Vanishing Theorems in Algebraic Geometry

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Sept 06
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X = smooth projective varieties $/\mathbb{C}$.
$\quad\quad ||$
projective cpx manifolds

Could think of $X \subseteq \mathbb{P}^N(\mathbb{C})$ (zeros of poly equation)

- Think of X as abstract variety, how to embed it in $\mathbb{P}^N$ effectively

Line bundles and sections.

<u>Basic facts</u> 1) To give a morphism $f: X \to \mathbb{P}^N$
$\Longleftrightarrow$ specify a line bundle L on X
$+ s_0, \dots, s_N$ sections of L
s.t. $\forall x \in X$
$\exists i$ s.t. $s_i(x) \neq 0$.

via sending $x \longmapsto [s_0(x) : \dots : s_N(x)]$.

2) To give an <u>embedding</u> $X \xhookrightarrow{f} \mathbb{P}^N \Longleftrightarrow$

$(L, s_0, \dots s_N)$ as before $+ \ \forall x, y \ \exists i$
s.t. $s_i(x) = 0$ and $s_i(y) \neq 0$

$+ \ \forall v \in T_x X$
"$\exists i$ s.t. jet of s_i at x is different from

A line bundle on X (vector bundle of rk 1) is a variety L with projection $p: L \to X$

+ open cover $(U_i)_{i \in I}$ of X with isomorphisms

$$\begin{tikzcd} p^{-1}(U_i) \arrow[r, "\psi_i", "\simeq"'] \arrow[d, "p_i"] & U_i \times \mathbb{C} \arrow[d] \\ U_i \arrow[r, equal] & U_i \end{tikzcd}$$

on $U_i \cap U_j$ $\quad \psi_i \circ \psi_j^{-1} : (U_i \cap U_j) \times \mathbb{C} \longrightarrow (U_i \cap U_j) \times \mathbb{C}$

$$(x, t) \longmapsto (x, g_{ij}(x)\, t)$$

nowhere vanishing regular function.

A section is a morphism $s: X \to L$ s.t. $p \circ s = \mathrm{id}$

Line bundles $\overset{1:1}{\longleftrightarrow}$ invertible sheaves.

$$L \longmapsto \mathcal{L}(U) = \left\{ \begin{array}{c} \text{sections of } L \\ \text{defined over } U \end{array} \right\}$$

global sections def over X

$H^0 L = \Gamma(\mathcal{L})$ global sections of $\mathcal{L}$

<u>Ex:</u> $\mathcal{O}_X$ = "functions on X"

$\mathcal{O}_X(U)$ = holomorphic functions on X.

$\Gamma(\mathcal{O}_X) = \mathcal{O}_X(X) = \mathbb{C}$.

$\mathbb{P}^n$ can be covered by $\left(U_i = \underset{\substack{\cong \\ \mathbb{C}^n}}{(x_i \neq 0)} \right)_{i=0,\ldots n}$

For any $d \in \mathbb{Z}$ can take lbd L
defined by

$$L(U_i) = \mathcal{O}_X(U_i) \quad \text{with} \quad g_{ij} = \left(\frac{x_j}{x_i}\right)^d$$

called $\mathcal{O}_{\mathbb{P}^n}(d)$

<u>Thm</u> These are all line bundles on $\mathbb{P}^n$.

Given poly $F \in \mathbb{C}[x_0, \ldots, x_N]$ homogeneous of deg d.

can associate a global section

$$S_F \in H^0(\mathcal{O}_{\mathbb{P}^n}(d))$$

$$\text{"} \frac{F(x_0, \ldots, x_N)}{x_i^d} \text{"} \qquad (\text{on } U_i)$$

In fact

<u>Prop</u> $H^0(\mathcal{O}_{\mathbb{P}^n}(d)) = \underbrace{\mathbb{C}[x_0, \ldots, x_N]_d}$
(note, no global sections if $d<0$)

poly of deg d.

$s \in H^0(L)$ global section $\rightsquigarrow Z(s)$ zero locus.

as a alg set of codim 1

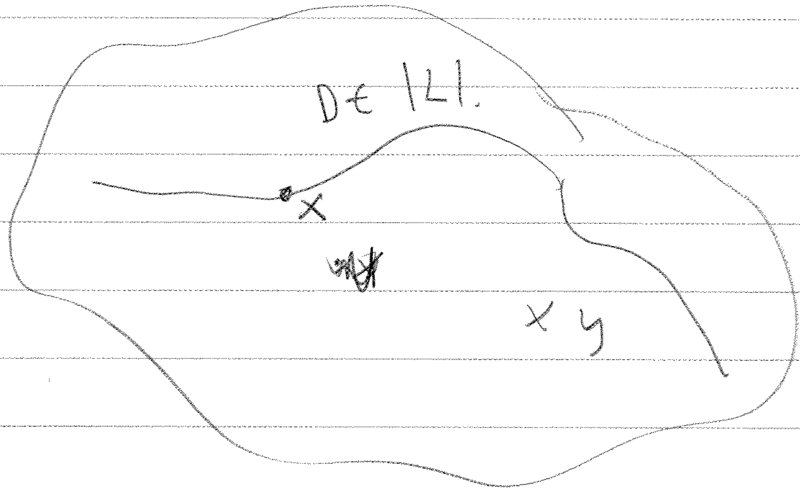
actually we get <u>divisor</u> = formal lin combination of
codim 1 sub[illegible]

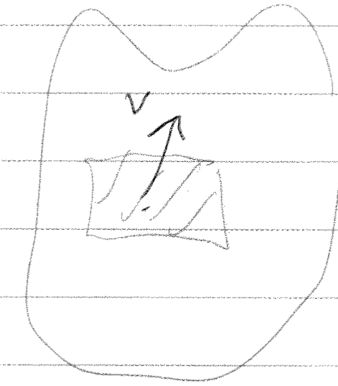
$D = Z(s)$

$L := \mathcal{O}_X(D)$.

$\mathbb{P} H^0(L) =: |L|$ the space of divisors assoc to L.

$v \in T_x X$





so to get inj need to separate points

This gives a 1-1 corresp. between embeddings $\longleftrightarrow$ line bundles which have property (2).

[Conversely pull back. $L = f^*(\mathcal{O}_{\mathbb{P}^n}(1))$]

If (1) holds ($\forall\ x \in X$) we say that L is globally generated (at $x \in X$)

If (2) holds we say that L is very ample

If $\exists\, m \geq 1$ s.t. $L^{\otimes m}$ is very ample L is called ample.

Kodaira (embedding thm): L is ample

$\Leftrightarrow$ as a cpx line bundle, L is positive

i.e. $\exists$ Hermitian metric h s.t. curvature form is positive definite

$$\left(\Theta(L,h) = -\partial\bar{\partial} \log |s|_h^2 \quad \text{on } U \subseteq X \right)$$

(one direction trivial, if has ample line bundle, pull back Fubini-Study from $\mathbb{P}^n$)

Nakai-Moishezon criterion.

L ample $\Leftrightarrow$ L is positive from viewpoint of intersection theory.

$\dim V = d$, $V \subseteq X$: $\underbrace{L \cdot L \cdots L}_{d\text{-times}} \cdot V > 0$

$$= \int_V c_1(L)^d > 0.$$

$= \Theta$ $\leftarrow$ form from before

[by theorem of Kleiman enough to intersect with $L = C$ curve] (E

$$L \text{ is } \underset{\text{numerically effective.}}{\underline{\text{nef}}} \iff \underbrace{L \cdot L \cdot \;\cdot L}_{d} \cdot V \geq 0 \quad \forall V.$$

<u>E.g.</u> If $f : X \to \mathbb{P}^n$ $\left(L = f^* \mathcal{O}_{\mathbb{P}^n}(1) \right)$.

~~thus~~ • L is always nef.

• L is ample $\iff$ f has finite fibers.

$$N = \dim H^0 L^{\otimes m}$$

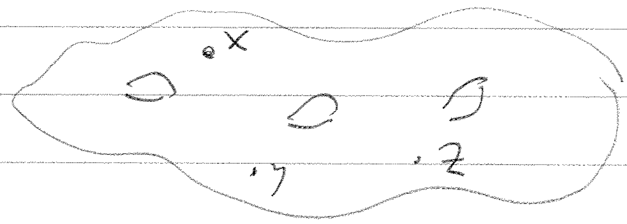
<u>Fact:</u> There does <u>not</u> exist m only as a function of $n = \dim X$ s.t. $L^{\otimes m}$ is ample.

<u>Riemann-Roch:</u> (L ample) $\chi(L^{\otimes m}) = \underset{\text{poly}}{P(m)}$

$$= \frac{L \cdot L \cdots L}{m!} m^n + \text{lower order terms}$$

self-intersection # $m \gg 0$

<u>For Curves:</u>



divisors have degrees. $D = x + 2y + 3z$

$\deg = 6$

Fact: • $\deg L > 0 \iff L$ is ample

$(\deg L \geq 2g-1 \Rightarrow h^1 L = 0)$

• $\deg \geq 2g \Rightarrow L$ is globally generated.

• $\deg \geq 2g+1 \Rightarrow L$ is very ample.

$\forall X$ $\exists$ natural bundle called canonical bundle $\omega_X = \mathcal{O}_X(K_X)$

$$\omega_X := \Lambda^n \Omega^1 X$$

in local coordinates gen by $dz_1 \wedge \dots \wedge dz_n$

$$2-2g = \chi(X) = c_1(T_X) = -\deg K_X$$

So $\deg K_X = 2g-2$.

$\deg L \geq 2g-1 \Rightarrow L = K_X + (\text{ample})$

additive notation

$\deg L \geq 2g \Rightarrow L = K_X + \text{ample of } \deg \geq 2 = \dim X + 1$.

$\deg \geq 2g+1 \Rightarrow L = K_X + \text{ample of } \deg \geq 3 = \dim X + 2$.

(Fujita's Conj)

<u>Conj</u>: $\forall X$ smooth proj of dim n, ~~and~~ any ample L

$\Rightarrow$ (a) $K_X + mL$ $(= \omega_X \otimes L^{\otimes m})$ is globally gen $\forall m \geq n+1$

(b) $K_X + mL$ is very ample $\forall m \geq n+$

"adjoint bundle" $:= K_X$ + ample

<u>Known</u> for (a) $n=2$ (Reider).
$n=3$ (Ein-Lazarsfeld).
$n=4$ (Kawamata

(b) $n=2$ (Reider)

Best known result for (a) $m \geq \binom{n+1}{2}$

(Angehrn-Siu, Helmke).

(also other results where you add $2K_X +$ (big number) L.

$\mathcal{F}$ coherent sheaf on $X \Rightarrow H^0\mathcal{F}, H^1\mathcal{F}, \ldots, H^n\mathcal{F}$ $\quad n = \dim X$.
$\quad \| $
$\Gamma(\mathcal{F})$

<u>Question</u>:

When do we have $h^i_{\mathbb{Z}}\mathcal{F} := \dim_{\mathbb{C}} H^i\mathcal{F} = 0$

$Y \subseteq X$ subvar $\Leftrightarrow$ $I_Y \subseteq \mathcal{O}_X$ (ideal sheaf)

Have seq $0 \to I_Y \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$

(global version of $0 \to I \to A \to A/I \to 0$).

D = divisor

$I_D = \mathcal{O}_X(-D)$ again a line bundle.

and we have

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0.$$

$$\mathcal{O}_X \hookrightarrow \mathcal{O}_X(D).$$

$x \in X$ $\left(0 \to I_{\{x\}} \to \mathcal{O}_X \to \mathcal{O}_{\{x\}}^{\;\mathbb{C}} \to 0 \right) \otimes \mathcal{O}_X(K_X+L)$

$$0 \to \mathcal{O}_X(K_X+L) \otimes I_{\{x\}} \to \mathcal{O}_X(K_X+L) \to \mathcal{O}_X(K_X+L) \otimes \mathcal{O}_{\{x\}} \to 0$$

$s \longmapsto s(x)$,

$$0 \to H^0(\mathcal{O}_X(K_X+L) \otimes I_{\{x\}}) \to H^0(\mathcal{O}_X(K_X+L)) \xrightarrow{\alpha_x} H^0(\mathcal{O}_X(K_X+L) \otimes \mathcal{O}_{\{x\}})$$

$$\to H^1(\mathcal{O}_X(K_X+L) \otimes I_{\{x\}}) \to \underset{\substack{\parallel \\ 0}}{H^1(\mathcal{O}_X(K_X+L))} \to \cdots$$

By def. α_x surjective $\Leftrightarrow$ K_X+L globally gen at x.

Hence globally gen at x

$$\Longleftrightarrow H^1(\mathcal{O}_X(K_X+L)\otimes I_{\{x\}})=0.$$

Basic Vanishing theorem

Thm (Kodaira) X sm projective
L = ample on X

$$\Rightarrow H^i(\mathcal{O}_X(K_X+L))=0 \quad \forall i>0$$

Before Serre - Cartan

Thm A: $\forall$ coherent sheaf $\mathcal{F}$ $\forall$ ample L

$$H^i(\mathcal{F}\otimes L^{\otimes m})=0, \quad \forall i>0 \ \forall m \gg 0$$

Thm B: $\mathcal{F}\otimes L^{\otimes m}$ globally generated $\forall m \gg 0$

(for curves use RR + Serre duality
in general use splitting principle)

note not effective

For Kodaira, theorem about solving $\bar{\partial}$-equation.

Hodge theorem X sm cpx.

$$H^j(X;\mathbb{C}) \simeq \bigoplus_{p+q=j} H^q(X;\Omega_X^p)$$

Simplifying assumption: $\exists$ smooth $D \in |L|$.

$L = \mathcal{O}_X(-D)$

<u>Want:</u> $H^i\, \mathcal{O}_X(K_X + L) = 0 \ \forall\, i > 0$

$\Leftrightarrow$ $H^j\, \mathcal{O}_X(-D) = 0 \ \forall\, j < n = \dim X$

Serre duality

$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0.$

(*) $H^{j-1}\mathcal{O}_X \to H^{j-1}\mathcal{O}_D \to H^j\mathcal{O}_X(-D) \to H^j(\mathcal{O}_X)$

$\to H^j(\mathcal{O}_D)$

<u>Hodge thm</u> $\Rightarrow$ $H^j\mathcal{O}_X$ direct summand of $H^j(X;\mathbb{C})$

<u>Lefschetz thm:</u> If $j \leq n-2 \Rightarrow H^j(X;\mathbb{C}) \to H^j(D;\mathbb{C})$

isomorphism

If $j = n-1$ — " — is injective

So Kodaira vanishing follows from (*)

using Hodge + Lefschetz

(~~under simplifying assumption, which one~~ can reduce to).

This proof is due to Ramanujan.