EXAMPLES OF 2-DIMENSIONAL, ODD GALOIS REPRESENTATIONS OF A_5-TYPE OVER $\mathbb{Q}$ SATISFYING THE ARTIN CONJECTURE

IAN KIMING, XIANGDONG WANG

1.

Let us retain the notation of the introduction of I: We are considering irreducible representations:

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{C}),$$

and we say that ρ is of dihedral-, A_4-, S_4-, or A_5-type, if the image of the projective representation associated with ρ is a dihedral group or isomorphic to A_4, S_4, A_5, respectively. Also, for such a representation ρ we denote by N and ε the Artin conductor and determinant character of ρ, respectively.

The purpose of this lecture is to provide new examples of odd representations of A_5-type which satisfy, together with all their twists by 1-dimensional characters, the Artin conjecture, i.e.: odd representations whose Artin L-series are L-series of cusp forms of weight 1 on some $\Gamma_0(N)$. For a general description of how such examples may be obtained, we refer to the introduction of I.

Concerning the choice of the examples, we remark the following. First we have to start with a Galois extension $K/\mathbb{Q}$ with Galois group isomorphic to A_5. Such a field gives us a projective representation:

$$\bar{\rho}_0 : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{PGL}_2(\mathbb{C})$$

by choosing 1 of the 2 embeddings $A_5 \hookrightarrow \mathrm{PGL}_2(\mathbb{C})$. Now, $\bar{\rho}_0$ has a lifting ρ_0 with some Artin conductor N and determinant character ε. We must of course demand that ε be odd. If we want to verify the Artin conjecture for ρ_0 we must be able to enumerate all Galois representations ρ with the same conductor and determinant. This forces us to require first that:

A. The field K should occur in table 1 (which was discussed in II). This means that the discriminant of a quintic subfield of K should be bounded by 2083^2.

Secondly, we must compute the dimension of the space $S_1(N, \varepsilon)$, and in order to keep the complexity of this computation within reasonable bounds, this imposes certain restrictions on N and ε: We require that $\dim S_1(N, \varepsilon)$ should be computable by the second method presented in I, which means that:

B. N should have form $2^\alpha \cdot n$, where n is odd and square free.

Further, we have found it necessary to require:

C. $\varepsilon^2 = 1$ and $\text{l.c.m}(N, 4N) \leq 10000$, again for the purpose of keeping the complexity manageable. The reason for the requirement $\text{l.c.m}(N, 4N) \leq 10^4$ instead of just $N \leq 10^4$ is that we may have to compute:

$$\dim S_1(4N, \varepsilon) \quad \text{instead of just} \quad \dim S_1(N, \varepsilon)$$

if N is odd. Summarizing, we have to look for A_5-fields K in table 1 such that a projective representation in $\text{PGL}_2(\mathbb{C})$ associated with K (and hence any such) has a lifting with Artin conductor N and odd determinant character ε, where (N, ε) satisfies the requirements **B** and **C**. An inspection of table 1 (using table 2 and the theory of I) shows that there are exactly 7 such fields: These are the fields with numbers $27, 69, 128, 135, 181, 191$, and 238 in table 1. (Had we just required ε odd with $\varepsilon^2 = 1$ and $N \leq 10000$, we would have found 6 additional fields, namely the fields with numbers $1, 16, 44, 46, 73$, and 88 in table 1). As was shown in section 5 of I, the involved pairs (N, ε) are (not respectively):

$$\begin{array}{rll} N = p, \quad \varepsilon = \chi_{-p} & \text{for} & p = 2083, \\ N = 4p, \quad \varepsilon = \chi_{-p} & \text{for} & p = 487, 751, 887, 919, \\ & \text{and} & \\ N = 2^5 p, \quad \varepsilon = \chi_{-p} & \text{for} & p = 73, 193, \end{array}$$

where χ_{-p} denotes the character of $\mathbb{Q}(\sqrt{-p})/\mathbb{Q}$. Note that the numbers 73, 193, 487, 751, 887, 919, 2083 are prime numbers. In this connection, we also found in section 5 of I the number of inequivalent irreducible representations ρ with conductor N and determinant ε for certain pairs (N, ε). Let us recall in the following table what these numbers are:

	Type:	Dihedral	A_4	S_4	A_5
N	ε				
487	χ_{-487}	3	0	0	0
751	χ_{-751}	7	0	2	0
887	χ_{-887}	14	0	0	0
919	χ_{-919}	9	0	0	0
2083	χ_{-2083}	3	0	0	4
$2^2 \cdot$ 487	χ_{-487}	0	0	0	4
$2^2 \cdot$ 751	χ_{-751}	0	0	0	4
$2^2 \cdot$ 887	χ_{-887}	0	0	0	4
$2^2 \cdot$ 919	χ_{-919}	0	0	0	4
$2^5 \cdot$ 73	χ_{-73}	0	0	0	8
$2^5 \cdot$ 193	χ_{-193}	0	0	0	8

Let us also recall that for each of the prime numbers $p = 73$ and $p = 193$, there is at least 1 representation ρ of dihedral type with $N = 2^2 p$ and $\varepsilon = \chi_{-p}$, cf. section 5.4 in I.

It follows that in order to verify by computational means the Artin conjecture for the above representations of A_5-type, we have to show computationally that:

$$\dim S_1(2083, \chi_{-2083}) \quad = \quad 7 \ ,$$

$$\dim S_1^{\mathrm{new}}(2^2 \cdot p, \chi_{-p}) \quad = \quad 4 \ , \quad \text{for} \quad p = 487, 751, 887, 919,$$

and

$$\dim S_1^{\mathrm{new}}(2^5 \cdot p, \chi_{-p}) \quad = \quad 8 \ , \quad \text{for} \quad p = 73, 193.$$

2. COMPUTATIONS.

2.1. Before describing the concrete computations, we want to start with a few general remarks.

Let k be a non-negative integer, N a natural number, ε a Dirichlet character mod N with $\varepsilon(-1) = (-1)^k$, and let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. We have an exact sequence of complex vector spaces:

$$0 \longrightarrow S_{k+2}(\Gamma_0(N), \varepsilon) \xrightarrow{\iota} \mathrm{H}^1(\Gamma, U_{k,\varepsilon})_+ \xrightarrow{r_+} \mathrm{H}^1(\Gamma_\infty, U_{k,\varepsilon})_+ \longrightarrow 0,$$

where $U_{k,\varepsilon}$ is a certain $\mathbb{C}\Gamma$-module which was described in V, Γ_∞ is the infinite cyclic group generated by:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \ ,$$

'+' means 'fixed points under the action of $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$', and r_+ is the restriction map.

Let:

$$X_k(N, \varepsilon)_+ = \mathrm{Ker}(r_+) \ .$$

For any $n \in \mathbb{N}$ we have the Hecke operator on $S_{k+2}(\Gamma_0(N), \varepsilon)$:

$$T_n = \sum_{l,m} T(l, m) \ ,$$

where the summation is over natural numbers l and m with $lm = n$, $l \mid m$ and $(l, N) = 1$, and:

$$T(l, m) = \Gamma_0(N) \begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix} \Gamma_0(N) \ ,$$

cf. [3], chap. 4. We learned in IV and V how to obtain a concrete model for the space $\mathrm{H}^1(\Gamma, U_{k,\varepsilon})_+$ and how to define Hecke operators $\tilde{T_n}$ on this space which are compatible with ι. For convenience we shall drop the ' ˜ ' and denote the cohomological Hecke operators also by T_n.

Suppose now that t is a divisor of N such that $(t, N/t) = 1$. We can then write:

$$\varepsilon = \varepsilon_1 \varepsilon_2 \ ,$$

where ε_1 and ε_2 are Dirichlet characters mod t and mod N/t, respectively. Choosing $a,b,c,d \in \mathbb{Z}$ such that:

$$adt - bc\frac{N}{t} = 1 \ ,$$

we have an isomorphism:

$$W_{N,t} : S_{k+2}\left(\Gamma_0(N), \varepsilon_1\varepsilon_2\right) \overset{\sim}{\longrightarrow} S_{k+2}\left(\Gamma_0(N), \bar{\varepsilon}_1\varepsilon_2\right) \ ,$$

where the 'Artin-Lehner operator' $W_{N,t}$ (cf. [1]) is given by:

$$W_{N,t} : f \mapsto f \mid W_{N,t} = f \mid_{k+2} \begin{pmatrix} at & b \\ Nc & dt \end{pmatrix} \ ,$$

for $f \in S_{k+2}(\Gamma_0(N), \varepsilon_1\varepsilon_2)$. This action does not depend on the choice of a,b,c,d. We learned in IV how to define $W_{N,t}$ on $\mathrm{H}^1(\Gamma, U_{k,\varepsilon})_+$ in a way which is compatible with ι.

Now, the following simple lemma is important to us. We formulate it for the sake of simplicity only in the case of trivial nebentypus; in case of a non-trivial nebentypus, one must identify (in an obvious way) the complex vector spaces $X_k(N, \varepsilon_1\varepsilon_2)_+$ and $X_k(N, \bar{\varepsilon}_1\varepsilon_2)_+$.

Lemma 1. *Consider the above situation with k even and $\varepsilon = 1$. Suppose that $x \in X_k(N,1)_+$, and that $\varphi \in \mathrm{Hom}(X_k(N,1)_+, \mathbb{C})$. Then, as we saw in IV, the function f given by:*

$$f(z) = \sum_{n=1}^{\infty} \varphi(T_n x) q^n \ , \quad q = e^{2\pi i z}, \quad \mathrm{Im}(z) > 0 \ ,$$

is the Fourier expansion at ∞ of an element in $S_{k+2}\left(\Gamma_0(N)\right)$. We have:

$$(f \mid W_{N,t})(z) = \sum_{n=1}^{\infty} \varphi(T_n W_{N,t} x) q^n.$$

Proof. Let us write $W = W_{N,t}$ for convenience and note that W commutes with all Hecke operators T_n. Let:

$$a_1 : S_{k+2}\left(\Gamma_0(N)\right) \longrightarrow \mathbb{C}$$

be the linear map 'coefficient of $e^{2\pi i z}$ in the Fourier expansion at ∞'; we then have:

$$g(z) = \sum_{n=1}^{\infty} a_1(g \mid T_n) q^n \ , \quad \text{for } g \in S_{k+2}\left(\Gamma_0(N)\right).$$

Define:

$$A_1 : X_k(N,1)_+ \longrightarrow \mathbb{C} \ ,$$

by $A_1 = a_1 \circ \iota^{-1}$. Then we have for $g \in S_{k+2}\left(\Gamma_0(N)\right)$ with $y = \iota g$:

$$\begin{aligned} g &= \sum a_1(g \mid T_n) q^n \\ &= \sum a_1(\iota^{-1} y \mid T_n) q^n \\ &= \sum a_1\left(\iota^{-1}(T_n y)\right) q^n \\ &= \sum A_1(T_n y) q^n \ . \end{aligned}$$

Now, there is $h \in S_{k+2}\left(\Gamma_0(N)\right)$ such that:

$$x = \iota h \ ;$$

then:

$$W x = \iota(h \mid W) \ ,$$
$$h = \sum A_1(T_n x) q^n \ ,$$

and:

$$h \mid W = \sum A_1(T_n W x) q^n.$$

Further, if $m \in \mathbb{N}$ we have:

$$\begin{aligned} h|T_m &= \sum_n A_1 \left(\sum_{\substack{d|(m,n) \\ (d,N)=1}} T_{mn/d^2} x \right) q^n \\ &= \sum_n A_1(T_n T_m x) q^n. \end{aligned}$$

There is an element T of the Hecke algebra $\mathbb{T} = \langle T_n \rangle$ such that:

$$\varphi(y) = A_1(Ty) \ , \quad \text{for} \quad y \in X_k(N,1)_+ \ ;$$

this follows from the analogous result on $S_{k+2}\left(\Gamma_0(N)\right)$. Then:

$$\begin{aligned} f &= \sum \varphi(T_n x) q^n \\ &= \sum A_1(T T_n x) q^n \\ &= \sum A_1(T_n T x) q^n \\ &= h \mid T \ , \end{aligned}$$

whence:

$$\begin{aligned} f \mid W &= (h \mid T) \mid W = (h \mid W) \mid T \\ &= \left(\sum A_1(T_n W x) q^n \right) \mid T \\ &= \sum A_1(T_n W T x) q^n \\ &= \sum A_1\left(T(T_n W x)\right) q^n \\ &= \sum \varphi(T_n W x) q^n. \end{aligned}$$

□

2.2. We want to compute $\dim S_1\left(\Gamma_0(2^{s+1}p), \chi_{-p}\right)$, where p is an odd prime number and:

$$s \geq \begin{cases} 2 \ , & \text{if } p \equiv 1 \quad (4) \\ 1 \ , & \text{if } p \equiv 3 \quad (4), \end{cases}$$

by the last-mentioned method in section 1 of I. In other words, we consider the theta-function:

$$\theta_2(z) = \sum_{\substack{n=-\infty \\ n \equiv 1 \quad (2)}}^{\infty} e^{\pi i n^2 z/4}, \quad \mathrm{Im}(z) > 0 \ ,$$

and

$$g_{s,p}(z) = \theta_2(2^s z)\theta_2(2^s pz) \ , \quad \mathrm{Im}(z) > 0 \ ,$$

and we have an embedding:

$$0 \longrightarrow S_1\left(\Gamma_0(2^{s+1}p), \chi_{-p}\right) \longrightarrow S_2\left(\Gamma_0(2^{s+1}p)\right)$$

given by multiplication with $g_{s,p}$; the image consists of forms $f \in S_2\left(\Gamma_0(2^{s+1}p)\right)$ such that the first $2^{s-3}(p+1)$ Fourier coefficients at ∞ (counting from the coefficient of $e^{2\pi iz}$) of both f and $f \mid W_{s,p}$ all vanish, where:

$$W_{s,p} = \begin{pmatrix} p & b \\ -2^{s+1}p & dp \end{pmatrix} \ ,$$

where b and d are integers with $pd + 2^{s+1}b = 1$.

Let us put $N = 2^{s+1}p$ and retain the notation of subsection 2.1. We have the following concrete model for the complex vector space $\mathrm{H}^1(\Gamma, U_{0,1})_+$: It is generated by elements:

$$(u, v) \in (\mathbb{Z}/\mathbb{Z}N)^2 \quad \text{of order } N \ ,$$

with the following relations:

$$\begin{aligned} (\lambda u, \lambda v) - (u, v) &= 0 \ , \quad \text{for } \lambda \in (\mathbb{Z}/\mathbb{Z}N)^\times \\ (u, v) + (u, -v) &= 0 \ , \\ (u, v) + (-v, u) &= 0 \ , \\ &\text{and} \\ (u, v) + (u - v, u) + (-v, u - v) &= 0. \end{aligned}$$

Let us write $\overline{(u, v)}$ for the class in $\mathrm{H}^1(\Gamma, U_{0,1})_+$ of (u, v). The action of $W_{s,p}$ on $\mathrm{H}^1(\Gamma, U_{0,1})_+$ can now be described in explicit terms as follows (cf. IV and [2]):

Suppose that $x = \overline{(u, v)}$, where g.c.d$(u, v) = 1$. Let $\alpha, \beta \in \mathbb{Z}$ be such that $\alpha v - \beta u = 1$. For an element $r \in \mathbb{P}^1(\mathbb{Q})$ define $\psi(r) \in \mathrm{H}^1(\Gamma, U_{0,1})_+$: If $r = \infty$, put $\psi(r) = 0$. Otherwise, consider the continued fractions expansion of r:

$$r = [c_0, \dots, c_n]$$

determined by the requirements:

$$rc_0 \geq 0 \ , \quad rc_i \geq 1 \quad \text{for} \quad i \geq 1 \ , \quad \text{and} \quad |c_n| \geq 2 \ , \text{if } n \geq 1.$$

Let ρ_i/σ_i for $i \geq 1$ be the convergents with the sign conventions:

$$\mathrm{sgn}(\sigma_{2i+1}) = (-1)^i \ , \quad \mathrm{sgn}(\sigma_{2i}) = (-1)^i \mathrm{sgn}(r).$$

Then:

$$\psi(r) = \sum_{k=1}^{n} \overline{(\sigma_{k-1}, \sigma_k)}.$$

Writing:

$$W_{s,p} \cdot \frac{\alpha}{u} = \eta \ , \quad W_{s,p} \cdot \frac{\beta}{v} = \zeta \ ,$$

we then have:

$$W_{s,p} \cdot x = \psi(\eta) - \psi(\zeta) \ ,$$

(up to a non-zero constant depending only on p).

Now the computation of $\dim S_1(N, \chi_{-p})$ is performed in the following steps.

1. Construct a $\mathbb{C}$-basis $\{b_i\}$ for $\mathrm{H}^1(\Gamma, U_{0,1})_+$ where each b_i is a $\mathbb{Z}$-linear combination of the above symbols $\overline{(u,v)}$. Let φ_i be the corresponding projections, i.e.:

$$x = \sum_i \varphi_i(x) b_i \ , \quad \text{for } x \in \mathrm{H}^1(\Gamma, U_{0,1})_+.$$

2. Choose a large prime number l ($\neq 2, 3$) and an element $x \in X_2(N,1)_+$, which is a $\mathbb{Z}$-linear combination of the symbols $\overline{(u,v)}$. If $\dim S_2(\Gamma_0(N)) = \dim \mathrm{H}^1(\Gamma, U_{0,1})_+$ we may choose any x in $\mathrm{H}^1(\Gamma, U_{0,1})_+$. In any case, it can always be determined whether a given x belongs to $X_2(N,1)_+ = \mathrm{Ker}(r_+)$, since the map r_+ can be determined explicitly.

Consider the matrix:

$$C(x) = \begin{pmatrix} \varphi_1(T_1 x) & \dots & \varphi_1(T_n x) \\ \vdots & & \vdots \\ \varphi_{d_1}(T_1 x) & \dots & \varphi_{d_1}(T_n x) \end{pmatrix} ,$$

where $d_1 = \dim \mathrm{H}^1(\Gamma, U_{0,1})_+$ and $n = \frac{1}{6}\mu(N)$ with $\mu(N) = [\Gamma : \Gamma_0(N)]$. The numbers $\varphi_i(T_j(x))$ are all in $\mathbb{Z}\left[\frac{1}{6}\right]$. We can thus reduce $C(x)$ mod l and compute its rank mod l. If this rank is not $d_0 = \dim S_2(\Gamma_0(N))$, choose another $x_1 \in X_2(N,1)_+$, and compute the rank of:

$$\begin{pmatrix} C(x) \\ \hline C(x_1) \end{pmatrix} .$$

If we do not obtain the rank d_0 after a reasonable amount of effort, choose another prime number l_1 and repeat the process with l_1.

This process must eventually terminate, and when it does, we have a prime number $l \neq 2, 3$, and a number of elements $x_1, x_2, \ldots \in X_2(N,1)_+$ such that the matrix:

$$C = \begin{pmatrix} C(x_1) \\ \hline C(x_2) \\ \hline \vdots \end{pmatrix}$$

has rank d_0 mod l.

Experimentally, it has been found that if one chooses l large enough (for example $l \sim 3 \cdot 10^4$) then one finds in most cases 1 or 2 elements $x_i \in X_2(N,1)_+$ such that this matrix C has rank d_0 mod l.

3. Compute the matrix:

$$D = \begin{pmatrix} D(x_1) \\ \hline D(x_2) \\ \hline \vdots \end{pmatrix} ,$$

where:

$$D(x_i) = \begin{pmatrix} \varphi_1(T_1 W_{s,p} x_i) & \dots & \varphi_1(T_n W_{s,p} x_i) \\ \vdots & & \vdots \\ \varphi_{d_1}(T_1 W_{s,p} x_i) & \dots & \varphi_{d_1}(T_n W_{s,p} x_i) \end{pmatrix}.$$

Then D has coefficients in $\mathbb{Z}\left[\frac{1}{6}\right]$.

4. Put $\delta_{s,p} = 2^{s-3}(p+1)$. Use Gauß-elimination on the matrix $(C\|D)$ to bring it in form:

$$(+) \qquad \left(\begin{array}{c|c||c} A & & \\ \cline{1-1}\cline{3-3} 0 & \cdots & \widetilde{D} \end{array}\right), \qquad \begin{array}{ccc} \underbrace{\qquad\quad}_{\delta_{s,p}} & \qquad & \underbrace{\qquad\quad}_{n} \end{array}$$

where A has 'echelon form':

$$\begin{pmatrix} *0\dots0 & & & \\ & *0\dots0 & & \\ & & \ddots & \\ & 0 & & *0\dots0 \end{pmatrix}$$

with non-zero $*$'s. Then use Gauss-elimination on $\widetilde{D}$ to bring it in form:

$$(++) \qquad \left(\begin{array}{c|c} A' & \\ \hline 0 & D' \end{array}\right), \qquad \underbrace{\qquad\quad}_{\delta_{s,p}}$$

again with A' in echelon form. The dimension of $S_1(\Gamma_0(2^{s+1}p), \chi_{-p})$ is then the rank of D'.

Let us observe the following simple fact: That if $l \neq 2, 3$ is a prime number such that the matrix C in step **2** has rank d_0 mod l and such that the $*$'s in (+), and hence in (++), are all invertible mod l, then $\dim S_1\left(\Gamma_0(2^{s+1}p), \chi_{-p}\right)$ is the rank of D' mod l. These conditions can be checked by computations modulo l if the upper left parts of (+) and (++) appear in upper triangular form:

$$\begin{pmatrix} * & & \\ & \ddots & \\ 0 & & * \end{pmatrix} ,$$

with non-zero $*$'s mod l; this turned out to be the case in all the cases considered below for the following choice of l:

$$l = 32749.$$

This circumstance of course reduced the computational complexity of step **4** significantly.

Recall our notation:

$$\begin{aligned} d_0 &= \dim S_2\left(\Gamma_0(2^{s+1}p)\right) , \\ d_1 &= \dim \mathrm{H}^1(\Gamma, U_{0,1})_+ , \end{aligned}$$

$x_1, x_2, \ldots :$ elements of $X_2(2^{s+1}p, 1)_+$ giving the matrix C in step 2,

and put:

$$\begin{aligned} t_1 &= \operatorname{rank}(\widetilde{D}) , \\ t_2 &= \operatorname{rank}(D') = \dim S_1\left(\Gamma_0(2^{s+1}p), \chi_{-p}\right) . \end{aligned}$$

The results of the computations are then given by the following table:

s	p	d_0	d_1	$x_1, x_2, \ldots$	t_1	t_2
1	487	242	242	$\overline{(1,19)}$	120	13
1	751	374	374	$\overline{(1,14)}$	186	31
1	887	442	442	$\overline{(1,14)}$	220	46
1	919	458	458	$\overline{(1,9)}$	228	31
1	2083	1040	1040	$\overline{(1,10)}$	519	28
2	73	71	71	$\overline{(1,17)}$	34	2
3	73	143	145	$\overline{(1,89)}$	69	5
4	73	289	293	$\overline{(1,89)}$	141	16
2	193	191	191	$\overline{(1,21)}$	94	2
3	193	383	385	$\overline{(1,25)}$	189	5
4	193	769	773	$\overline{(1,3)}$ $\overline{(1,11)}$	381	16

The computing time was in no case greater than for the case $(s, p) = (1, 2083)$, which required about 40 hours of computation.

Let us then verify the Artin conjecture for the representations of A_5-type (as well as for their twists by characters of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$) listed in section 1.

2.3. $N = 2^2 p$, $\varepsilon = \chi_{-p}$ with $p \in \{487, 751, 887, 919\}$:

In each of these cases the information in the table of section 1 allows us immediately to write down the dimension of:

$$S_1\left(\Gamma_0(p), \chi_{-p}\right) ,$$

since there are no representations of A_5-type with conductor p and determinant character χ_{-p} in any of these cases. We find that $\dim S_1\left(\Gamma_0(p), \chi_{-p}\right)$ is $3, 9, 14, 9$

for $p = 487, 751, 887, 919$, respectively. Since, as is easily seen,

$$\dim S_1^{\text{new}}\left(\Gamma_0(2p), \chi_{-p}\right) = 0 \ ,$$

for any prime p, we have:

$$\dim S_1^{\text{new}}\left(\Gamma_0(4p), \chi_{-p}\right) = \dim S_1\left(\Gamma_0(4p), \chi_{-p}\right) - 3 \cdot \dim S_1\left(\Gamma_0(p), \chi_{-p}\right) .$$

Hence we find:

$$\dim S_1^{\text{new}}\left(\Gamma_0(4p), \chi_{-p}\right) = 4 \ , \quad \text{for} \quad p = 487, 751, 887, 919.$$

2.4. $N = 2083$, $\varepsilon = \chi_{-2083}$:

Here we note that we obtain the Fourier expansions of $g_{s,p} \cdot f_i$, where (f_i) is a basis of $V = S_1\left(\Gamma_0(4 \cdot 2083), \chi_{-2083}\right)$, from the matrix D' at the end of step 4. Dividing these by the Fourier expansion of:

$$g_{1,2083}(z) = \theta_2(4z)\, \theta_2(4 \cdot 2083z)$$

at ∞, we obtain the Fourier expansion of the f_i's. This permits the determination of the subspace of V consisting of forms $f \in V$ such that $f(4z) \in V$, and this subspace is precisely $S_1\left(\Gamma_0(2083), \chi_{-2083}\right)$. We find that:

$$\dim S_1\left(\Gamma_0(2083), \chi_{-2083}\right) = 7 \ .$$

2.5. $N = 2^5 \cdot p$, $\varepsilon = \chi_{-p}$ with $p \in \{73, 193\}$:

If p is one of the prime numbers 73 or 193, we introduce the following notation:

$$\alpha_i = \dim S_1^{\text{new}}\left(\Gamma_0(2^i p), \chi_{-p}\right) \quad \text{for} \quad i = 2, 3, 4, 5.$$

Then from the table of section 2.2, we obtain in each of the cases:

$$\begin{aligned}
\dim S_1\left(\Gamma_0(2^2 p), \chi_{-p}\right) &= \alpha_2 \ , \\
\dim S_1\left(\Gamma_0(2^3 p), \chi_{-p}\right) &= 2\alpha_2 + \alpha_3 = 2 \ , \\
\dim S_1\left(\Gamma_0(2^4 p), \chi_{-p}\right) &= 3\alpha_2 + 2\alpha_3 + \alpha_4 = 5 \ , \\
\dim S_1\left(\Gamma_0(2^5 p), \chi_{-p}\right) &= 4\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 = 16 \ .
\end{aligned}$$

Recall the remark at the end of section 1 that:

$$\dim S_1^{\text{new}}\left(\Gamma_0(2^2 p), \chi_{-p}\right) > 0 \quad \text{for} \quad p = 73 \quad \text{and} \quad p = 193 \ .$$

Thus $\alpha_2 \geq 1$ and we then find successively $\alpha_2 = 1$, $\alpha_3 = 0$, $\alpha_4 = 2$, and finally:

$$\dim S_1^{\text{new}}\left(\Gamma_0(2^5 p), \chi_{-p}\right) = \alpha_5 = 8 \ ,$$

for both $p = 73$ and $p = 193$.

2.6. The results of sections 2.3, 2.4, 2.5 together with the table of section 1 and the discussion in the introduction of I, give us a verification of the Artin conjecture for all the representations listed in section 1 and for the twists of these by characters of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

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kiming@math.ku.dk

DEPT. OF MATH., UNIV. OF COPENHAGEN, UNIVERSITETSPARKEN 5, 2100 COPENHAGEN Ø, DENMARK.