NEW MODELS FOR THE ACTION OF HECKE OPERATORS IN SPACES OF MAASS WAVE FORMS.

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ABSTRACT. Utilizing the theory of the Poisson transform, we develop some new concrete models for the Hecke theory in a space $M_\lambda(N)$ of Maass forms with eigenvalue $1/4-\lambda^2$ on a congruence subgroup $\Gamma_1(N)$. We introduce the field $F_\lambda = \mathbb{Q}(\lambda, \sqrt{n}, n^{\lambda/2} \mid n \in \mathbb{N})$ so that F_λ consists entirely of algebraic numbers if $\lambda = 0$.

The main result of the paper is the following. For a packet $\Phi = (\nu_p \mid p \nmid N)$ of Hecke eigenvalues occurring in $M_\lambda(N)$ we then have that either every ν_p is algebraic over F_λ, or else Φ will - for some $m \in \mathbb{N}$ - occur in the first cohomology of a certain space $W_{\lambda,m}$ which is a space of continuous functions on the unit circle with an action of $\mathrm{SL}_2(\mathbb{R})$ well-known from the theory of (non-unitary) principal representations of $\mathrm{SL}_2(\mathbb{R})$.

1. Introduction

We shall fix the following notation throughout the paper: N is a natural number, p always denotes a prime number, Γ is the congruence subgroup $\Gamma_1(N) \leq \mathrm{SL}_2(\mathbb{Z})$, $G := \mathrm{SL}_2(\mathbb{R})$, $\mathbf{T} := \mathbb{R}/\pi\mathbb{Z}$, λ is a complex number, and $M_\lambda(N)$ denotes the complex vector space of Maass forms g on Γ with eigenvalue $\frac{1}{4}-\lambda^2$ for the Laplacian ([8]), i.e. real-analytic functions $g(x,y)$ on the upper half plane $\mathfrak{H} := \{x+iy \mid y > 0\}$ with the following 3 properties:

$$(i) \qquad -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) g = \left(\frac{1}{4}-\lambda^2\right) g \ ,$$

$$(ii) \qquad g(\gamma.z) = g(z),\ z = x+iy,\ \text{for all } \gamma \in \Gamma,$$

where $\gamma.$ denotes the usual action of $\gamma \in \Gamma$ on $\mathfrak{H}$,

(iii) There is a positive number d such that

$$g(x+iy) = O(y^d) \text{ for } y \to +\infty$$

and

$$g(x+iy) = O(y^{-d}) \text{ for } y \to 0+,$$

uniformly in x.

As usual, an element $g \in M_\lambda(N)$ is said to be a cusp form, if it is bounded as a function on $\mathfrak{H}$.

Let $\Delta_1(N)$ be the set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a,b,c,d \in \mathbb{Z}$, $a \equiv 1 \ (N)$, $c \equiv 0 \ (N)$, and positive determinant. Then for $\alpha \in \Delta_1(N)$ we have a Hecke operator T_α acting on $M_\lambda(N)$: Suppose that $\Gamma\alpha\Gamma = \bigcup_i \Gamma\alpha_i$ as a disjoint union, and that $g \in M_\lambda(N)$. Then $g \mid T_\alpha$ is the form given by:

$$(g \mid T_\alpha)(z) := (\det \alpha)^{-1/2} \sum_i g(\alpha_i.z) \ .$$

For primes $p \nmid N$ we shall denote as usual by T_p the Hecke operator belonging to $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$. Also, $\mathbb{T} = \mathbb{T}_N$ will denote the Hecke algebra generated over $\mathbb{C}$ by the operators T_p, $p \nmid N$, acting on $M_\lambda(N)$. Thus, $\mathbb{T}$ is a commutative $\mathbb{C}$-algebra.

If W is a complex vector space which is also a $\mathbb{T}$-module, and if $\Phi = (\nu_p \mid p \nmid N)$ is a system of complex numbers, we say that Φ occurs in W, if $\mathbb{T}$ has a (non-zero) eigenvector in W such that ν_p equals the eigenvalue corresponding to T_p, for all $p \nmid N$.

From an arithmetical point of view, the interest in the Hecke theory of the spaces $M_\lambda(N)$ centers around the case $\lambda = 0$, because a part of 'Langlands' philosophy' predicts that a system $\Phi = (\nu_p \mid p \nmid N)$ occurs as a system of Hecke eigenvalues in the subspace of cusp forms in $M_0(N)$, precisely if there exists an irreducible Galois representation $\rho : \operatorname{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \operatorname{GL}_2(\mathbb{C})$ with finite image, Artin conductor dividing N, $(\det \rho)(\text{complex conjugation}) = 1$, and such that $\nu_p = \operatorname{Tr}\rho(\text{Frobenius at } p)$, $\forall p \nmid N$. A consequence of this prediction is the conjecture that for every such system Φ, every ν_p is an algebraic integer. In [2], [3], an interesting attempt was made to prove this conjecture; however, as discovered by Henniart [7], the approach was unfortunately irreparably flawed. Thus, the conjecture is - in its full generality - still completely open.

It seems clear that any progress in connection with this problem must involve the development of new and more structured models for the Hecke theory in the spaces $M_\lambda(N)$. The purpose of the present paper is to suggest and initiate the study of some new such models. Here is a rough description of the ideas involved. First we utilize the Poisson transform to transform the space $M_\lambda(N)$ to a space of distributions on the unit circle $\mathbf{T}$, - equipped with a certain on λ depending action of $\operatorname{GL}_2^+(\mathbb{R})$. The basic idea is then - roughly speaking - to study 'point values' - and the action of Hecke operators on them - of these distributions for points in a certain countable dense subset $\Xi \subseteq \mathbf{T}$. The actual realization of this idea is a bit technical: By representing the space of distributions in question as derivatives of continuous functions, one ultimately ends up - for some $k \in \mathbb{N}_0$ - with a very short exact sequence of (cohomological) Hecke modules

$$(*) \qquad H^0(\Gamma, U_{\lambda,k}) \longrightarrow M_\lambda(N) \longrightarrow H^1(\Gamma, V_{\lambda,k}) \ ;$$

here, $U_{\lambda,k}$ is the space of $(k+1)$-tuples of continuous functions on $\mathbf{T}$ with absolutely converging Fourier series and equipped with a certain $\operatorname{GL}_2^+(\mathbb{R})$-action, and $V_{\lambda,k}$ is a subspace of $U_{\lambda,k}$ that is shown to have a natural filtration whose successive quotients are isomorphic to spaces $W_{\lambda,m}$, $1 \le m \le k$, introduced in definition 3 below: These are spaces consisting of continuous functions on $\mathbf{T}$ with a certain growth condition on their Fourier coefficients and equipped with an - on (λ, m) depending - action of $\operatorname{GL}_2^+(\mathbb{R})$ that is easily recognizable from the theory of (non-unitary) principal series representations (of $\operatorname{SL}_2(\mathbb{R})$). Apart from some remarks at the end of the paper, we shall make no further references to representation theory of $\operatorname{SL}_2(\mathbb{R})$ as we shall introduce all concepts in a completely explicit and self-contained manner.

Using $(*)$ combined with a study of the evaluation of elements of $H^0(\Gamma, U_{\lambda,k})$ at points of the above $\Xi \subseteq \mathbf{T}$, and by using the Ash-Stevens 'lifting lemma' for packages of Hecke eigenvalues ([1]), we are then able to prove the following theorem.

Theorem 1. *Let $\Phi = (\nu_p \mid p \nmid N)$ be a system of Hecke eigenvalues occurring in $M_\lambda(N)$. Then either*

(1) *Every ν_p is algebraic over the field $\mathbb{Q}(\lambda, \sqrt{n}, n^{\lambda/2} \mid \ n \in \mathbb{N})$,*

or

(2) *Φ occurs in $H^1(\Gamma, W_{\lambda,m})$ for some $m \in \mathbb{N}$.*

2.

In this section we prove Theorem 1. We proceed first by recalling some facts about the action of Hecke operators on cohomology of Hecke-modules, then with various preliminary constructions before introducing the spaces $W_{\lambda,m}$ and proving Theorem 1.

2.1. In the terminology of [1], section 1.1, we will be concerned with the Hecke algebra $\mathcal{H}$ of the Hecke pair $(\Gamma, \Delta_1(N))$, so that our $\mathbb{T}$ introduced above is a subalgebra of $\mathcal{H} \otimes \mathbb{C}$. So, if M is a right $\mathbb{C}\Delta_1(N)$-module, we have a natural right action of $\mathbb{T}$ on the cohomology groups $H^i(\Gamma, M)$, $i \geq 0$, cf. *loc. cit.*. In particular, we have a right action of $\mathbb{T}$ on the Γ-fixed points of M that we consistently denote M^Γ, and also on $H^1(\Gamma, M)$: Explicitly, if $\alpha \in \Delta_1(N)$ with $\Gamma\alpha\Gamma = \cup_i \Gamma\alpha_i$ (disjoint), the action of the Hecke operator T_α on a homogeneous 1-cocycle $c\colon \Gamma \times \Gamma \to M$ is given by

$$(c \mid T_\alpha)(\gamma_0, \gamma_1) := \sum_i c(t_i(\gamma_0), t_i(\gamma_1)) \mid \alpha_i \ ,$$

where $t_i\colon \Gamma \to \Gamma$ is the map determined by the requirements $\Gamma\alpha_i\gamma = \Gamma\alpha_j$, for some j depending on i and γ, and $\alpha_i\gamma = t_i(\gamma)\alpha_j$.

Let us also recall that the right action of $\mathbb{T}$ commutes with the long exact cohomology sequence associated with a short exact sequence of $\mathbb{C}\Delta_1(N)$-modules.

2.2. In this subsection, we shall consider the Poisson-transformation associated with the upper half plane $\mathfrak{H}$ interpreted as the symmetric space $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$, use this to pull back the action of $\mathrm{GL}_2^+(\mathbb{R})$ on Δ-eigenfunctions on $\mathfrak{H}$ to the space of distributions on $\mathbf{T} = \mathbb{R}/\pi\mathbb{Z}$, and establish an isomorphism as Hecke-modules between $M_\lambda(N)$ and a certain space of distributions on $\mathbf{T}$. Let us first introduce the appropriate actions:

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$ and $\theta \in \mathbf{T}$, we put:

$$j(\gamma, \theta) := \left((a\cos\theta + b\sin\theta)^2 + (c\cos\theta + d\sin\theta)^2\right)^{1/2} \ ,$$

so that for fixed γ, $j(\gamma, \cdot)$ is a C^∞ function on $\mathbf{T}$. Secondly, we define $\gamma.\theta \in \mathbf{T}$ by the requirement

$$(\cos\gamma.\theta, \sin\gamma.\theta) = \pm\left(\frac{a\cos\theta + b\sin\theta}{j(\gamma,\theta)}, \frac{c\cos\theta + d\sin\theta}{j(\gamma,\theta)}\right) \ ,$$

where the sign is chosen such that $\gamma.\theta \in [0, \pi[$. One immediately verifies that $(\gamma, \theta) \mapsto \gamma.\theta$ actually defines an action of $\mathrm{GL}_2^+(\mathbb{R})$ on $\mathbf{T}$, and that we have

$$j(\gamma_1\gamma_2, \theta) = j(\gamma_1, \gamma_2.\theta) j(\gamma_2, \theta) \ .$$

With this, we see that we have a right action $|_\lambda$ of $\mathrm{GL}_2^+(\mathbb{R})$ on C^∞-functions on the torus $\mathbf{T}$ given by:

$$(\varphi \mid_\lambda \gamma)(\theta) := (\det\gamma)^{1+\lambda/2} j(\gamma, \theta)^{-1-\lambda} \cdot \varphi(\gamma.\theta) \ , \quad \text{for } \varphi \in C^\infty(\mathbf{T}) \ .$$

Definition 1. *Define $\mathcal{D}_\lambda$ to be the complex vector space of distributions on $\mathbf{T}$ with the (on λ depending) right action of $\mathrm{GL}_2^+(\mathbb{R})$ given by:*

$$(\Lambda \mid_\lambda \gamma)\varphi := \Lambda(\varphi \mid_\lambda \gamma^{-1})$$

for $\Lambda \in \mathcal{D}_\lambda$, $\varphi \in C^\infty(\mathbf{T})$, and $\gamma \in \mathrm{GL}_2^+(\mathbb{R})$.

With this definition, the space $\mathcal{D}_\lambda^\Gamma$ is also endowed with the structure of a $\mathbb{T}$-module. Explicitly, if $\Lambda \in \mathcal{D}_\lambda^\Gamma$ and $\alpha \in \Delta_1(N)$ with $\Gamma\alpha\Gamma = \bigcup_i \Gamma\alpha_i$ as a disjoint union, then

$$\Lambda \mid T_\alpha := \sum_i (\Lambda \mid_\lambda \alpha_i) \ .$$

Proposition 1. *Suppose that* $\mathrm{Re}(\lambda) \geq 0$*. Then*

$$\mathcal{D}_\lambda^\Gamma \cong M_\lambda(N)$$

as $\mathbb{T}$-modules.

Proof. Consider the maximal compact subgroup

$$K := \mathrm{SO}_2(\mathbb{R}) = \left\{ r(\theta) := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mid \ \theta \in]-\pi, \pi] \right\}$$

in G. We identify $\mathfrak{H}$ with the symmetric space G/K; explicitly, $x + iy \in \mathfrak{H}$ is identified with the coset $g_{x,y}K$ where

$$g_{x,y} := \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix} \ .$$

We shall now utilize the Poisson transform and the Helgason isomorphism associated with this situation. We shall use the particular version given in [6], Chap. IV, Theorem 5, and proceed now with introducing the necessary notation.

Consider the following standard subgroups of G:

$$A := \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid \ a \in \mathbb{R}_+ \right\} \ ,$$

whose Lie algebra $\mathfrak{a}$ we identify with $\mathbb{R}$, and

$$N := \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mid \ t \in \mathbb{R} \right\}, \quad M := \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \right\},$$

so that we have the Iwasawa decomposition $G = KAN$, and can identify K/M with $\mathbf{T}$. Explicitly, $r(\theta)M$ is identified with

$$\theta \pmod{\pi} \in \mathbf{T} \ ,$$

and the Iwasawa decomposition of an element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ has the shape

$$g = r(\theta) \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \ ,$$

with $u := (a^2 + c^2)^{1/2}$ and $\theta \in]-\pi, \pi]$ determined by $\cos\theta = a/u$, $\sin\theta = c/u$. The map $h \colon G \longrightarrow \mathfrak{a}$ is the map uniquely determined by the requirement

$$g \in K \exp(h(g))N, \quad \text{for } g \in G \ .$$

Explicitly, one finds (with the identification of $\mathfrak{a}$ with $\mathbb{R}$)

$$h\left(\left(\begin{array}{cc} a & b \\ c & d \end{array}\right)\right) = \log(a^2+c^2)^{1/2} \ .$$

Finally, define for $g \in G$:

$$|g| := \mathrm{trace}_{\mathfrak{g}}(\mathrm{ad}(gg^t)) \ ,$$

where g^t is the transpose of g, and $\mathfrak{g}$ the Lie algebra of G. One shows that $|k_1 g k_2| = |g|$ for $k_1, k_2 \in K$, so that it makes sense to define the notion of *at the most exponential growth* for functions f on G/K by the requirement

$$|f(g)| \le a|g|^b$$

with some constants $a \in \mathbb{R}_+$, $b \in \mathbb{R}$.

Now we specialize Chap. IV, 2 of [6], in particular Theorem 5 (ii), to our present situation: We identify elements λ of the space of linear forms on the complexified Lie algebra $\mathfrak{a}_{\mathbb{C}}$ with complex numbers. The Poisson transform $\mathcal{P}_\lambda$ associated with λ is then a linear map from the space $\mathcal{D}(K/M)$ of distributions on K/M to the space E_λ of C^∞ functions on G/K with at the most exponential growth and are eigenfunctions for the Laplace-Beltrami operator on G/K with an eigenvalue that we specify below; $\mathcal{P}_\lambda$ is given explicitly by

$$(*) \qquad (\mathcal{P}_\lambda\Lambda)(gK) := \Lambda\left(kM \mapsto e^{(-\lambda-1)\cdot h(g^{-1}k)} \ , \ k \in K\right) \ ,$$

for $\Lambda \in \mathcal{D}(K/M)$. Furthermore, Theorem 5 (ii) of *loc. cit.* implies that $\mathcal{P}_\lambda$ is an *isomorphism*, provided that $\mathrm{Re}(\lambda) \ge 0$ which is insured in the present case by assumption. Using the above identification of K/M with $\mathbf{T}$ and of G/K with $\mathfrak{H}$ via $x+iy \leftrightarrow g_{x,y}K$, it is straightforward to verify that we can rewrite $(*)$ as

$$(**) \qquad (\mathcal{P}_\lambda\Lambda)(x+iy) = \Lambda\left(\theta \mapsto b(x,y,\theta)^{\frac{-1-\lambda}{2}} \ , \ \theta \in \mathbf{T}\right),$$

where

$$b(x,y,\theta) := y^{-1}\left(\cos^2\theta + x^2\sin^2\theta - 2x\cos\theta\sin\theta\right) + y\sin^2\theta \ ,$$

which arises because of

$$h(g_{x,y}^{-1}r(\theta)) = \frac{1}{2}\log b(x,y,\theta) \ .$$

From this, one readily checks that elements of E_λ viewed as C^∞ functions on $\mathfrak{H}$ have Δ-eigenvalue $1/4-\lambda^2$. Furthermore, it is tedious but straightforward to check that we have defined the action of $\mathrm{GL}_2^+(\mathbb{R})$ on $\mathcal{D}$ precisely so as to make $\mathcal{P}_\lambda$ equivariant w.r.t. to $\mathrm{GL}_2^+(\mathbb{R})$-action. Thus, we may conclude that

$$\mathcal{D}^\Gamma \cong E_\lambda^\Gamma \ ,$$

as $\mathbb{T}$-modules. Hence the proof is concluded by showing that

$$M_\lambda(N) = E_\lambda^\Gamma \ .$$

Now, the growth condition on elements of $f \in E_\lambda$ viewed as functions on $\mathfrak{H}$ requires the existence of constants $a \in \mathbb{R}_+$ and $b \in \mathbb{R}$ (depending on f) such that

$$|f(x+iy)| \le a|g_{x,y}|^b \ ,$$

and as a simple computation shows that

$$|g_{x,y}| = 1 + 2x^2y^{-2} + (y + x^2y^{-1})^2 + y^{-2} \ge \max\{y^2, y^{-2}\} \ ,$$

it is immediately clear that $M_\lambda(N) \le E_\lambda^\Gamma$. On the other hand, one sees that $|g_{x,y}| \le (\text{const.}) \cdot (y^2 + y^{-2})$ for $x \in [0,1]$, so that if $f \in E_\lambda^\Gamma$ then certainly for some positive d we have

$$f(x+iy) = O(y^d) \quad \text{as } y \to +\infty$$

and

$$f(x+iy) = O(y^{-d}) \quad \text{as } y \to 0+$$

uniformly in $x \in [0,1]$. But since f is invariant under the substitution $x \mapsto x+1$, this holds uniformly in $x \in \mathbb{R}$. Thus, $f \in M_\lambda(N)$. □

If f is a continuous complex-valued function on $\mathbf{T}$, we shall write

$$f \sim \sum_{n \in \mathbb{Z}} a_n e^{2in\theta}$$

(only) to signal that the Fourier coefficients of f are being denoted by a_n, $n \in \mathbb{Z}$.

If $\Lambda \in \mathcal{D}(\mathbf{T})$, the Fourier coefficients of Λ are

$$a_n := \Lambda(e^{-2in\theta}),$$

and as is well-known, a given sequence (a_n) of complex numbers is the sequence of Fourier coefficients of some distribution $\Lambda \in \mathcal{D}(\mathbf{T})$ if and only if there exists an integer $m \ge 0$ such that

$$\sum_n (1+n^2)^{-m} |a_n|^2 < \infty \ ;$$

if this is the case, there is then an integer $s \ge 0$ such that the series

$$\sum_n (1+2in)^{-s} \cdot a_n$$

converges absolutely. Then

$$f(\theta) := \sum_n (1+2in)^{-s} a_n e^{2in\theta}$$

defines a continuous function on $\mathbf{T}$, and with the notation

$$\partial := 1 + \frac{d}{d\theta} \ ,$$

we have $\Lambda = \partial^s f$ in the usual sense, i.e.

$$\Lambda\varphi = \frac{1}{\pi} \int_{\mathbf{T}} f(\theta)(\partial^s \varphi)(\theta) d\theta$$

for $\varphi \in C^\infty(\mathbf{T})$. It will be convenient for us to consider distributions representable in slightly more general form

$$(\dagger) \qquad \Lambda = \sum_{j=0}^{s} \partial^j f_j \ ,$$

where the f_j are continuous functions on $\mathbf{T}$ *with absolutely converging Fourier series*. We now define some auxiliary objects, and then the spaces occurring in Theorem 1.

Definition 2. *For $s \in \mathbb{N}_0$, denote by $\mathcal{D}_{\lambda,s}$ the subspace of $\mathcal{D}(\mathbf{T})$ consisting of distributions Λ representable in the form (†) and with the right $\mathrm{GL}_2^+(\mathbb{R})$-action given in Definition 1. Define also $U_{\lambda,s}$ to be the complex vector space consisting of $(s+1)$-tuples $(f_0,\dots,f_s)$ of continuous functions on $\mathbf{T}$ with absolutely converging Fourier series, and define $V_{\lambda,s}$ to be the subspace of $U_{\lambda,s}$ consisting of tuples $(f_0,\dots,f_s)$ with*

$$\sum_j \partial^j f_j = 0$$

in the distribution sense. Because of Proposition 1 and [8], Satz 5, the space $\mathcal{D}_\lambda^\Gamma$ is finite-dimensional. Consequently, there exists a non-negative integer k such that

$$\mathcal{D}_\lambda^\Gamma \le \mathcal{D}_{\lambda,k} \ ,$$

so that obviously,

$$\mathcal{D}_{\lambda,k}^\Gamma = \mathcal{D}_\lambda^\Gamma \ .$$

Definition 3. *Let $m \in \mathbb{N}$. We define $W_{\lambda,m}$ as the complex vector space consisting of continuous functions $f \sim \sum_{n\in\mathbb{Z}} a_n e^{2in\theta}$ on $\mathbf{T}$ with*

$$\sum_{n\in\mathbb{N}} |n||a_n| < \infty \ ,$$

and with the following on (λ, m) depending right action of $\mathrm{GL}_2^+(\mathbb{R})$:

$$(f\,|_{\lambda,m}\,\gamma)(\theta) := f(\gamma.\theta)(\det\gamma)^{-m-\lambda/2} j(\gamma,\theta)^{(2m-1)+\lambda} \ .$$

Using the fact that $j(\gamma,\cdot) \in C^\infty(\mathbf{T})$, we have $j(\gamma,\cdot) \in W_{\lambda,m}$ for every $m \in \mathbb{N}$, and it is easily verified that $\cdot\,|_{\lambda,m}\,\gamma$ actually maps $W_{\lambda,m}$ into itself so that the definition makes sense.

The definitions of $U_{\lambda,s}$ and $V_{\lambda,s}$ are such that we have a natural exact sequence of complex vector spaces

$$0 \longrightarrow V_{\lambda,s} \longrightarrow U_{\lambda,s} \longrightarrow \mathcal{D}_{\lambda,s} \longrightarrow 0 \ ,$$

where the map $U_{\lambda,s} \longrightarrow \mathcal{D}_{\lambda,s}$ is given by $(f_0,\dots,f_s) \mapsto \sum_j \partial^j f_j$. In the next subsection we lift the action of $\mathrm{GL}_2^+(\mathbb{R})$ on $\mathcal{D}_{\lambda,s}$ to the space $U_{\lambda,s}$ so that this sequence becomes an exact sequence of $\Delta_1(N)$-modules, and we show that the $W_{\lambda,m}$ appear as subquotients of the modules $V_{\lambda,s}$. In subsection 2.4, we prove a theorem concerning eigenvalues of Hecke operators acting on the space $U_{\lambda,s}^\Gamma$ of Γ-fixed points of $U_{\lambda,s}$. With this preparation, we then proceed in subsection 2.5 to finish the proof of Theorem 1.

2.3.

Lemma 1. *For each $s \in \mathbb{N}_0$ and to each $\gamma \in \mathrm{GL}_2^+(\mathbb{R})$ there exist on λ depending, uniquely determined C^∞ functions $u_\gamma^{s,j}(\theta)$, $j \in \mathbb{Z}$, on $\mathbf{T}$ with $u_\gamma^{s,j} = 0$ for $j < 0$ and for $j > s$, and with the following properties.*

(1) *For all $\varphi \in C^\infty(\mathbf{T})$ we have*

$$\partial^s(\varphi\,|_\lambda\,\gamma)(\theta) = (\det\gamma)^{1+\lambda/2} \sum_{j=0}^s u_\gamma^{s,j}(\theta)\cdot(\partial^j\varphi)(\gamma.\theta) \ .$$

(2) *For all s,*

$$u_\gamma^{s,s}(\theta) = j(\gamma,\theta)^{-1-\lambda-2s}\cdot(\det\gamma)^s.$$

(3) *For all s and $j = 0, \dots s$, the function $j(\gamma,\theta)^\lambda u_\gamma^{s,j}(\theta)$ is a polynomial in the functions*

$$\frac{d^\ell}{d\theta^\ell} j(\gamma,\theta) \quad , \ \ell = 0, \dots, s-j,$$

with coefficients in $\mathbb{Z}[\lambda, (\det\gamma)]$.
(4) *For $\gamma_1, \gamma_2 \in \mathrm{GL}_2^+(\mathbb{R})$,*

$$u_{\gamma_1\gamma_2}^{s,j}(\theta) = \sum_{\ell=j}^{s} u_{\gamma_2}^{s,\ell}(\theta) \cdot u_{\gamma_1}^{\ell,j}(\gamma_2.\theta) \ .$$

Proof. This is a completely trivial exercise in differential calculus so we shall be very brief. Define first $u_\gamma^{s,j}(\theta) := 0$ for $j < 0$ and for $j > s$. Put $u_\gamma^{0,0}(\theta) := j(\gamma,\theta)^{-1-\lambda}$, and recursively (w.r.t. s)

$$u_\gamma^{s+1,j}(\theta) := \partial u_\gamma^{s,j}(\theta) + (\det\gamma) \cdot j(\gamma,\theta)^{-2} \left(u_\gamma^{s,j-1}(\theta) - u_\gamma^{s,j}(\theta) \right),$$

for $j = 0, \dots s+1$. Then (2) and (3) are immediately clear, and (1) is easily proved by induction on s using the relation $\partial(fg) = -fg + f(\partial g) + (\partial f)g$ and that

$$\frac{d(\gamma.\theta)}{d\theta} = (\det\gamma) \cdot j(\gamma,\theta)^{-2} \ .$$

Notice that the case $s = 0$ in (1) is merely the definition of the action $|_\lambda \gamma$ on C^∞ functions. Uniqueness for a given γ of functions $u_\gamma^{s,j}(\theta)$ with property (1), is shown by assuming functions $w_\gamma^{s,j}(\theta)$ given with

$$0 = \sum_{j=0}^{s} w_\gamma^{s,j}(\theta) \cdot (\partial^j \varphi)(\gamma.\theta)$$

for all $\varphi \in C^\infty(\mathbf{T})$. Using this on test functions $\varphi(\theta) = e^{2im\theta}$, $m \in \mathbb{N}$, changing variables $\theta \mapsto \gamma^{-1}.\theta$, and letting $m \to \infty$, one obtains successively $w_\gamma^{s,s} = 0, \dots, w_\gamma^{s,0} = 0$.

Finally, uniqueness for each fixed γ of functions $u_\gamma^{s,j}$ with (1) proves (4): Applying ∂^s to the a function $\varphi |_\lambda \gamma_1\gamma_2 = (\varphi |_\lambda \gamma_1) |_\lambda \gamma_2$, we see that (1) holds for $\gamma = \gamma_1\gamma_2$ if $u_{\gamma_1\gamma_2}^{s,j}$ is replaced by the function

$$\sum_{\ell=0}^{s} u_{\gamma_2}^{s,\ell}(\theta) \cdot u_{\gamma_1}^{\ell,j}(\gamma_2.\theta) = \sum_{\ell=j}^{s} u_{\gamma_2}^{s,\ell}(\theta) \cdot u_{\gamma_1}^{\ell,j}(\gamma_2.\theta) \ .$$

By uniqueness, (4) follows. □

Definition 4. *Define for $s \in \mathbb{N}_0$ and $\gamma \in \mathrm{GL}_2^+(\mathbb{R})$ a matrix function (depending on λ) of size $(s+1) \times (s+1)$ by*

$$A_\gamma^{(s)}(\theta) := \Big((\det\gamma)^{-\lambda/2} j(\gamma,\theta)^{-2} u_{\gamma^{-1}}^{\mu,\nu}(\gamma.\theta) \Big)_{0 \le \mu,\nu \le s} \ ,$$

$\theta \in \mathbf{T}$, where the $u_{\gamma^{-1}}^{\mu,\nu}$ are the uniquely determined functions from Lemma 1.

Define also for $(f_0, \dots, f_s) \in U_{\lambda,s}$

$$(f_0, \dots, f_s) |_\lambda \gamma = (\tilde{f}_0, \dots, \tilde{f}_s) \ ,$$

where

$$(\tilde{f}_0(\theta), \dots, \tilde{f}_s(\theta) := (f_0(\gamma.\theta), \dots, f_s(\gamma.\theta)) \cdot A_\gamma^{(s)}(\theta) \ ,$$

and the linear map $h_s\colon U_{\lambda,s} \longrightarrow \mathcal{D}_{\lambda,s}$ *by*

$$h_s(f_0,\ldots,f_s) := \sum_j \partial^j f_j \ .$$

Proposition 2. (1) *The map* $|_\lambda\ \gamma$ *defines a right* $\mathrm{GL}_2^+(\mathbb{R})$*-action on* $U_{\lambda,s}$ *such that* h_s *is a homomorphism of* $\mathrm{GL}_2^+(\mathbb{R})$*-modules.*

(2) *If* $s>0$*, the natural injection* $\iota_{s-1}\colon U_{\lambda,s-1} \longrightarrow U_{\lambda,s}$ *given by*

$$(f_0,\ldots,f_{s-1}) \mapsto (f_0,\ldots,f_{s-1},0)$$

is a $\mathrm{GL}_2^+(\mathbb{R})$*-homomorphism that injects* $V_{\lambda,s-1}$ *into* $V_{\lambda,s}$*.*

(3) *For* $s>0$*, if* $V_{\lambda,s-1}$ *is viewed as a* $\mathrm{GL}_2^+(\mathbb{R})$*-submodule of* $V_{\lambda,s}$*, we have*

$$V_{\lambda,s}/V_{\lambda,s-1} \cong W_{\lambda,s}$$

as $\mathrm{GL}_2^+(\mathbb{R})$*-modules, and hence also as* $\mathbb{T}$*-modules.*

Proof. (1) Notice first that $A_\gamma^{(s)}(\theta)$ has C^∞ coefficients so that $|_\lambda\ \gamma$ actually maps $U_{\lambda,s}$ into itself. That we have thus defined a right $\mathrm{GL}_2^+(\mathbb{R})$-action follows from the formula

$$A^{(s)}_{\gamma_1\gamma_2}(\theta) = A^{(s)}_{\gamma_1}(\gamma_2.\theta)A^{(s)}_{\gamma_2}(\theta)$$

which is easily verified on the basis of Lemma 1, (4).

To see that h_s commutes with the action of $\mathrm{GL}_2^+(\mathbb{R})$, suppose that

$$F := (f_0,\ldots,f_s) \in U_{\lambda,s}\ ,$$

put $\Lambda := h_s(F)$, and suppose that $\varphi \in C^\infty(\mathbf{T})$. Then by definition of $|_\lambda\ \gamma$ and $A_\gamma^{(s)}(\theta)$, the function $h_s(F\ |_\lambda\ \gamma)\varphi$ equals

$$\frac{1}{\pi}\int_{\mathbf{T}} \sum_{\ell,j} \Big(f_j(\gamma.\theta)\cdot(\det\gamma)^{-\lambda/2} j(\gamma,\theta)^{-2} u^{j,\ell}_{\gamma^{-1}}(\gamma.\theta)\Big)\cdot(\partial^\ell\varphi)(\theta)d\theta,$$

where the summation is over $\ell, j = 0,\ldots,s$. Making the change of variables $\theta \mapsto \gamma^{-1}.\theta$, using $\frac{d(\gamma.\theta)}{d\theta} = (\det\gamma)\cdot j(\gamma,\theta)^{-2}$, and remembering the properties of the $u^{j,\ell}_{\gamma^{-1}}$, we find that this integral equals

$$\Lambda(\varphi\ |_\lambda\ \gamma^{-1}) = (\Lambda\ |_\lambda\ \gamma)\varphi\ .$$

(2) That ι_{s-1} is a $\mathrm{GL}_2^+(\mathbb{R})$-homomorphism is a consequence of the definition of the action of $\mathrm{GL}_2^+(\mathbb{R})$ combined with the observation that the matrix $A^{(s)}_\gamma(\theta)$ is an upper triangular $(s+1)\times(s+1)$ matrix whose upper left $s\times s$ minor coincides with $A^{(s-1)}_\gamma(\theta)$. That ι_{s-1} maps $V_{\lambda,s-1}$ into $V_{\lambda,s}$ is trivial.

(3) Suppose that $s>0$ and that $(f_0,\ldots,f_s)\in V_{\lambda,s}$. Denote the Fourier coefficients of f_j by $a_n(f_j)$, $n\in\mathbb{Z}$. By definition of $V_{\lambda,s}$, the distribution $\Lambda = \sum_j \partial^j f_j \in \mathcal{D}(\mathbf{T})$ is 0, and hence every Fourier coefficient $a_n(\Lambda)$ of Λ vanishes. But we have

$$a_n(\Lambda) = \sum_{j=0}^{s} (1+2in)^j a_n(f_j)\ ,$$

hence

$$(1+2in)a_n(f_s) = -\sum_{j=0}^{s-1}(1+2in)^{j-s+1}a_n(f_j)$$

for each $n \in \mathbb{Z}$. As each of the series $\sum_n a_n(f_j)$ is absolutely convergent, the same holds for the series $\sum_n (1+2in)a_n(f_s)$. It follows that f_s lies in the space $W_{\lambda,s}$. Consequently, we can define a linear map $\psi_s \colon V_{\lambda,s} \longrightarrow W_{\lambda,s}$ by

$$\psi_s(f_0,\ldots,f_s) := f_s \ .$$

Then clearly the kernel of ψ_s is $V_{\lambda,s-1}$ viewed as a subspace of $V_{\lambda,s}$ via ι_{s-1}. We claim that ψ_s is surjective. Suppose that $f \in W_{\lambda,s}$ with Fourier coefficients $a_n(f)$. Then if we define the numbers

$$a_n := -(1+2in)a_n(f)$$

for $n \in \mathbb{Z}$, the series $\sum_n a_n$ converges absolutely. There is thus a continuous function g on $\mathbf{T}$ with Fourier coefficients $a_n(g) = a_n$, and $F := (0,\ldots,0,g,f)$ is an element of $U_{\lambda,s}$. The Fourier coefficients of the distribution

$$\partial^{s-1} g + \partial^s f$$

are all 0, so F is in fact an element of $V_{\lambda,s}$. But $\psi_s(F) = f$.

It remains to be seen that ψ_s commutes with the action of the group $\mathrm{GL}_2^+(\mathbb{R})$. Let $\mathbf{f} := (f_0,\ldots,f_s) \in V_{\lambda,s}$ and let $\gamma \in \mathrm{GL}_2^+(\mathbb{R})$. By the definition of $\mathbf{f} \mid_\lambda \gamma$ we find

$$\mathbf{f} \mid_\lambda \gamma = (\ldots, f_s(\gamma.\theta)(\det\gamma)^{-\lambda/2} j(\gamma,\theta)^{-2} u_{\gamma^{-1}}^{s,s}(\gamma.\theta)) \ .$$

Together with Lemma 1 (2) and $j(\gamma^{-1},\gamma.\theta) = j(\gamma,\theta)^{-1}$, this gives

$$(\psi_s(\mathbf{f} \mid_\lambda \gamma))(\theta) = f_s(\gamma.\theta)(\det\gamma)^{-s-\lambda/2} j(\gamma,\theta)^{2s-1+\lambda} \ ,$$

which is precisely the definition of $f_s \mid_{\lambda,s} \gamma$ for $f_s \in W_{\lambda,s}$. □

2.4. The purpose of this subsection is to prove a statement concerning possible eigenvalues of Hecke operators acting on the spaces $U_{\lambda,s}^\Gamma$. We proceed with some preparations.

Let us denote by F_λ the field occurring in Theorem 1, (1), i.e.,

$$F_\lambda := \mathbb{Q}(\lambda, \sqrt{n}, n^{\lambda/2} \mid \ n \in \mathbb{N}) \ .$$

Denote also by Ξ the subset of $\mathbf{T} = \mathbb{R}/\pi\mathbb{Z}$ consisting of those $\theta \in \mathbf{T}$ for which $\cot\theta \in \mathbb{P}^1(\mathbb{Q})$, i.e.,

$$\Xi := \{0\} \cup \{0 \neq \theta \in \mathbf{T} \mid \cot\theta \in \mathbb{Q}\} \ .$$

Lemma 2. (1) *Ξ is dense in $\mathbf{T}$ and stable under the action on $\mathbf{T}$ of the group $\mathrm{GL}_2^+(\mathbb{Q})$.*

(2) *Suppose that $\theta_0 \in \Xi$, $\ell \in \mathbb{N}_0$, and $\gamma \in \mathrm{GL}_2^+(\mathbb{Q})$. Then*

$$\frac{d^\ell}{d\theta^\ell} j(\gamma,\theta)_{|\theta=\theta_0} \in F_0 \ .$$

(3) *Suppose that $\theta_0 \in \Xi$, $s \in \mathbb{N}_0$, and $\gamma \in \mathrm{GL}_2^+(\mathbb{Q})$. Then the matrix*

$$A_\gamma^{(s)}(\theta_0)$$

(definition 4) is a $(s+1)\times(s+1)$ matrix with coefficients in F_λ.

Proof. (1) The first statement is clear, and the second follows from the formula

$$\cot \gamma.\theta = \frac{a\cot\theta + b}{c\cot\theta + d}$$

for $\gamma \in \mathrm{GL}_2^+(\mathbb{R})$.

(2) Notice first that $\cos\theta, \sin\theta \in F_0$ if $\theta \in \Xi$. Now, if $\ell = 0$, the statement is clear if $\sin\theta_0 = 0$, and also if $\sin\theta_0 \neq 0$ since then

$$j(\gamma, \theta_0) = \pm \sin\theta_0 \left((a\cot\theta_0 + b)^2 + (c\cot\theta_0 + d)^2\right)^{1/2} \in F_0 \ .$$

For $\ell \geq 1$, the statement follows from this by induction on ℓ when one shows by induction on $\ell \geq 1$ that

$$\frac{d^\ell}{d\theta^\ell} j(\gamma,\theta) = j(\gamma,\theta)^{-\ell} \cdot p_{\gamma,\ell}(\theta) \ ,$$

where $p_{\gamma,\ell}(\theta)$ is a polynomial with rational coefficients in $\cos\theta$, $\sin\theta$ and the $\frac{d^\mu}{d\theta^\mu} j(\gamma,\theta)$, $\mu = 0, \ldots, \ell - 1$.

(3) This follows from (2), the definition of $A_\lambda^{(s)}(\theta)$, and from Lemma 1, (3). □

Now let $s \in \mathbb{N}_0$ and $\alpha \in \Delta_1(N)$. We shall consider the action of the Hecke operator T_α on a space $U_{\lambda,s}^\Gamma = H^0(\Gamma, U_{\lambda,s})$. The action of T_α on this space is as a cohomological Hecke operator which we can describe explicitly as follows. Let

$$\Gamma\alpha\Gamma = \cup_{\mu=1}^r \Gamma\alpha_\mu \ , \quad \alpha_\mu \in \Delta_1(N) \ ,$$

as a disjoint union. Then if $\mathbf{f} = (f_0, \ldots, f_s) \in U_{\lambda,s}^\Gamma$, we have

$$\begin{aligned}(\mathbf{f} \mid_\lambda T_\alpha)(\theta) :=& \quad \textstyle\sum_{\mu=1}^r ((f_0, \ldots, f_s) \mid_\lambda \alpha_\mu)(\theta) \\ =& \quad \textstyle\sum_{\mu=1}^r (f_0(\alpha_\mu.\theta), \ldots, f_s(\alpha_\mu.\theta)) A_{\alpha_\mu}^{(s)}(\theta).\end{aligned}$$

Let us also as usual denote by $\Gamma_\infty := \langle \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \rangle$ the stabilizer in Γ of the cusp $\infty \in \mathbb{P}^1(\mathbb{Q})$. Notice that if Γ is viewed as acting on $\mathbf{T}$ then Γ_∞ is precisely the stabilizer of $0 \in \mathbf{T}$. Choose a decomposition of $\mathrm{SL}_2(\mathbb{Z})$ in double cosets w.r.t. (Γ, Γ_∞):

$$\mathrm{SL}_2(\mathbb{Z}) = \cup_{\nu=1}^m \Gamma\gamma_\nu\Gamma_\infty \ ,$$

so that the set $\{\gamma_\nu\}$ is in 1-1 correspondence with the cusps w.r.t. Γ. We define a linear map ϕ_s of $U_{\lambda,s}^\Gamma$ into $\mathrm{Mat}_{m,s+1}(\mathbb{C}) \cong \mathbb{C}^{m(s+1)}$ by

$$\phi_s(f_0, \ldots, f_s) := \begin{pmatrix} f_0(\gamma_1.0) & \ldots & f_s(\gamma_1.0) \\ \vdots & & \vdots \\ f_0(\gamma_m.0) & \ldots & f_s(\gamma_m.0) \end{pmatrix} \ .$$

Theorem 2. *The space $U_{\lambda,s}^\Gamma$ is finite-dimensional and for any $\alpha \in \Delta_1(N)$, the eigenvalues of the Hecke operator T_α acting on $U_{\lambda,s}^\Gamma$ are algebraic over the field F_λ.*

Proof. Retain the above notation. We first show that the linear map ϕ_s is injective which will prove the first part of the theorem. So, suppose that $(f_0, \ldots, f_s) \in U_{\lambda,s}^\Gamma$ with

$$\phi_s(f_0, \ldots, f_s) = 0 \ .$$

We must show $f_0 = \ldots = f_s = 0$. View $\mathrm{SL}_2(\mathbb{Z})$ as acting on $\mathbf{T}$. One finds that $\mathrm{SL}_2(\mathbb{Z}).0 = \Xi$ which is dense in $\mathbf{T}$ (Lemma 2 (1)). As the f_j are continuous it thus suffices to show that $f_0(g.0) = \ldots = f_s(g.0) = 0$ for all $g \in \mathrm{SL}_2(\mathbb{Z})$. Let then $g \in \mathrm{SL}_2(\mathbb{Z})$ and write

$$g = \gamma \cdot \gamma_\nu \cdot \gamma_\infty \ ,$$

with $1 \le \nu \le m$, $\gamma \in \Gamma$, and $\gamma_\infty \in \Gamma_\infty$. Then,

$$\begin{aligned} (f_0(g.0), \ldots, f_s(g.0)) &= (f_0(\gamma.(\gamma_\nu.0)), \ldots, f_s(\gamma.(\gamma_\nu.0))) \\ &= ((f_0, \ldots, f_s) \mid_\lambda \gamma)(\gamma_\nu.0) \cdot A_\gamma^{(s)}(\gamma_\nu.0)^{-1} \\ &= (f_0(\gamma_\nu.0), \ldots, f_s(\gamma_\nu.0)) A_\gamma^{(s)}(\gamma_\nu.0)^{-1} \\ &= (0, \ldots, 0). \end{aligned}$$

Secondly we show the existence of an endomorphism t_α of $\mathbb{C}^{m(s+1)}$ with the following 2 properties:

(i) The diagram

$$\begin{array}{ccc} U_{\lambda,s}^\Gamma & \xrightarrow{\phi_s} & \mathbb{C}^{m(s+1)} \\ \downarrow T_\alpha & & \downarrow t_\alpha \\ U_{\lambda,s}^\Gamma & \xrightarrow{\phi_s} & \mathbb{C}^{m(s+1)} \end{array}$$

is commutative.

(ii) t_α is defined over the field F_λ (i.e. given by a matrix with coefficients in F_λ).

Together with the injectivity of ϕ_s, this will then prove the rest of the theorem.

Let $\mu \in \{1, \ldots, r\}$ and $\nu \in \{1, \ldots, m\}$. Now, $\alpha_\mu \gamma_\nu.0 \in \Xi \subseteq \mathbf{T}$, and as $\Xi = \mathrm{SL}_2(\mathbb{Z}).0$ there is $g_{\mu,\nu} \in \mathrm{SL}_2(\mathbb{Z})$ such that $g_{\mu,\nu}.0 = \alpha_\mu \gamma_\nu.0$. We can write

$$g_{\mu,\nu} = \beta_{\mu,\nu} \gamma_{\xi_\mu(\nu)} \gamma_\infty \ ,$$

where $\beta_{\mu,\nu} \in \Gamma$, ξ_μ is some map of $\{1, \ldots, m\}$ into itself, and $\gamma_\infty \in \Gamma_\infty$. Then

$$\alpha_\mu \gamma_\nu.0 = \beta_{\mu,\nu} \gamma_{\xi_\mu(\nu)}.0 \ .$$

Define then the endomorphism t_α of $\mathrm{Mat}_{m,s+1}(\mathbb{C})$ by

$$t_\alpha \begin{pmatrix} x_{1,0} & \ldots & x_{1,s} \\ \vdots & & \vdots \\ x_{m,0} & \ldots & x_{m,s} \end{pmatrix} :=$$

$$\sum_{\mu=1}^{r} \begin{pmatrix} (x_{\xi_\mu(1),0}, \ldots, x_{\xi_\mu(1),s}) A_{\beta_{\mu,1}}^{(s)}(\gamma_{\xi_\mu(1)}.0)^{-1} A_{\alpha_\mu}^{(s)}(\gamma_1.0) \\ \vdots \\ (x_{\xi_\mu(m),0}, \ldots, x_{\xi_\mu(m),s}) A_{\beta_{\mu,m}}^{(s)}(\gamma_{\xi_\mu(m)}.0)^{-1} A_{\alpha_\mu}^{(s)}(\gamma_m.0) \end{pmatrix} .$$

Then claim (ii) above is clear because of Lemma 2 (3). We proceed to show (i). So, let $\mathbf{f} = (f_0, \dots, f_s) \in U^\Gamma_{\lambda,s}$. Then,

$$\phi_s(\mathbf{f} \mid_\lambda T_\alpha) = \sum_{\mu=1}^{r} \phi_s(\mathbf{f} \mid_\lambda \alpha_\mu)$$

$$= \sum_{\mu=1}^{r} \begin{pmatrix} \vdots \\ (f_0(\alpha_\mu\gamma_\nu.0), \dots, f_s(\alpha_\mu\gamma_\nu.0))A^{(s)}_{\alpha_\mu}(\gamma_\nu.0) \\ \vdots \end{pmatrix}$$

$$= \sum_{\mu=1}^{r} \begin{pmatrix} \vdots \\ (f_0(\beta_{\mu,\nu}\gamma_{\xi_\mu(\nu)}.0), \dots, f_s(\beta_{\mu,\nu}\gamma_{\xi_\mu(\nu)}.0))A^{(s)}_{\alpha_\mu}(\gamma_\nu.0) \\ \vdots \end{pmatrix} .$$

As $\mathbf{f} \in U^\Gamma_{\lambda,s}$ we have

$$(f_0(\beta_{\mu,\nu}.\theta), \dots, f_s(\beta_{\mu,\nu}.\theta)) = (f_0(\theta), \dots, f_s(\theta))A^{(s)}_{\beta_{\mu,\nu}}(\theta)^{-1} ,$$

for all $\theta \in \mathbf{T}$. Consequently,

$$\phi_s(\mathbf{f} \mid_\lambda T_\alpha) =$$

$$\sum_{\mu=1}^{r} \begin{pmatrix} \vdots \\ (f_0(\gamma_{\xi_\mu(\nu)}.0), \dots, f_s(\gamma_{\xi_\mu(\nu)}.0))A^{(s)}_{\beta_{\mu,\nu}}(\gamma_{\xi_\mu(\nu)}.0)^{-1}A^{(s)}_{\alpha_\mu}(\gamma_\nu.0) \\ \vdots \end{pmatrix}$$

$$= t_\alpha \begin{pmatrix} \vdots \\ f_0(\gamma_\nu.0), \dots, f_s(\gamma_\nu.0) \\ \vdots \end{pmatrix} = t_\alpha \phi_s(\mathbf{f}) ,$$

as desired. □

2.5. *Proof of Theorem 1:* Fix $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) \geq 0$. Recall (Definition 2) that we have fixed a non-negative integer k such that

$$\mathcal{D}^\Gamma_{\lambda,k} = \mathcal{D}^\Gamma_\lambda \cong M_\lambda(N) ,$$

where the isomorphism is as $\mathbb{T}$-modules, cf. Proposition 1. Let $\Phi = (\nu_p \mid p \nmid N)$ be a system of Hecke eigenvalues occurring in $M_\lambda(N)$ and hence also in $\mathcal{D}^\Gamma_{\lambda,k}$. Let $0 \neq v \in \mathcal{D}^\Gamma_{\lambda,k}$ be a corresponding $\mathbb{T}$-eigenvector, i.e. $T_p v = \nu_p v$ for $p \nmid N$. Recall (Definition 2 and Proposition 2 (1)) that we have a short exact sequence of $\Delta_1(N)$-modules

$$0 \longrightarrow V_{\lambda,k} \longrightarrow U_{\lambda,k} \longrightarrow \mathcal{D}_{\lambda,k} \longrightarrow 0 ,$$

which gives rise to the very short exact sequence of $\mathbb{T}$-modules

$$U^\Gamma_{\lambda,k} \xrightarrow{\alpha} \mathcal{D}^\Gamma_\lambda \xrightarrow{\beta} H^1(\Gamma, V_{\lambda,k}) ,$$

with $\mathbb{T}$ acting as an algebra of cohomological Hecke operators. Suppose that $\beta v = 0$ so that $v \in \mathrm{Im}(\alpha)$. Then Φ occurs in $\mathrm{Im}(\alpha)$, and because of Theorem 2 the tautological homomorphism

$$\alpha\colon U^\Gamma_{\lambda,k} \longrightarrow \mathrm{Im}(\alpha)$$

is a surjective homomorphism of $\mathbb{T}$-modules that are finite-dimensional as complex vector spaces. Using Proposition 1.2.2 of [1], we may then conclude that Φ occurs in $U^{\Gamma}_{\lambda,k}$. According to Theorem 2 all eigenvalues ν_p are then algebraic over the field

$$F_{\lambda} := \mathbb{Q}(\lambda, \sqrt{n}, n^{\lambda/2} \mid \ n \in \mathbb{N}) \ .$$

We then suppose that $\beta v \neq 0$ and will show that Φ occurs in $H^1(\Gamma, W_{\lambda,m})$ for some $0 \leq m \leq k$. This then finishes the proof of Theorem 1.

Recall that according to Proposition 2 (3) we may consider the spaces $V_{\lambda,m}$, $0 \leq m \leq k$, as a filtration

$$0 = V_{\lambda,0} \leq \ldots V_{\lambda,m-1} \leq V_{\lambda,m} \ldots \leq V_{\lambda,k}$$

of $\mathbb{T}$-submodules of $V_{\lambda,k}$ where the successive quotients are isomorphic to the $W_{\lambda,m}$ as $\mathbb{T}$-modules. Now, our assumption $\beta v \neq 0$ implies that Φ occurs in a finite-dimensional sub-$\mathbb{T}$-module of $H^1(\Gamma, V_{\lambda,k})$ namely $\beta(\mathcal{D}^{\Gamma}_{\lambda})$; notice, that we must have $k \geq 1$. We shall assume that Φ occurs in some finite-dimensional, sub-$\mathbb{T}$-module X of $H^1(\Gamma, V_{\lambda,m})$ for some m with $1 \leq m \leq k$ and will show that then either Φ occurs in $H^1(\Gamma, W_{\lambda,m})$ or else in some finite-dimensional, sub-$\mathbb{T}$-module of $H^1(\Gamma, V_{\lambda,m-1})$. By induction on k, this gives the desired conclusion as $V_{\lambda,0} = 0$.

Consider the short exact sequence of $\mathbb{T}$-modules

$$0 \longrightarrow V_{\lambda,m-1} \longrightarrow V_{\lambda,m} \longrightarrow W_{\lambda,m} \longrightarrow 0$$

coming from Proposition 2 (3). This gives rise to a long exact sequence of $\mathbb{T}$-modules:

$$W^{\Gamma}_{\lambda,m} \longrightarrow H^1(\Gamma, V_{\lambda,m-1}) \stackrel{\epsilon}{\longrightarrow} H^1(\Gamma, V_{\lambda,m}) \stackrel{\eta}{\longrightarrow} H^1(\Gamma, W_{\lambda,m}) \ .$$

Now, it is easy to see that the space $W^{\Gamma}_{\lambda,m}$ is 0: Suppose that $f \in W^{\Gamma}_{\lambda,m}$. As f is continuous, f is bounded. However, the definition of $j(\gamma, \theta)$ shows that $j(\gamma_n, \theta)$, $n \in \mathbb{N}$, is unbounded if $0 \neq \theta \in \mathbf{T}$ where γ_n denotes the matrix

$$\gamma_n := \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \in \Gamma \ .$$

So, the definition of the Γ-structure on $W_{\lambda,m}$ implies that $f(\theta)$ vanishes for every $\theta \neq 0$ and hence for all θ.

Thus, ϵ is an injection and the very short exact sequence

$$\epsilon^{-1}(X) \longrightarrow X \longrightarrow \eta(X)$$

is an exact sequence of finite-dimensional $\mathbb{T}$-modules. Applying Proposition 1.2.2 of [1] as above, we conclude that either Φ occurs in $\epsilon^{-1}(X)$, and thus a fortiori in $H^1(\Gamma, V_{\lambda,m-1})$, or else in $\eta(X)$ and so also in $H^1(\Gamma, W_{\lambda,m})$.

Remarks: The reader will notice that our use of continuous coefficients in the above – as opposed to L^2-coefficients – is necessitated by the use we made of the evaluation maps ϕ_s in the proof of Theorem 2 above. Thus, the use of continuous coefficients is indispensable for our approach. However, we wish to remark here that the there is a certain price to be paid for this, notably the following.

The author does not have concrete examples of triples (Γ, λ, s) where he can prove that the space $U^{\Gamma}_{\lambda,s}$ is actually non-zero. However, given the injectivity of the evaluation map ϕ_s, the proof of Theorem 2 shows that any eigenvalue of a Hecke operator T_{α} acting on the space $U^{\Gamma}_{\lambda,s}$ is also an eigenvalue of the linear operator t_{α} acting on $\mathbb{C}^{m(s+1)}$; moreover, the eigenvalues of t_{α} can – for any concretely given triple (Γ, λ, s) – be computed numerically. Such numerical experiments seem to indicate that Hecke eigenvalues on the spaces $U^{\Gamma}_{\lambda,s}$ are probably not very interesting, and that one could at the most retrieve packages of Hecke eigenvalues which are readily recognizable as belonging to certain standard Eisenstein series. Thus, it would appear that the interesting packages of Hecke eigenvalues should be the ones occurring in the spaces $H^1(\Gamma, W_{\lambda,m})$.

There are certain reasons that make it not wholly unreasonable to venture the conjecture that the spaces $H^1(\Gamma, W_{\lambda,m})$ are in fact finite-dimensional. For instance, the methods of the papers [5] and more specifically [4], might show the way towards analyzing this question. It will be seen however, that because the $W_{\lambda,m}$ are spaces of *continuous* functions, an attempt to use the methods of these papers to approach the question of finite-dimensionality of the $H^1(\Gamma, W_{\lambda,m})$ will quickly lead to some serious analytical difficulties.

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