n, \dots) \quad \text{and} \quad \beta = (y_1, y_2, \dots, y_n, \dots).$$

We start by defining $x_0 = x_1 := x$ and $y_0 = y_1 := y$. If we have defined x_n, y_n we proceed by defining x_{n+1}, y_{n+1} as follows:

$$(x_{n+1}, y_{n+1}) := \begin{cases} (z, y_n) & \text{if there is some } z \in A \text{ with } \frac{x_n+y_n}{2} < z \leq y_n \\ (x_n, \frac{x_n+y_n}{2}) & \text{if there is no } z \in A \text{ with } \frac{x_n+y_n}{2} < z \leq y_n. \end{cases}$$

We claim that the sequences have the following properties:

- $x_n \in A$ and $x_{n-1} \leq x_n$,
- y_n is an upper bound of A and $y_n \leq y_{n-1}$,
- $0 \leq y_n - x_n \leq \frac{1}{2^{n-1}} \cdot \eta$.

This is proved by induction on $n \in \mathbb{N}$. For $n = 1$ the claim is readily checked. We do then the induction step: assume the properties for a certain $n \in \mathbb{N}$. We must show that there are valid for the value $n + 1 \in \mathbb{N}$.

Suppose first that $(x_{n+1}, y_{n+1}) = (z, y_n)$ where $z \in A$ is such that

$$\frac{x_n + y_n}{2} < z \leq y_n.$$

Then clearly $x_{n+1} = z \in A$. Also, $(x_n + y_n)/2 \geq x_n$ since $y_n \geq x_n$ by the induction hypothesis. Hence, $x_{n+1} = z \geq x_n$.

We also have $y_{n+1} = y_n$, so certainly $y_{n+1} \leq y_n$ and y_{n+1} is an upper bound of A since y_n is. We have $y_{n+1} - x_{n+1} = y_n - z \geq 0$, and

$$y_{n+1} - x_{n+1} = y_n - z \leq y_n - \frac{x_n + y_n}{2} = \frac{y_n - x_n}{2};$$

since $y_n - x_n \leq \frac{1}{2^{n-1}} \cdot \eta$ by the induction hypothesis we can also conclude that

$$y_{n+1} - x_{n+1} \leq \frac{1}{2^n} \cdot \eta.$$

The other possibility is that $(x_{n+1}, y_{n+1}) = (x_n, \frac{x_n+y_n}{2})$ and there is no $z \in A$ with $\frac{x_n+y_n}{2} < z \leq y_n$. Then clearly $x_{n+1} \geq x_n$ and $x_{n+1} \in A$.

Suppose that y_{n+1} were not an upper bound of A; then there would exist $\xi \in A$ such that $\xi > y_{n+1}$; on the other hand, by the induction hypothesis, y_n is an upper bound of A; so we would certainly have $y_n \geq \xi$; thus, there would be an element ξ of A with $\frac{x_n+y_n}{2} = y_{n+1} < \xi \leq y_n$; we would then have a contradiction, and can conclude that y_{n+1} is in fact an upper bound of A.

We have $y_{n+1} = \frac{x_n+y_n}{2} \leq y_n$ since $x_n \leq y_n$ by the induction hypothesis and furthermore,

$$y_{n+1} - x_{n+1} = \frac{y_n - x_n}{2} \leq \frac{1}{2^n} \cdot \eta,$$

again by using the induction hypothesis.

We have now sequences $\alpha = (x_1, x_2, \ldots, x_n, \ldots)$ and $\beta = (y_1, y_2, \ldots, y_n, \ldots)$ with the above properties. In particular, they satisfy:

$$x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots,$$

and

$$y_1 \geq y_2 \geq \ldots \geq y_n \geq \ldots,$$

and we have $|y_n - x_n| \leq \frac{1}{2^{n-1}} \cdot \eta$ for all $n \in \mathbb{N}$. This last property implies that the sequence:

$$(y_1 - x_1, y_2 - x_2, \ldots, y_n - x_n, \ldots)$$

converges to 0: For let $\epsilon > 0$ be arbitrary; by Theorem 3, *(1)*, there is an $N \in \mathbb{N}$ such that $N \cdot \epsilon > \eta$. Then $n \cdot \epsilon > \eta$ for all $n \geq N$. Since one easily proves by induction on k that $2^k > k$ for all $k \in \mathbb{N}$ we can conclude that $2^n \cdot \epsilon > \eta$ for all $n \geq N$. But then $|y_n - x_n| < \epsilon$ for all $n \geq N + 1$.

Now we can apply Lemma 4 to conclude that the sequences α and β are both convergent with the same limit. Call this common limit a.

We claim that a is the least upper bound of A. So we must show that a is an upper bound of A, and that it is smallest among all upper bounds of A.

Let $z \in A$ be any element. Suppose that we had $z > a$. Then $\epsilon := z - a$ would be a positive real number. As the sequence β converges to a there would then certainly be some $n \in \mathbb{N}$ such that $|a - y_n| < \epsilon$. But then we would have $y_n - a < \epsilon = z - a$ and hence $y_n < z$; since this contradicts the fact that y_n is an upper bound of A we can conclude that $z \leq a$.

Hence, a is an upper bound of A.

Let now b be any upper bound of A. We must prove that $b \geq a$. Suppose to the contrary that we had $b < a$. Then $\epsilon := a - b$ would be a positive real number. As the sequence α converges to a there would then certainly be some $n \in \mathbb{N}$ such that $|a - x_n| < \epsilon$. But then we would have $a - x_n < \epsilon = a - b$ and hence $b < x_n$; since $x_n \in A$ this would contradict the fact that b is an upper bound of A. We can conclude that $a \leq b$.

Hence, a is the least upper bound of A. □

4. Final remarks.

In the previous sections we constructed a set $\mathbb{R}$ with certain properties: $\mathbb{R}$ is a field with a total order $\leq$ for which the following holds: If $x, y, z \in \mathbb{R}$ then:

$$x \leq y \Rightarrow x + z \leq y + z$$

and

$$(0 \leq x \wedge 0 \leq y) \Rightarrow 0 \leq xy.$$

A field with a total order satisfying these conditions is called an ordered field. Notice that the statement of Theorem 6 makes sense for any ordered field; if the Theorem is true for a given ordered field one then naturally says that that ordered field has the *least upper bound property*.

So we can say that the set $\mathbb{R}$ that we constructed is an ordered field with the least upper bound property.

One can prove the following *uniqueness theorem*: If L is an ordered field with the least upper bound property then L is isomorphic to $\mathbb{R}$. This means that there is a bijective map $\mathbb{R} \to L$ which respects in the natural way the field operations as well as the order relations.

The uniqueness theorem means that we can speak of the set of real numbers without ambiguity: we may have other constructions of this set (such as the construction via "Dedekind cuts"), but the theorems we can prove about the system of real numbers are independent of which construction we choose.

So what is the advantage of choosing the construction of the previous sections via Cauchy sequences of rational numbers?

The answer is that this construction is one that generalizes to other interesting situations. Thus, in analysis you will later learn about so–called "metric spaces" which are sets equipped with a measure of distance between points. For metric spaces one has a construction called "completion" and this construction is a direct generalization of the above process going from $\mathbb{Q}$ to $\mathbb{R}$.

The construction is also important in number theory: Looking at the field $\mathbb{Q}$ of rational numbers we could define the distance between two numbers a and b as the rational number $|a-b|$. This notion of distance is the so–called archimedean metric (it is thus called because the distance between 0 and a natural number n goes to ∞ with n, – cf. Theorem 3). You can see that this notion of distance is occurring in the definitions of Cauchy sequences, convergence, null–sequences, etc.

But there are other measures of distance between rational numbers than the archimedean metric: in fact, there is a metric called the p-adic metric attached to any prime number p. It is defined as follows: Given $a, b \in Q$ we can write:

$$a - b = p^s \cdot \frac{m}{n}$$

where $s \in \mathbb{Z}$ and $m, n \in \mathbb{Z}$ are both not divisible by p. One then defines:

$$|a - b|_p := p^{-s}.$$

Thus, in the p-adic metric the numbers close to 0 are those that are divisible by high powers of p.

Now, if one goes through with the construction via Cauchy sequences, null–sequences, etc., replacing everywhere $|\cdot|$ by $|\cdot|_p$ one obtains from $\mathbb{Q}$ not the real numbers but another field $\mathbb{Q}_p$ called the field of p-adic numbers. It is not an ordered field and so one can not speak about the least upper bound property and so on. But apart from that the field $\mathbb{Q}_p$ has some of properties of $\mathbb{R}$; for instance, sequences in $\mathbb{Q}_p$ converge if and only if they are Cauchy sequences. The fields $\mathbb{Q}_p$ also have some other, more complicated structures.

The point of these new fields $\mathbb{Q}_p$ is that they can be used to build important theories in number theory and arithmetic geometry that can be used to study such diverse topics as solutions to polynomial equations, for instance Fermat's last theorem, algebraic numbers, etc. etc..

So if you want to study any of those topics you have made a good investment of your time learning the material of the previous sections.

5. Additional exercises

Exercise 1. *Prove that any finite decimal number (i.e., for instance* 2.1415*) is a rational number.*

Exercise 2. *Give several examples of sequences of rational numbers that are not Cauchy sequences. You should prove rigorously that your examples are not Cauchy sequences.*

Exercise 3. *Construct via the following steps a Cauchy sequence of rational numbers that is not convergent in* $\mathbb{Q}$*:*
(1). For any $n \in \mathbb{N}$ *the integer* $2^{2n+5} - 2^{n+4}$ *is positive. Let* k_n *be the smallest integer* ≥ 0 *such that*

$$k_n^2 \geq 2^{2n+5} - 2^{n+4}.$$

Thus, we have in fact $k_n \geq 1$*. Show that we must have* $k_n \leq 2^{n+3}$*. (Hint: use the minimality of* k_n*).*

Use this to prove that $k_n^2 \leq 2^{2n+5}$*. (Hint: otherwise, we could show that* $(k_n - 1)^2 > 2^{2n+5} - 2^{n+4}$ *and obtain a contradiction).*
(2). For $n \in \mathbb{N}$ *let* a_n *denote the rational number* $k_n/2^{n+2}$*:*

$$a_n := \frac{k_n}{2^{n+2}}.$$

Show that:

$$(\dagger) \qquad 2 - \frac{1}{2^n} \leq a_n^2 \leq 2 \quad \text{for all} \;\; n \in \mathbb{N}.$$

(3). Show that $\alpha := (a_1, a_2, \ldots, a_n, \ldots)$ *is a Cauchy sequence of rational numbers: First use* $(\dagger)$ *to conclude that* $a_n \geq 1$ *for all* $n \in \mathbb{N}$*. Then use* $(\dagger)$ *to show:*

$$-\frac{1}{2^m} \leq a_m^2 - a_n^2 \leq \frac{1}{2^n}$$

for all $m, n \in \mathbb{N}$*. Write* $a_m^2 - a_n^2 = (a_m + a_n)(a_m - a_n)$ *and conclude that:*

$$-\frac{1}{2^{m+1}} \leq a_m - a_n \leq \frac{1}{2^{n+1}}$$

for all $m, n \in \mathbb{N}$*. Use this to prove the desired.*
(4). Show that α *is not convergent in* $\mathbb{Q}$*. (Hint: Show that if* α *converged to a rational number* q *then we would be able to prove* $q^2 = 2$*. However, it is known that there exists no rational number with this property).*
(5). Consider the real number $x := \bar{\alpha}$*. Show that* $x^2 = 2$ *in* $\mathbb{R}$*. Thus, we have proved the existence of a square root of* 2 *in* $\mathbb{R}$*.*

Exercise 4. *Let* $\alpha = (x_1, x_2, \ldots, x_n, \ldots)$ *and* $\beta = (y_1, y_2, \ldots, y_n, \ldots)$ *be sequences of real numbers and define in a natural way the sequences* $\alpha + \beta$ *and* $\alpha\beta$*:*

$$\alpha + \beta := (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n, \ldots),$$
$$\alpha\beta := (x_1 y_1, x_2 y_2, \ldots, x_n y_n, \ldots).$$

Suppose that the sequences α *and* β *are both convergent with limits* a *and* b*, respectively.*

Show that the sequences $\alpha + \beta$ *and* $\alpha\beta$ *are then also convergent with limits* $a + b$ *and* ab*, respectively.*

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