# A NOTE ON A THEOREM OF A. GRANVILLE AND K. ONO.

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ABSTRACT. It was recently proved by A. Granville and K. Ono, confer [2],[3], that if $t \in \mathbb{N}$, $t \geq 4$ then every natural number has a $t$-core partition. The essence of the proof consists in showing this assertion for $t$ prime, $t \geq 11$. We give an alternative, short proof for these cases.

Suppose that $n \in \mathbb{N}$ and that $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a partition of $n$, i.e. $\lambda_1, \lambda_2, \ldots$ is a sequence of non-negative integers of which only finitely many are $\neq 0$, and such that:

$$\lambda_1 \geq \lambda_2 \geq \ldots \qquad \text{and} \qquad n = \sum_{i=1}^{\infty} \lambda_i \ .$$

Then the partition $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ associated with $\lambda$ is defined by

$$\lambda'_i = \#\{j \in \mathbb{N} | \lambda_j \geq i\},$$

and $\lambda'$ is also a partition of $n$. If $(i,j) \in \mathbb{N}^2$ such that $\lambda_i \lambda'_j > 0$ we have the $(i,j)$'th hook number of $\lambda$:

$$h_{i,j}(\lambda) := \lambda_i + \lambda'_j - i - j + 1.$$

If $t \in \mathbb{N}$ and if none of the hook numbers of $\lambda$ is divisible by $t$, one says that $\lambda$ is a $t$-core partition of $n$.

The '$t$-core partition conjecture' asserts that if $t \in \mathbb{N}$, $t \geq 4$, then every $n \in \mathbb{N}$ has a $t$-core partition. This conjecture is proved by A. Granville and K. Ono in the papers [2], [3]. The conjecture had attracted some interest since it has implications in the representation theory of symmetric and alternating groups: It implies that if $p$ is a prime number $\geq 5$ then for every $n \in \mathbb{N}$ both the symmetric group $S_n$ and the alternating group $A_n$ has a $p$-block of defect zero. Together with previously known results this further implies that if $p$ is prime $\geq 5$ then every finite simple group has a $p$-block of defect zero. We refer to [2] for a survey of these results.

Now let $n \in \mathbb{N}$. Then $n$ has a $t$-core partition if and only if the equations:

$$n = \sum_{i=0}^{t-1} (\frac{t}{2} \cdot x_i^2 + i x_i) \quad \text{and} \quad \sum_{i=0}^{t-1} x_i = 0 \ , \tag{0}$$

have a (simultaneous) integral solution; in fact, it was proved in [1] that the number of integral solutions to (0) equals the number of $t$-core partitions of $n$.

As explained in [3], the $t$-core partition conjecture is proved if it is shown that (0) has for all $n \in \mathbb{N}$ an integral solution in each of the cases: $t = 4, 5, 6, 7, 9$ and ($t$ prime, $t \geq 11$). The cases $t = 4, 6, 9$ were handled in [3] by special arguments. The cases $t = 5, 7, 11, 13$ can be handled by using modular forms: K. Ono informed me that the cases $t = 5, 7$ and $t = 11$ respectively were done in unpublished notes by A. O. L. Atkin and himself respectively. Finally, in [2] a special argument using modular forms for the case $t = 13$, and a general argument for $t \geq 17$ were given.

We shall give an alternative, simple proof of the fact that (0) has an integral solution for all $n \in \mathbb{N}$ if $t$ is prime, $t \geq 11$; in particular, our proof avoids computation with modular forms. As in [2], the proof has two parts: One for 'large' $n$ and one for 'small' $n$. We pay for the simplicity of our argument for 'large' $n$ by having to prove the following lemma, which requires a little amount of computation (which however is not more than the amount of computation which was needed for the case $t = 13$ in [2]).

**Lemma.** *Suppose that $t$ is prime, $t \geq 11$ and $n \leq t(t^2-1)/4 + (t-1)$.*

*Then (0) has an integral solution $(x_0, \dots, x_{t-1})$.*

*Proof.* Put $n_0 = t(t^2-1)/4 + (t-1)$. Suppose that $s \in \{1, \dots, (t-1)/2\}$, put:

$$m_s := \frac{t+1}{2} - s \ , \qquad c_0(n) := n,$$

define the integers $y_i(n),\ c_i(n)$ for $i = 1, \dots, s$ successively:

$$y_i(n) := [\left(-1 + \sqrt{1 + 4tc_{i-1}(n)}\right)/2t], \quad c_i(n) := c_{i-1}(n) - (ty_i(n)^2 + y_i(n)),$$

so that $c_i(n) \geq 0$ for all $i$, and define the integers $x_{m_s}(n), \dots, x_{t-m_s}(n)$ :

$$x_{m_s+2i-2}(n) := -y_i(n), \quad x_{m_s+2i-1}(n) := y_i(n) \qquad \text{for} \quad i = 1, \dots, s.$$

Then we have:

$$c_s(n) = n - \sum_{i=m_s}^{t-m_s} (\frac{t}{2} \cdot x_i(n)^2 + ix_i(n)) \quad \text{and} \quad \sum_{i=m_s}^{t-m_s} x_i(n) = 0.$$

We conclude that the proof is finished if for some $s \in \{1, \dots, (t-1)/2\}$ the equations:

$$r = \left(\sum_{i=0}^{m_s-1} + \sum_{i=t-m_s+1}^{t-1}\right)(\frac{t}{2} \cdot x_i^2 + ix_i) \quad \text{and} \quad \left(\sum_{i=0}^{m_s-1} + \sum_{i=t-m_s+1}^{t-1}\right) x_i = 0 \tag{1}$$

have an integral solution for $r = c_s(n)$, for all $n \leq n_0$.

Now let us notice that the proof of lemma 1 in [2] actually shows that if $r \leq m_s(m_s-1)$ then (1) has an integral solution (with $x_i \in \{0, \pm 1\}$ for all $i$).

So, the proof is finished if for some $s \in \{1, \dots, (t-1)/2\}$ we have:

$$c_s(n) \leq (\frac{t+1}{2} - s)(\frac{t-1}{2} - s) \qquad \text{for all} \quad n \leq n_0. \tag{2}$$

Now, $y_i(n) \in \mathbb{Z}$ is largest possible such that $ty_i(n)^2 + y_i(n) \leq c_{i-1}(n)$, and so

$$c_i(n) = c_{i-1}(n) - (ty_i(n)^2 + y_i(n)) \leq 2ty_i(n) + t + 1 \leq t + \sqrt{1 + 4tc_{i-1}(n)} \ ;$$

consequently, if $s \in \{1, \dots, (t-1)/2\}$, $j \in \{0, \dots, s-1\}$ and $f_j(x)$ is a polynomial, then the condition

$$(4t)^{2^j-1} c_{s-j}(n) \leq f_j(t) \qquad \text{for all} \quad n \leq n_0$$

is implied by:

$$(4t)^{2^{j+1}-1} c_{s-j-1}(n) \leq f_{j+1}(t) \qquad \text{for all} \quad n \leq n_0,$$

where

$$f_{j+1}(x) := \left(f_j(x) - x \cdot (4x)^{2^j-1}\right)^2 - (4x)^{2^{j+1}-2}, \tag{3}$$

provided that

$$g_j(t) := f_j(t) - t \cdot (4t)^{2^j - 1} \geq 0. \tag{4}$$

Hence, if we define $f_0(x) := ((x+1)/2 - 4)((x-1)/2 - 4)$, $g_0(x) := f_0(x) - x$, and $f_i(x), g_i(x)$ for $i = 1, 2, 3, 4$ in accordance with (3) and (4), we have that (2) holds for the case $s = 4$, if

$$g_0(t), g_1(t), g_2(t), g_3(t) \geq 0 \quad \text{and} \quad (\ (4t)^{15} c_0(n) \leq f_4(t) \qquad \text{for all} \quad n \leq n_0\ ). \tag{5}$$

But since we have $c_0(n) = n \leq t(t^2-1)/4 + (t-1)$, we see that (5) holds if

$$g_0(t), \ldots, g_3(t), f(t) \geq 0, \tag{6}$$

where

$$f(x) := f_4(x) - (4x)^{15}(x(x^2-1)/4 + x - 1).$$

Now, $g_0(x), \ldots, g_3(x), f(x)$ are certain polynomials with rational coefficients which can easily be computed by using (for example) MAPLE. For example, one finds:

$$f(x) = 2^{-32}(x^{32} - 320x^{31} + 48496x^{30} - 4639040x^{29} + \ldots)\ .$$

Again, using MAPLE we can compute approximations to the real roots of these polynomials and thus verify that (6) holds for $t \geq 43$, i.e. (2) holds for $s = 4$ if $t \geq 43$.

On the other hand, when $t \in \{37, 41\}$, (2) holds in the case $s = 4$ since a direct computation shows that for all $n \leq n_0$ we have:

$$(c_4(n) \leq 206 \quad \text{if} \quad t = 41), \qquad (c_4(n) \leq 186 \quad \text{if} \quad t = 37);$$

hence we may assume $t \leq 31$.

In the remaining cases $t = 11, 13, 17, 19, 23, 29, 31$ we finish the proof by stating the following facts which can be checked in a few minutes on a machine.

If $t \in \{11, 13\}$ and $n \leq n_0$, (0) has a solution with

$$x_1, x_2, x_3, x_4, x_6 \in \{0, \pm 1\} \quad \text{and} \quad x_{t-4}, x_{t-3}, x_{t-2}, x_{t-1} \in \{0, \pm 1, \pm 2\} \tag{7}$$

and

$$x_5 = 0 \quad \text{and} \qquad x_j = 0 \quad \text{for} \quad 7 \leq j \leq t-5. \tag{8}$$

For $t = 17, 19, 23, 29, 31$ we have $c_1(n) \leq c_1 := 290, 362, 530, 842, 962$ respectively, and (1) has for $s = 1$ and all $r \leq c_1$ a solution with (7) and (8). □

We can now give our alternative proof of the following theorem due to A. Granville and K. Ono, cf. [2],[3]. In their proof of existence of integral solutions to (0) when $t \geq 17$, they exploited the theorem of Lagrange on the representation of natural numbers as sums of 4 squares. In our proof of existence for $t \geq 11$ ($t$ prime) we use Gauss' three square theorem.

**Theorem.** *Suppose that $t$ is prime, $t \geq 11$.*

*Then for all $n \in \mathbb{N}$, (0) has an integral solution.*

*Proof.* Write $n = tm' + r'$ with $0 \leq r' \leq t-1$. Because of the lemma we may and will assume $n \geq t(t^2-1)/4 + (t-1)$, hence $m' \geq (t^2-1)/4$. Define the integers $m$ and $r$ as follows:

$$(m, r) := \begin{cases} (m', r') & \text{if } m' \equiv 1\ (2),\ r' \not\equiv 0\ (4) \\ (m'+2, r'-2t) & \text{if } m' \equiv 1\ (2),\ r' \equiv 0\ (4) \\ (m' \mp 1, r' \pm t) & \text{if } m' \equiv 0\ (2),\ r' \equiv \pm t\ (4) \\ (m'+1, r'-t) & \text{if } m' \equiv r' \equiv 0\ (2). \end{cases}$$

Then $n = tm + r$ and either

(I) $$m \equiv r \equiv 1\ (2) \quad \text{and} \quad 4m \geq r^2,$$

or

(II) $$m \equiv 1\ (2),\ r \equiv 2\ (4) \quad \text{and} \quad 16m \geq r^2.$$

If case (I) prevails, we have that $4m - r^2$ is odd and $\not\equiv -1\ (8)$. Hence, by Gauss' three square theorem we have

$$4m - r^2 = a^2 + b^2 + c^2$$

for certain integers $a, b, c$ which must all be odd (since $4m - r^2 \equiv 3\ (4)$). We may then assume that $r + a + b + c$ is divisible by 4, and we define the integers:

$$\alpha = (r+a+b+c)/4,\ \beta = (r-a-b+c)/4,\ \gamma = (r-a+b-c)/4,\ \delta = (r+a-b-c)/4,$$

$$x_0 = -\alpha,\ x_1 = \alpha,\ x_2 = -\beta,\ x_3 = \beta,\ x_4 = -\gamma,\ x_5 = \gamma,\ x_6 = -\delta,\ x_7 = \delta,$$

and $x_i = 0$ for $i \geq 8$. Then $x_0 + \ldots + x_{t-1} = 0$ and:

$$\sum_{i=0}^{t-1} (\frac{t}{2} \cdot x_i^2 + i x_i) = t \cdot \frac{1}{4}(r^2 + a^2 + b^2 + c^2) + r = tm + r = n.$$

In case (II), use the same arguments and definitions of $\alpha, \beta, \gamma, \delta$ with $r$ replaced by $r/2$. Then put

$$x_0 = -\alpha,\ x_1 = -\beta,\ x_2 = \alpha,\ x_3 = \beta,\ x_4 = -\gamma,\ x_5 = -\delta,\ x_6 = \gamma,\ x_7 = \delta,$$

and $x_i = 0$ for $i \geq 8$. □

**Remark.** *The above lemma could of course be checked for all odd $t$ in the range $9 \leq t \leq 41$. This would prove the lemma and the above theorem for odd $t$ with $t \geq 9$.*

## References

[1] F. Garvan, D. Kim, D. Stanton: 'Cranks and $t$-cores.' Invent. Math. **101** (1990), 1–17.

[2] A. Granville, K. Ono: 'Defect zero $p$-blocks for finite simple groups.' Trans. Amer. Math. Soc. **348** (1996), 331–347.

[3] K. Ono: 'On the positivity of the number of $t$-core partitions.' Acta Arith. **66** (1994), 221–228.

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