ON THE EXISTENCE OF $\bar{p}$-CORE PARTITIONS OF NATURAL NUMBERS.

IAN KIMING

Abstract. Extending previous work of J. B. Olsson, cf. [7], [8], and of K. Erdmann and G. O. Michler, cf. [2], on the number of $p$-spin blocks of defect zero ($p$ prime) of a double covering group of the symmetric group $S_n$, we prove that this number is positive for all $n$ whenever $p \geq 7$. More precisely, it is shown that $s_p(n) > 0$ if $p \geq 7$, where $s_p(n)$ denotes the number of bar partitions of $n$ which are $\bar{p}$-cores.

1. Introduction.

Everywhere in this article the following notation is used: $p$ denotes an odd prime number, $n$ a natural number, and we put: $t := (p - 1)/2$.

If $G$ is a finite group, there is some interest in the question whether $G$ has a $p$-block of defect zero, since the existence of such a block means the existence of an irreducible, projective $G$-module in characteristic $p$. In general, it is a difficult problem to decide whether $G$ has a $p$-block of defect zero.

It was recently proved, cf. [3] (see also [4] for an alternative proof), that if $p \geq 5$ then for every $n$ the symmetric group $S_n$ (and the alternating group $A_n$) has a $p$-block of defect zero.

On the other hand, the representation theory of ‘double covering groups’ $\hat{S}_n$ of $S_n$ has also been studied intensively, cf. [5], [6], [7], [8]; by a ‘double covering group’ of $S_n$ we shall understand any of the groups $R_n$ and $T_n$ given by generators $a_1, \ldots, a_{n-1}, z$ and defining relations:

For $R_n$:
\[ z^2 = 1, \quad a_i^2 = (a_i a_{i+1})^3 = z, \quad \text{and} \quad [a_i, a_j] = z \quad \text{for} \quad |i - j| \geq 2, \]

and:
\[ z^2 = a_i^2 = (a_i a_{i+1})^3 = [a_i, z] = 1, \quad \text{and} \quad [a_i, a_j] = z \quad \text{for} \quad |i - j| \geq 2, \]

for $T_n$. Thus, if $n \geq 4$, $n \neq 6$, $R_n$ and $T_n$ are the 2 representation groups of $S_n$, whereas for $n = 6$, $R_n$ and $T_n$ are both isomorphic to the unique representation group of $S_6$; cf. [9], pp. 355–357.

Denoting by $\hat{S}_n$ any of the groups $R_n$ and $T_n$, it is natural to ask similarly whether $\hat{S}_n$ has a spin character, i.e. a faithful, irreducible character, of $p$-defect zero. This is known (A. O. Morris) to be the case if and only if $n$ has a bar partition which is a $\bar{p}$-core, cf. [8], p. 190; see [7] or below for the definition of ‘$\bar{p}$-core partition’. Hence, if we denote by $s_p(n)$ the number of $\bar{p}$-core partitions of $n$, the question of interest is to determine the pairs $(p, n)$ for which $s_p(n) > 0$.

The first study of the numbers $s_p(n)$ was by K. Erdmann and G. O. Michler, cf. [2], who studied the situation for $p = 5$ and $p = 7$: They proved that $s_5(n) > 0$ for all $n$ and gave an explicit criterion for $s_5(n) > 0$ to hold.
We show in the theorem below that $s_p(n) > 0$ for all $n$ if $p \geq 7$.

The structure of the proof is this: For ‘small’ $n$ the assertion is proved more or less directly, and for ‘large’ $n$ the problem is by use of a certain trick reduced to the question of representing integers by the quadratic form:

$$2x^2 + 2y^2 + z^2.$$ 

2.

We shall now recall from [7], pp. 233–237, some facts concerning $\bar{p}$-core partitions. Recall that a bar partition of $n$ is a partition $\lambda = (a_1, \ldots, a_m)$ with $a_1 > \ldots > a_m > 0$. Such a partition is represented on the \('p\)-abacus’ which has $p$ runners numbered $0, 1, \ldots, p-1$ going from north to south and has its rows numbered by the non-negative integers: In the representation of $\lambda$ on the $\bar{p}$-abacus there is a bead in the $i$'th runner and $j$'th row if and only if $j \in X_i(\lambda)$ where:

$$X_i(\lambda) := \{ a \in \mathbb{N}_0 \mid \exists k \in \{1, \ldots, m\} : a_k = ap + i \}.$$ 

Then $\lambda$ is a $\bar{p}$-core if and only if there are no beads on the 0'th runner, and for each $i$ with $1 \leq i \leq p-1$ the $i$'th runner contains

$$\ell_i := \max(|X_i(\lambda)| - |X_{p-i}(\lambda)|, 0)$$

beads in the first $\ell_i$ rows.

Consequently, if $\lambda$ is a $\bar{p}$-core then:

$$n = \sum_{i=1}^{p-1} (p \cdot \frac{1}{2} \ell_i(\ell_i - 1) + i\ell_i), \quad \text{where} \quad \ell_i\ell_{p-i} = 0 \quad \text{for} \quad i = 1, \ldots, p-1.$$ 

Conversely, a $(p-1)$-tuple $(\ell_1, \ldots, \ell_{p-1})$ of non-negative integers with (1) gives rise to a unique $\bar{p}$-core partition of $n$. Thus, $s_p(n)$ is the number of such $(p-1)$-tuples, – a fact which was also noted in [2].

From this we conclude that $s_p(n)$ is the number of $t$-tuples $(y_1, \ldots, y_t) \in \mathbb{Z}^t$ with:

$$n = \sum_{i=1}^{t} (p \cdot \frac{1}{2} y_i(y_i - 1) + iy_i);$$

in fact the maps: $(\ell_1, \ldots, \ell_{p-1}) \mapsto (y_1, \ldots, y_t)$ given by:

$$y_i = \ell_i - \ell_{p-i} \quad \text{for} \quad i = 1, \ldots, t,$$

and $(y_1, \ldots, y_t) \mapsto (\ell_1, \ldots, \ell_{p-1})$ given by:

$$(\ell_i, \ell_{p-i}) = (y_i, 0) \quad \text{if} \quad y_i \geq 0, \quad \text{and} \quad (\ell_i, \ell_{p-i}) = (0, -y_i) \quad \text{if} \quad y_i < 0,$$

are seen to give mutually inverse bijections between the set of $(\ell_1, \ldots, \ell_{p-1}) \in \mathbb{N}_0^{p-1}$ with (1), and the set of $(y_1, \ldots, y_t) \in \mathbb{Z}^t$ with (2).

Using

$$\sum_{i=1}^{t} (2i - p)^2 = \frac{1}{6} p(p-1)(p-2),$$

we have:

$$s_p(n) = \sum_{i=1}^{t} (2i - p)^2.$$
and writing \( x_i = (2i - p) + 2py_i \), we then also conclude that \( s_p(n) \) is the number of \( t \)-tuples \((x_1, \ldots, x_t) \in \mathbb{Z}^t\) such that:

\[
(3) \quad n = \frac{1}{8p} \sum_{i=1}^t x_i^2 - \frac{(p-1)(p-2)}{48}, \quad \text{where} \quad x_i \equiv 2i - p \quad (2p)
\]

for \( i = 1, \ldots, t \).

Let us now briefly indicate a proof of one of the results in [2], namely that this last equation is solvable for all \( n \in \mathbb{N} \), if \( p = 7 \):

Putting \( N := 7 \cdot (8n + 5) \), we have that \( s_7(n) \) is the number of integral solutions to:

\[
(*) \quad N = x^2 + y^2 + z^2,
\]

with \((x, y, z) \equiv (-5, -3, -1) \quad (14)\). Now one notices that any solution to (*) satisfies either \((x, y, z) \equiv (0, 0, 0) \quad (7)\) or \((x, y, z) \equiv (\pm 5, \pm 3, \pm 1) \quad (7)\) up to permutation of \(x, y, z\). In the latter case one has in fact \((x, y, z) \equiv (\pm 5, \pm 3, \pm 1) \quad (14)\) up to permutation, since \(x, y, z\) must all be odd since \( N \equiv 3 \quad (4)\). From this, one easily concludes that \( s_7(n) \) equals \( a_3(N)/48 \) if \( n \neq 2 \quad (7)\), and equals \((a_3(N) - a_3(N/7^2))/48 \) if \( n \equiv 2 \quad (7)\), where for \( M \in \mathbb{N} \), \( a_3(M) \) denotes the number of representations of \( M \) as a sum of 3 squares. Now, from the classical formulas for the numbers \( a_3(M) \), one then easily derives even explicit formulas for \( s_7(n) \). Suppose for example \( n \neq 2 \quad (7)\), and write \( N = p_1^{2a_1} \cdots p_v^{2a_v} \cdot r \), where \( p_1, \ldots, p_v \) are distinct odd primes and \( r \) is square free and \( \equiv 3 \quad (8)\); then if \( h \) denotes the class number of \( \mathbb{Q}(\sqrt{-r}) \) and \( \chi_{-r} \) the Dirichlet character belonging to this field, so that

\[
\chi_{-r}(x) = \left( \frac{-r}{x} \right) \quad \text{for} \quad (x, r) = 1,
\]

one has:

\[
s_7(n) = \frac{1}{2} h \cdot \Pi_{i=1}^v \left( 1 + (p_i - \chi_{-r}(p_i)) \cdot \frac{p_i^{a_i} - 1}{p_i - 1} \right).
\]

Similar formulas are obtained for \( n \equiv 2 \quad (7)\). In particular, one has \( s_7(n) > 0 \) for all \( n \in \mathbb{N} \).

The interpretation of \( s_p(n) \) as the number of solutions to (3) above suggests a connection to modular forms. In fact the numbers \( s_p(n) \) for fixed \( p \) are related to the Fourier coefficients of a certain product of classical theta series. One can use this to obtain asymptotic formulas for \( s_p(n) \) for fixed \( p \) and \( n \to \infty \), at least for \( p \equiv 1 \quad (4) \). We shall report on this elsewhere.

3.

Let us define the functions \( f_i \) for \( i = 1, \ldots, t = \frac{p-1}{2} \):

\[
f_i(y) = p \cdot \frac{1}{2} y(y - 1) + iy.
\]

Hence, in order to show that \( s_p(n) > 0 \) we must show that the equation:

\[
n = \sum_{i=1}^t f_i(y_i),
\]

has a solution in integers \( y_1, \ldots, y_t \).
Lemma 1. Suppose that \( p \geq 11 \) and that \( n \in \mathbb{N} \) with \( n \leq p \cdot \frac{1}{2}(p-1)^2 + (p-1) \). Then there exist \( y_i \in \mathbb{Z}, i = 1, \ldots, t \) such that:

\[
 n = \sum_{i=1}^{t} f_i(y_i).
\]

Proof. Put \( n_0 := p(p-1)^2/2 + p - 1 \).

Notice first that if \( m, s \in \mathbb{N} \) then there are \( y_i \in \{0,1\} \) for \( i = 1, \ldots, s \) such that:

\[
 m = \sum_{i=1}^{s} iy_i,
\]

if and only if \( m \leq \frac{1}{2}s(s+1) \) (use induction on \( s \)). Since \( f_i(0) = 0, f_i(1) = i \), we see that it suffices to find \( s \in \{0, \ldots, t\} \) and \( y_i \in \mathbb{Z} \) for \( s < i \leq t \) such that:

\[
 n - \sum_{s<i\leq t} f_i(y_i) \leq \frac{1}{2}s(s+1).
\]

So, suppose that \( s \in \{0, \ldots, t-1\} \), put \( c_0(n) := n \), and define the integers \( y_{t-i}(n) \) and \( c_{i+1}(n) \) for \( i = 0, \ldots, t-s-1 \) successively:

\[
y_{t-i}(n) := \left[ (2t+1) + \sqrt{(2t+1)^2 + 8pc_i(n)} / 2p \right],
\]

and \( c_{i+1}(n) := c_i(n) - f_{t-i}(y_{t-i}(n)) \), so that \( c_i(n) \geq 0 \) for \( i = 0, \ldots, t-s \). Then we have

\[
c_{t-s}(n) = n - \sum_{s<i\leq t} f_i(y_i).
\]

Hence the proof is finished if

\[
 (1) \quad c_{t-s}(n) \leq \frac{1}{2}s(s+1) \quad \text{for all } n \leq n_0.
\]

Now, since \( y_{t-i}(n) \in \mathbb{Z} \) is largest possible such that \( f_{t-i}(y_{t-i}(n)) \leq c_i(n) \) we have:

\[
c_{i+1}(n) \leq py_{t-i}(n) + t - i \leq \frac{1}{2} p + \sqrt{(2t+1)^2 + 8pc_i(n)};
\]

hence, if \( i \in \{1, \ldots, t-s\} \) and \( G_i(x) \) is a polynomial, the condition

\[
 (8p)^{2^{s-i-1}}c_i(n) \leq G_i(p) \quad \text{for all } n \leq n_0
\]

is seen to be implied by

\[
 (8p)^{2^{s-i-1}+1}c_{i-1}(n) \leq G_{i-1}(p) \quad \text{for all } n \leq n_0,
\]

where

\[
 (2) \quad G_{i-1}(x) := (2G_i(x) - x \cdot (8x)^{2^{s-i-1}-1})^2 - (2i - 1)^2 \cdot (8x)^{2^{s-i-1}-2},
\]

provided that \( H_i(p) \geq 0 \) where

\[
 (3) \quad H_i(x) := 2G_i(x) - x \cdot (8x)^{2^{s-i}-1}.
\]

Hence, if we define \( G_4(x) := \frac{1}{2}((x-1)/2-4)((x-1)/2-3) \), \( H_4(x) := 2G_4(x) - x \), and \( G_i(x), H_i(x) \) for \( i = 3, 2, 1, 0 \) in accordance with (2) and (3), we have that (1) holds for the case \( s = t-4 \), if

\[
 (4) \quad H_1(p), \ldots, H_4(p) \geq 0 \quad \text{and} \quad (8p)^{15}c_0(n) \leq G_0(p) \quad \text{for all } n \leq n_0.
\]
But as \( c_0(n) = n \leq p(p-1)^2/2 + p - 1 \), (4) holds if
\[
H_1(p), \ldots, H_4(p), G(p) \geq 0,
\]
where
\[
G(x) := G_0(x) - (8x)^{15}(x(x-1)^2/2 + x - 1).
\]
Now, \( H_1(x), \ldots, H_4(x), G(x) \) are polynomials with rational coefficients which can be computed by using (for example) MAPLE. For example, one finds:
\[
G(x) := 2^{-18}(x^{32} - 320x^{31} + 48496x^{30} - \ldots + 5871615^8).
\]
Using MAPLE again, we can compute approximations to the real roots of these polynomials and thus verify that (5) holds if \( p > 43 \), i.e. (1) holds for the case \( s = t - 4 \) if \( p > 43 \).

In the remaining cases \( 11 \leq p \leq 43 \) one may verify the lemma by direct computation on a machine. We leave this to the reader indicating only that one finds in each of these cases solutions with \( y_i = 0 \) for \( 5 < i < t - 5 \). □

**Theorem 1.** Suppose that \( p \geq 7 \) and that \( n \in \mathbb{N} \). Then there exists a \( \bar{p} \)-core partition of \( n \).

Hence, a double covering group \( \hat{S}_n \) of \( S_n \) has a spin character of \( p \)-defect zero.

**Proof.** The last statement follows from the first as explained in the introduction.

By [2], or by the remarks made above, we may assume that \( p \geq 11 \).

We must show the existence of integers \( y_1, \ldots, y_t \) such that:
\[
n = \sum_{i=1}^{t} f_i(y_i).
\]
Because of the lemma, we may and will assume:
\[
n \geq p \cdot \frac{1}{2}(p-1)^2 + (p-1).
\]
Hence, if we write
\[
n = pm + r \quad \text{with} \quad m \in \mathbb{N}_0, \quad |r| \leq p - 1 \quad \text{and} \quad r \equiv 0 \pmod{2},
\]
then \( m \geq r^2/2 \) so that the number \( N := 48m - 24r^2 + 5 \) is a natural number.

Now, we know, confer [1], that the quadratic form \( 2x^2 + 2y^2 + z^2 \) represents every natural number not of the form \( 4^s(8l + 7) \). Thus there exist \( u, v, w \in \mathbb{Z} \) such that:
\[
N = 2u^2 + 2v^2 + w^2;
\]
in fact, the solvability of \((*)\) in integers follows easily from the fact that \( N \) is a sum of 3 squares.

Since \( r \) is even, we have \( N \equiv 5 \pmod{16} \); a consideration of \((*)\) mod 16 then shows that \( u \) and \( v \) are odd and \( w \equiv \pm1, \pm7 \pmod{16} \). We may then assume
\[
w \equiv -1 \pmod{6}.
\]

Considering \((*)\) mod 3 and exchanging \( u \) and \( v \) if necessary, we find
\[
( u^2 \equiv 1 \pmod{3}, \quad v^2 \equiv w^2 \equiv 0 \pmod{3} ) \quad \text{or} \quad ( u^2 \equiv v^2 \equiv w^2 \equiv 1 \pmod{3} ).
\]
Since \( u, v, w \) are all odd, we may then assume
\[
u \equiv -1 \pmod{6} \quad \text{and} \quad v \equiv w \pmod{6}.
\]
Then the number \( v + 1 - (w + 1)/4 \) is divisible by 6, and we define the integers:

\[
a := \frac{u + 1}{6}, \quad b := -\frac{r}{2} + \frac{1}{6} \left( v + 1 - \frac{w + 1}{4} \right), \quad c := \frac{r}{2} + \frac{w + 1}{8},
\]

and

\[
y_1 := a + b, \quad y_2 := a + c, \quad y_3 := -a + b + c, \quad y_4 := -r - b, \quad y_5 := r - c,
\]

and \( y_i = 0 \) for \( i \geq 6 \). Then we have:

\[
\sum_{i=1}^{5} f_i(y_i) = \sum_{i=1}^{5} \left( p \cdot \frac{1}{2} y_i(y_i - 1) + iy_i \right) = p \cdot \frac{1}{48} (2u^2 + 2v^2 + w^2 + 24v^2 - 5) + r
\]

\[
= pm + r = n.
\]

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References


kiming@math.ku.dk

Dept. of math., Univ. of Copenhagen, Universitetsparken 5, 2100 Copenhagen Ø, Denmark.