ON THE ASYMPTOTICS OF THE NUMBER OF $\bar{p}$-CORE PARTITIONS OF INTEGERS.

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Abstract. Let p be an odd prime and let $n \in \mathbb{N}$. The so-called '$\bar{p}$-core partitions' of n arise naturally in the study of the modular representation theory of covering groups of the symmetric group S_n, cf. [11, 12]. In fact, the number of such partitions is closely related to the number of spin characters of p-defect zero of such a covering group. It was recently proved that this number is always positive if $p \geq 7$: cf. [2] resp. [7] for the cases $p = 7$ and $p \geq 11$ respectively. It is natural to ask for asymptotic formulae (for a fixed p) for the number of $\bar{p}$-core partitions of natural numbers n. We use modular forms to derive such asymptotic formulae in the cases $p \equiv 1$ (4), $p > 5$.

1. Introduction.

In the following the symbol p always denotes an odd prime number and n a natural number.

In the representation theory of the symmetric groups S_n there is some interest in the question of determining the $n \in \mathbb{N}$ for which S_n has a p-block of defect zero, since the existence of such a block means the existence of an irreducible, projective module in characteristic p. The question turns out to be not quite trivial and equivalent to the question of determining those $n \in \mathbb{N}$ which have a so-called 'p-core partition' (for a definition, see [3]). The work [3] turned the question into an arithmetical one, and using this it was recently proved, cf. [13, 4], that if $p \geq 5$ then every $n \in \mathbb{N}$ has a 'p-core partition' (see also [6] for an alternative proof). This result is optimal in the sense that the statement is false for $p = 3$.

On the other hand, if one wishes to study *projective* representations of the symmetric group S_n, then by Schur's theory [15], this is equivalent to the study of ordinary representations of any 'representation group' $\hat{S}_n$ of S_n. Here, 'representation group' is to be understood in the sense of Schur, i.e. $\hat{S}_n$ is a central extension of S_n with the property that any projective representation of S_n lifts to an ordinary representation of $\hat{S}_n$, and such that $\hat{S}_n$ has order equal to $n! = |S_n|$ times the order of the Schur multiplier of S_n, which is 1 for $n = 1, 2, 3$ and 2 for $n \geq 4$. All possibilities for $\hat{S}_n$ have been determined by Schur in [15] : For $n \geq 4$, $\hat{S}_n$ is isomorphic to one of the groups R_n or T_n given generators $a_1, \ldots, a_{n-1}, z$ and defining relations:

$$z^2 = 1,\ a_i^2 = (a_i a_{i+1})^3 = z,\ \text{and} \quad [a_i, a_j] = z \quad \text{for } |i - j| \geq 2,$$

for R_n, and:

$$z^2 = a_i^2 = (a_i a_{i+1})^3 = [a_i, z] = 1,\ \text{and} \quad [a_i, a_j] = z \quad \text{for } |i - j| \geq 2,$$

for T_n. For $n \geq 4$, $n \neq 6$, R_n and T_n are non-isomorphic, whereas R_6 is isomorphic to T_6, cf. [15], pp. 355–357.

Thus we denote in the following for $n \geq 4$ by $\hat{S}_n$ anyone of the groups R_n or T_n above. If $n \in \{1, 2, 3\}$, the following theory is not very interesting, but for practical reasons we shall redefine $\hat{S}_n$ in these cases to be also anyone of R_n or T_n.

So, $\hat{S}_n$ is a double covering group of S_n. The representation theory of these double covers has been studied intensively, cf. [8, 10, 11, 12]. In the general modular representation theory of finite groups the question of existence of a character of p-defect zero is a fundamental and difficult problem. Thus, for $\hat{S}_n$, one of the natural problems is to determine those $n \in \mathbb{N}$ for which $\hat{S}_n$ has a spin character, i.e. a faithful, irreducible character, of p-defect zero. This question turns out to be equivalent to the determination of those $n \in \mathbb{N}$ which have a so-called '$\bar{p}$-core partition', cf. [12, 11]; see below in section 1 for the definition of a $\bar{p}$-core partition of n. In fact, the number of $\bar{p}$-core partitions of n is closely related to the number of spin characters of p-defect zero of $\hat{S}_n$; more precisely, the $\bar{p}$-core partitions of n can be used as labels for such spin characters: A $\bar{p}$-core partition λ labels either 1 or 2 spin characters of p-defect zero depending on a certain sign attached to λ; cf. [12].

In [7] we proved that every $n \in \mathbb{N}$ has a $\bar{p}$-core partition if $p \geq 7$; see also [2] for the case $p = 7$. This is also an optimal result. It has some strong consequences for the representation theory of $\hat{S}_n$, for example the following (see [12]): If $p \geq 7$, and $m, n \in \mathbb{N}$ with $pm \leq n$, then $\hat{S}_n$ has a spin block whose defect group is isomorphic to a p-Sylow subgroup of S_{pm}.

Thus, $\bar{p}$-core partitions seem to be fundamental combinatorial objects, and in this article we study them for their own sake. We shall focus on a connection to modular forms and use this in section 2 below to give for $p > 5$, $p \equiv 1$ (4) asymptotic formulae for the number $s_p(n)$ of $\bar{p}$-core partitions of n. The reason for our restriction to the cases $p \equiv 1$ (4) is that we relate $s_p(n)$ to the Fourier coefficients of a certain modular form of weight $(p-1)/4$; for $p \equiv 3$ (4) we would thus have to deal with modular forms of half-integral weight, and this would in fact complicate the discussion considerably.

In order to find an asymptotic formula for the numbers $s_p(n)$ (p fixed) we proceed as follows. Based on the reinterpretation in the next section of $s_p(n)$ as the number of solutions to a certain quadratic diophantine equation, we construct in section 2 a modular form

$$f_p(z) = \sum_{m=0}^{\infty} b(m, f_p) \cdot e^{2\pi i m z} \qquad \text{for} \quad \mathrm{Im}(z) > 0 \ ,$$

on a certain congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, with the property that the numbers $s_p(n)$ occur among the Fourier coefficients $b(\cdot, f_p)$ of f_p; for example, one will have

$$s_p(n) = b(n + \frac{1}{48} \cdot (p-1)(p-2), f_p) \qquad \text{if} \quad p \equiv 1 \ (16) \ .$$

An asymptotic formula for $s_p(n)$ is then obtained by using the following principle first made explicit by Hecke (see [5]): First we split off an Eisenstein part e_p of f_p, i.e. we determine a linear combination e_p of standard Eisenstein series with the property that $f_p - e_p$ is a *cusp* form. The determination of e_p requires the knowledge of the constant terms in the Fourier expansions of f_p and standard Eisenstein series around various cusps. Our situation is complicated by the fact that the level of

f_p is not square free for all p, so that these constant terms can not in all cases be computed by using Atkin-Lehner involutions. In the proof of the theorem below we describe the principles used in computing the constant terms, but we shall leave most of the explicit computations to the reader. The proof of the asymptotic formulae for $s_p(n)$ is then finished by computing explicitly the Fourier coefficients of the form e_p and then employing known estimates on the Fourier coefficients of cusp forms on congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$.

2.

Now we recall from [11], pp. 233–237, the definition of a '$\bar{p}$-core partition' of n, and derive from this an interpretation of the number $s_p(n)$ of such partitions as the number of solutions to a certain diophantine equation.

A *bar partition* of n is a partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ of n with $\lambda_1 > \ldots > \lambda_m > 0$. The parts $\lambda_1, \ldots, \lambda_m$ of λ are represented as beads on the 'p-abacus', which is an abacus with p runners going from north to south and numbered $0, 1, \ldots, p-1$. The rows are numbered $0, 1, 2, \ldots$. The part λ_s is represented by a bead in the j'th row of the i'th runner where i and j are determined by:

$$0 \leq i \leq p-1 \quad \text{and} \quad \lambda_s = pj + i\ .$$

Thus, there is at most one bead in each position of the p-abacus. The bar partition λ is then called a $\bar{p}$-core if and only if the following conditions are satisfied:

(i) The 0'th runner contains no beads,

(ii) No bead can be pushed up its runner, i.e. for any i, if the i'th runner contains ℓ_i beads then these are positioned in the first ℓ_i rows,

(iii) For each $i \in \{1, \ldots, p-1\}$, at least one of the i'th and the $(p-i)$'th runner is empty.

From this we easily deduce that the number $s_p(n)$ is equal to the number of $(p-1)$-tuples $(\ell_1, \ldots, \ell_{p-1})$ of non-negative integers with

$$n = \sum_{i=1}^{p-1} (p \cdot \frac{1}{2}\ell_i(\ell_i - 1) + i\ell_i) \qquad \text{and} \qquad \ell_i \ell_{p-i} = 0 \quad \text{for all} \quad i\ .$$

Putting $t := (p-1)/2$, this means that $s_p(n)$ is the number of t-tuples $(y_1, \ldots, y_t) \in \mathbb{Z}^t$ with

$$n = \sum_{i=1}^{t} (p \cdot \frac{1}{2} y_i(y_i - 1) + i y_i)$$

(consider $y_i \longleftrightarrow \ell_i - \ell_{p-i}$). Diagonalizing this last expression, we then finally conclude that $s_p(n)$ is the number of integral solutions to:

$$\begin{cases} n = \frac{1}{8p} \sum_{i=1}^{t} x_i^2 \ - \frac{(p-1)(p-2)}{48} \\[2ex] x_i \equiv 2i - p \ (2p),\ \forall i\ ; \end{cases}$$

(use that

$$\sum_{i=1}^{t} (2i-p)^2 = \frac{1}{6} p(p-1)(p-2)\ ).$$

This is the interpretation of the numbers $s_p(n)$ that we shall now use to find an asymptotic formula for them.

3.

We fix the following notation: p is a prime number > 5 and $\equiv 1\ (4)$, $t := (p-1)/2$ as above, and

$$k := \frac{p-1}{4} \ ,$$

so that k is an integer ≥ 3.

The symbol χ denotes the Dirichlet character belonging to the field $\mathbb{Q}(\sqrt{-1})$, so that

$$\chi(x) = (-1)^{\frac{x-1}{2}} \qquad \text{for odd } x \in \mathbb{Z} \ .$$

Further, if $n \in \mathbb{N}$ we denote by $N = N(n)$ the integer

$$N := 4n + \frac{(p-1)(p-2)}{12} \ .$$

If $K \in \mathbb{N}$ and ϵ is a Dirichlet character mod K, we denote as usual by $M_k(K,\epsilon)$ the space of holomorphic modular forms of weight k on $\Gamma_0(K)$ with nebentypus ϵ. Also, $S_k(K,\epsilon)$ denotes the corresponding subspace of cusp forms. If $f \in M_k(K,\epsilon)$, we denote by

$$b(n,f)$$

the n'th Fourier coefficient of f at ∞.

For $h \in \mathbb{Z}$ we consider the following classical theta series:

$$\theta_{3,0}(z,h,2p) := \sum_{\substack{x\in\mathbb{Z} \\ x\equiv h(2p)}} e^{2\pi i z\cdot \frac{x^2}{4p}} \ ,$$

for z in the upper half plane, and define

$$f_p(z) := \begin{cases} \prod_{i=1}^{t} \theta_{3,0}(z/2, 2i-p, 2p) & \text{if} \quad p \equiv 1\ (16) \\ \prod_{i=1}^{t} \theta_{3,0}(z, 2i-p, 2p) & \text{if} \quad p \equiv 9\ (16) \\ \prod_{i=1}^{t} \theta_{3,0}(2z, 2i-p, 2p) & \text{if} \quad p \equiv 5\ (8) \ , \end{cases}$$

for $\mathrm{Im}(z) > 0$.

We shall also need the following Hecke-Eisenstein series:

$$G_k(z;a,b;M) := \sum_{\substack{(m,n)\equiv(a,b)\ (M) \\ (m,n)\neq(0,0)}} (mz+n)^{-k} \qquad \text{for} \quad \mathrm{Im}(z) > 0 \ ,$$

where $M \in \mathbb{N}$, $a, b \in \mathbb{Z}$. We define:

$$G_k(z) := (2\zeta(k))^{-1} G_k(z;0,1;1) \qquad \text{for} \quad p \equiv 1\ (8) \ ,$$

where ζ is Riemann's zeta function, and further

$$E_k(z) := L(k,\chi)^{-1} G_k(z;0,1;4) \qquad \text{for} \quad p \equiv 5\ (8) \ ,$$

$$F_k(z) := -2 \cdot i^{-k} \cdot L(k,\chi)^{-1} G_k(4z;1,0;4) \qquad \text{for} \quad p \equiv 5\ (8) \ ,$$

where $L(s,\chi)$ is the L-series of χ. Finally, if $\ell \in \mathbb{N}$ we denote by $G_k^{(\ell)}$, $E_k^{(\ell)}$, $F_k^{(\ell)}$ respectively the function $G_k(\ell z)$, $E_k(\ell z)$, $F_k(\ell z)$ respectively.

Theorem 1. *For $n \in \mathbb{N}$ let*

$$N := 4n + \frac{(p-1)(p-2)}{12} .$$

I. *Suppose that $p \equiv 1$ (16). Then $f_p \in M_k(2p,1)$ and*

$$f_p - \frac{2^k}{(2^k-1)(p^k-1)} \left(G_k^{(2p)} - G_k^{(p)} - G_k^{(2)} + G_k^{(1)} \right) \in S_k(2p,1) .$$

For $n \in \mathbb{N}$ we have that $N/4 \in \mathbb{N}$ and

$$s_p(n) = b(N/4, f_p) ;$$

if $N/4 = 2^r p^s m$ with $(m,2p) = 1$, then

$$s_p(n) = -\frac{2k}{B_k} \cdot \frac{2^k}{(2^k-1)(p^k-1)} \cdot N^{k-1} \sum_{d|m} d^{1-k} + O(n^{\frac{k-1}{2}+\varepsilon})$$

for all $\varepsilon > 0$. Here B_k is the k'th Bernoulli number.

II. *Suppose that $p \equiv 9$ (16). Then $f_p \in M_k(4p,1)$ and*

$$f_p - \frac{2^k}{(2^k-1)(p^k-1)} \Big(G_k^{(4p)} - (2^{1-k}+1) G_k^{(2p)} + 2^{1-k} G_k^{(p)} - G_k^{(4)} + (2^{1-k}+1) G_k^{(2)} - 2^{1-k} G_k^{(1)} \Big) \in S_k(4p,1) .$$

For $n \in \mathbb{N}$ we have that $N/2$ is an odd integer and

$$s_p(n) = b(N/2, f_p) ;$$

if $N/2 = p^s m$ with $(m,p) = 1$, then

$$s_p(n) = \frac{2k}{B_k} \cdot \frac{2}{(2^k-1)(p^k-1)} \cdot N^{k-1} \sum_{d|m} d^{1-k} + O(n^{\frac{k-1}{2}+\varepsilon})$$

for all $\varepsilon > 0$.

III. *Suppose that $p \equiv 5$ (8). Then $f_p \in M_k(8p,\chi)$ and*

$$f_p - \frac{1}{p^k-1} \Big(E_k^{(2p)} - E_k^{(p)} - E_k^{(2)} + E_k^{(1)} + 2^{k-1} F_k^{(2p)} - F_k^{(p)} - 2^{k-1} F_k^{(2)} + F_k^{(1)} \Big) \in S_k(8p,\chi) .$$

For $n \in \mathbb{N}$ we have that N is an odd integer and

$$s_p(n) = b(N, f_p) ;$$

if $N = p^s m$ with $(m,p) = 1$, then

$$s_p(n) = (-1)^{\frac{k+1}{2}} \cdot \frac{2k}{B_{k,\chi}} \cdot \frac{2}{p^k-1} \cdot N^{k-1} \sum_{d|m} \chi(d) d^{1-k} + O(n^{\frac{k-1}{2}+\varepsilon})$$

for all $\varepsilon > 0$. Here $B_{k,\chi}$ is the k'th Bernoulli number belonging to the character χ.

Proof. We prove only part **III.** The proofs of parts **I** and **II** are similar but simpler. So, suppose that $p \equiv 5\ (8)$.

(a) First we use the transformation formula for the theta series $\theta_{3,0}$: Suppose that $h \in \mathbb{Z}$, use the notation

$$\zeta_m := e^{\frac{2\pi i}{m}} \ ,$$

and let

$$L = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(4p) \ .$$

Then the transformation formula on p. 223 in [14] states that

$$\theta_{3,0}(z,h,2p) \mid_{1/2} L = \sigma_{\gamma,\delta} \cdot \left(\frac{2p\gamma}{|\delta|} \right) \cdot \zeta_8^{\delta-1} \zeta_{4p}^{\alpha\beta h^2} \cdot \theta_{3,0}(z,\alpha h, 2p) \ ,$$

where $\sigma_{\gamma,\delta}$ is -1 if both γ and δ are negative and is 1 otherwise, and where we used the usual notation

$$f(z) \mid_s L := (\gamma z + \delta)^{-s} f\left(\frac{\alpha z + \beta}{\gamma z + \delta} \right)$$

for holomorphic functions f on the upper half plane and $s \in \frac{1}{2}\mathbb{Z}$ (with the standard branch of the holomorphic square root if s is half-integral). Now, from the definitions of $\theta_{3,0}(z,h,2p)$ and f_p we see that if I is a set of t integers such that the numbers $\pm i$, $i \in I$ form a system of representatives of the invertible residues modulo $2p$, then the product

$$\prod_{i \in I} \theta_{3,0}(z,i,2p)$$

is independent of I and equals $f_p(z/2)$. Since α is prime to $2p$, we can then conclude:

$$f_p(z/2) \mid_k L = (-1)^{k \cdot \frac{\delta-1}{2}} \cdot (-1)^{\alpha\beta \cdot \frac{(p-1)(p-2)}{12}} f_p(z/2) \ , \tag{1}$$

where we used that t is even, $k = t/2$ and that $\sum_{i=1}^{t} (2i-p)^2 = p(p-1)(p-2)/6$.

Since k is odd, (1) implies

$$f_p(z/2) \mid_k L = \chi(\delta) f_p(z/2) \ ,$$

if

$$L = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma(4p,2) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0\ (4p), b \equiv 0\ (2) \right\} \ .$$

Since

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \Gamma(4p,2) \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \Gamma_0(8p) \ ,$$

we then deduce $f_p \in M_k(8p,\chi)$.

(b) We have the following Fourier expansion of the Hecke-Eisenstein series $G_k(z;a,b;M)$, cf. [5]:

$$G_k(z;a,b;M) =$$
$$\delta\left(\frac{a}{M}\right) \sum_{\substack{\ell \equiv b\ (M) \\ \ell \neq 0}} \ell^{-k} + \frac{(-2\pi i)^k}{M^k (k-1)!} \sum_{\substack{mn>0 \\ n \equiv a\ (M)}} m^{k-1} \mathrm{sgn}(m) e^{\frac{2\pi i}{M} \cdot bm} e^{\frac{2\pi i}{M} \cdot mnz},$$

where $\delta(x)$ is 1 or 0 according to whether x is an integer or not. Using the fact that k is odd and that

$$L(k,\chi) = (-1)^{\frac{k+1}{2}} \cdot \left(\frac{\pi}{2}\right)^k \cdot \frac{B_{k,\chi}}{k!} \ ,$$

one then finds the following Fourier expansion of the function E_k:

$$\begin{aligned} E_k(z) &= 1 + L(k,\chi)^{-1} \cdot \frac{(-2\pi i)^k}{4^k (k-1)!} \sum_{n=1}^{\infty} \left(\sum_{d|n} d^{k-1} (i^d - i^{-d}) \right) e^{2\pi i n z} \\ &= 1 + \frac{k}{B_{k,\chi}} \cdot (-1)^{\frac{k+1}{2}} \cdot (-1)^k \cdot i^k \cdot (2i) \cdot \sum_{n=1}^{\infty} \left(\sum_{d|n} \chi(d) d^{k-1} \right) e^{2\pi i n z} \\ &= 1 - \frac{2k}{B_{k,\chi}} \sum_{n=1}^{\infty} \left(\sum_{d|n} \chi(d) d^{k-1} \right) e^{2\pi i n z} \ . \end{aligned}$$

Similarly, one finds:

$$F_k(z) = (-1)^{\frac{k+1}{2}} \cdot \frac{2k}{B_{k,\chi}} \sum_{n=1}^{\infty} \left(\sum_{d|n} \chi(\frac{n}{d}) d^{k-1} \right) e^{2\pi i n z} \ .$$

So, we conclude, cf. for example [9], Theorem 4.7.1, p. 177, that $E_k, F_k \in M_k(4,\chi)$. It follows that

$$E_k^{(\ell)}, F_k^{(\ell)} \in M_k(8p,\chi) \qquad \text{for} \quad \ell = 1, 2, p, 2p \ .$$

We define the element $U_p \in M_k(8p,\chi)$:

$$U_p := \frac{1}{p^k - 1} \left(E_k^{(2p)} - E_k^{(p)} - E_k^{(2)} + E_k^{(1)} + 2^{k-1} F_k^{(2p)} - F_k^{(p)} - 2^{k-1} F_k^{(2)} + F_k^{(1)} \right) \ .$$

In order to show that $f_p - U_p$ is a cusp form, it suffices to show that $V(c, f_p) = V(c, U_p)$ for $c \in \mathbb{N}$, $c \mid 8p$, where for $f \in M_k(8p,\chi)$ and $c \in \mathbb{Z}$ we define

$$V(c,f) := \lim_{z \to i\infty} \left(f \mid_k \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \right) (z) \ ;$$

this follows because the numbers c^{-1} for $c \in \mathbb{N}$, $c \mid 8p$ form a system of representatives of the cusps with respect to $\Gamma_0(8p)$. In order to compute the numbers $V(c, f_p)$ and $V(c, U_p)$ we first recall the following trick (cf. for example [14] , p. 248):

Suppose that $f, g \in M_k(K,\epsilon)$, that $c, \ell \in \mathbb{N}$ and that

$$g(z) = f(\ell z) \ .$$

Choose $x, y \in \mathbb{Z}$ such that:

$$xc - y\ell = -(c,\ell) \ , \tag{2}$$

and put:

$$A = \begin{pmatrix} \ell/(c,\ell) & x \\ c/(c,\ell) & y \end{pmatrix} \ ,$$

so that $A \in \mathrm{SL}_2(\mathbb{Z})$. Then:

$$V(c,g) = \left(\frac{\ell}{(c,\ell)} \right)^{-k} \lim_{z \to i\infty} (f \mid_k A)(z) \ , \tag{3}$$

which we see as follows:

$$\begin{aligned}
V(c,g) &= \lim_{z\to i\infty} \left(\ell^{-k/2} f \mid_k \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix}\right) \mid_k \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} (z) \\
&= \ell^{-k/2} \lim_{z\to i\infty} \left(f \mid_k A \begin{pmatrix} (c,\ell) & -x \\ 0 & \ell/(c,\ell) \end{pmatrix}\right)(z) \\
&= \ell^{-k/2} \lim_{z\to i\infty} \ell^{k/2} \cdot \left(\frac{\ell}{(c,\ell)}\right)^{-k} (f \mid_k A) \left(\frac{(c,\ell)^2}{\ell} z - \frac{x(c,\ell)}{\ell}\right) \\
&= \left(\frac{\ell}{(c,\ell)}\right)^{-k} \lim_{z\to i\infty} (f \mid_k A)(z) \ .
\end{aligned}$$

Recall also (cf. [5]) the following two facts:

If

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

then

$$G_k(z;a,b;M) \mid_k \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = G_k(z;\alpha a+\gamma b, \beta a+\delta b; M) \ ;$$

we have

$$\lim_{z\to i\infty} G_k(z;a,b;M) = \delta\left(\frac{a}{M}\right) \sum_{\substack{m\equiv b \ (M) \\ m\neq 0}} m^{-k} \ ,$$

where as above $\delta(x)$ is 1 or 0 according to whether x is an integer or not.

Using these facts and (3) above, one then computes for $c,\ell \in \mathbb{N}$ with $\ell \mid 2p$, $c \mid 8p$:

$$V(c,E_k^{(\ell)}) = \left(\frac{\ell}{(c,\ell)}\right)^{-k} L(k,\chi)^{-1} \delta\left(\frac{c}{4(c,\ell)}\right) \cdot \sum_{\substack{m\equiv y \ (4) \\ m\neq 0}} m^{-k} \ ,$$

if $(x,y)\in\mathbb{Z}^2$ is chosen such that (2) above holds. Then, if $c/(c,\ell)$ is divisible by 4 we have that both y and $\ell/(c,\ell)$ are odd, and so:

$$L(k,\chi)^{-1} \sum_{\substack{m\equiv y \ (4) \\ m\neq 0}} m^{-k} = \chi(y) = \chi\left(\frac{\ell}{(c,\ell)}\right) = 1 \ ,$$

where the last equality follows because $\ell/(c,\ell)$ is a divisor of p (since $\ell \mid 2p$ and $\ell/(c,\ell)$ is odd), and because $\chi(p)=1$ since $p\equiv 1$ (4).

Hence,

$$V(c,E_k^{(\ell)}) = \delta\left(\frac{c}{4(c,\ell)}\right) \cdot \left(\frac{\ell}{(c,\ell)}\right)^{-k} \ . \tag{4}$$

Similarly, by choosing x,y according to (2) above with ℓ replaced by 4ℓ, we find:

$$V(c,F_k^{(\ell)}) = -2\cdot(4i)^{-k} \left(\frac{\ell}{(c,4\ell)}\right)^{-k} L(k,\chi)^{-1} \delta\left(\frac{\ell}{(c,4\ell)}\right) \sum_{\substack{m\equiv x \ (4) \\ m\neq 0}} m^{-k} \ .$$

If $(c, 4\ell)$ divides ℓ then x and $c/(c, 4\ell)$ are both odd, and so:

$$-L(k,\chi)^{-1} \sum_{\substack{m \equiv x\ (4) \\ m \neq 0}} m^{-k} = -\chi(x) = \chi\left(\frac{c}{(c,4\ell)}\right) = 1 \ ,$$

where the last equality follows because $c/(c, 4\ell)$ is a divisor of p (since $c \mid 8p$ and $c/(c, 4\ell)$ is odd).

Hence,

$$V(c, F_k^{(\ell)}) = \delta\left(\frac{\ell}{(c,4\ell)}\right) \cdot 2 \cdot (4i)^{-k} \cdot \left(\frac{\ell}{(c,4\ell)}\right)^{-k} \ . \tag{5}$$

Now we compute the numbers $V(c, f_p)$ for $c \in \mathbb{N}$, $c \mid 8p$. Using as above the notation

$$\zeta_m := e^{\frac{2\pi i}{m}}$$

for $m \in \mathbb{N}$, and

$$W(h, 2p, a, c) := \sum_{\substack{j \ mod \ 2pc \\ j \equiv h \ (2p)}} \zeta_{4pc}^{aj^2}$$

for integers h, a, c with $c > 0$ and $(a, c) = 1$, we have according to (A.14) on p. 220 in [14] that

$$\theta_{3,0}(z,h,2p) \mid_{1/2} S = (\zeta_8\sqrt{2pc})^{-1} \sum_{j \ mod \ 2p} \zeta_{4p}^{-bj(2h+dj)} W(h+dj, 2p, a, c)\theta_{3,0}(z, j, 2p) \ ,$$

if

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \qquad \text{with} \quad c > 0 \ .$$

Using this and the fact that

$$\lim_{z \to i\infty} \theta_{3,0}(z,h,2p) = \begin{cases} 0 & , \text{ if } h \not\equiv 0 \ (2p) \\ 1 & , \text{ if } h \equiv 0 \ (2p) \ , \end{cases}$$

we find

$$\lim_{z \to i\infty} \theta_{3,0}(z,h,2p) \mid_{1/2} S = (\zeta_8\sqrt{2pc})^{-1} W(h, 2p, a, c) \ ,$$

and so:

$$\lim_{z \to i\infty} \prod_{j=1}^{t} \theta_{3,0}(z, 2j-p, 2p) \mid_k S = (2pic)^{-k} \prod_{j=1}^{t} W(2j-p, 2p, a, c) \ .$$

With this, we deduce from the definition of f_p and from (3) above that:

$$V(c, f_p) = \left(\frac{4pic}{(c,2)^2}\right)^{-k} \Pi_{j=1}^{t} W\left(2j-p, 2p, \frac{2}{(c,2)}, \frac{c}{(c,2)}\right)$$

for $c \in \mathbb{N}$. From this, one easily computes the following explicit values:

$$V(1, f_p) = (4pi)^{-k} \ , \quad V(2, f_p) = -(2pi)^{-k} \ , \quad V(4, f_p) = p^{-k} \ ,$$

$$V(c, f_p) = 0 \qquad \text{for} \quad c = 8, p, 2p, 4p, 8p$$

(here $V(8p, f_p) = 0$ also follows directly from the definition of f_p because $f_p \in M_k(8p, \chi)$). Let us for example consider the computation of $V(4, f_p)$: We have

$$\prod_{j=1}^{t} W(2j-p, 2p, 1, 2) = \prod_{j=1}^{t} \sum_{\substack{r \bmod 4p \\ r \equiv 2j-p \ (2p)}} \zeta_{8p}^{r^2} = \prod_{j=1}^{t} \left(\zeta_{8p}^{4j^2+p^2-4pj} + \zeta_{8p}^{4j^2+p^2+4pj} \right)$$

$$= \prod_{j=1}^{t} 2 \cdot \zeta_8^p \cdot \zeta_{2p}^{j^2+pj} = 2^t \zeta_8^{pt} \cdot \zeta_{2p}^{\frac{p(p-1)(p+1)}{6}} = 4^k \cdot i^{pk} = (4i)^k \ ,$$

where we used that $p \equiv 1 \ (4)$; hence, $V(4, f_p) = p^{-k}$.

Using (4) and (5) above one then verifies that

$$V(c, U_p) = V(c, f_p) \qquad \text{for} \quad c = 1, 2, 4, 8, p, 2p, 4p, 8p \ .$$

Hence, $f_p - U_p \in S_k(8p, \chi)$.

(c) The relation

$$s_p(n) = b(N, f_p) \ ,$$

where $N = 4n + (p-1)(p-2)/12$, follows directly from the definition of f_p and the fact discussed in section 1 above that $s_p(n)$ is the number of integral solutions $(x_1, \ldots, x_t)$ to

$$n = \frac{1}{8p} \sum_{i=1}^{t} x_i^2 \ - \frac{(p-1)(p-2)}{48}$$

with $x_i \equiv 2i - p \ (2p)$ for $i = 1, \ldots t$.

Now, from **(b)** above and from the Ramanujan-Petersson conjecture for elements in $S_k(8p, \chi)$, which is proved by Deligne, cf. [1], Th. (8.2), p. 302, it follows that

$$b(r, f_p) = b(r, U_p) + O(r^{\frac{k-1}{2}+\varepsilon})$$

for all $\varepsilon > 0$. Hence we can finish the proof by showing that

(6) $$b(N, U_p) = (-1)^{\frac{k+1}{2}} \cdot \frac{2k}{B_{k,\chi}} \cdot \frac{2}{p^k - 1} \cdot N^{k-1} \sum_{d \mid m} \chi(d) d^{1-k} \ ,$$

if $N = p^s m$ with $(m, 2p) = 1$. If we use the notations

$$\varphi(M) := \sum_{d \mid M} \chi(d) d^{k-1} \ , \qquad \psi(M) := \sum_{d \mid M} \chi(M/d) d^{k-1} \ ,$$

for $M \in \mathbb{N}$ and $\varphi(x) = \psi(x) = 0$ for $x \notin \mathbb{N}$, we obtain from the definition of U_p together with the Fourier expansions of E_k and F_k :

$$\begin{aligned} b(N, U_p) = \ & \tfrac{1}{p^k-1} \left(-\tfrac{2k}{B_{k,\chi}} (\varphi(N) - \varphi(N/p)) + (-1)^{\frac{k+1}{2}} \cdot \tfrac{2k}{B_{k,\chi}} (\psi(N) - \psi(N/p)) \right) \\ = \ & (-1)^{\frac{k+1}{2}} \cdot \tfrac{2k}{B_{k,\chi}} \cdot \tfrac{1}{p^k-1} \cdot ((-1)^{\frac{k-1}{2}} \chi(p)^s p^{s(k-1)} \varphi(m) + p^{s(k-1)} \psi(m)) \\ = \ & (-1)^{\frac{k+1}{2}} \cdot \tfrac{2k}{B_{k,\chi}} \cdot \tfrac{1}{p^k-1} \cdot p^{s(k-1)} \cdot (\chi(k)\varphi(m) + \psi(m)), \end{aligned}$$

where we used that k and N are odd, and that $\chi(p) = 1$. Now, if we notice that

$$\chi(M)\varphi(M) = \psi(M) \qquad \text{for odd } M \in \mathbb{N} \ ,$$

and that

$$\chi(m) = \chi(p^s m) = \chi(N) = \chi\left(\frac{(p-1)(p-2)}{12} \right) = \chi(k) \ ,$$

because $p-2 \equiv 3$ (4), the equality (6) then follows immediately. $\square$

Remarks: The formulae for $s_p(n)$ in the above theorem are really asymptotic formulae, i.e., in each of the cases I, II, III, the main term of the formula grows faster with n as does the O-term: This is clear in cases I and II, and in case III it follows if we note that for odd m, the number

$$\sum_{d|m} \chi(d) d^{1-k}$$

is bounded below by $\zeta(k-1)^{-1}$, as is easily seen.

We also see that we obtain asymptotic formulae even if we use weaker estimates for the Fourier coefficients of cusp forms than the theorem of Deligne on the Ramanujan-Petersson conjecture. For example, if one replaces the O-terms in the theorem above with $O(n^{k/2})$, then this weaker theorem is proved by the above and with reference to Hecke's result in [5]: This result, which can be proved by elementary means, states precisely that the Fourier coefficients of cusp forms of weight k on any congruence subgroup $\Gamma_0(M)$ can be estimated by $O(n^{k/2})$.

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