Interactions between complex dynamics and potential theory (in $\mathbb{C}$)

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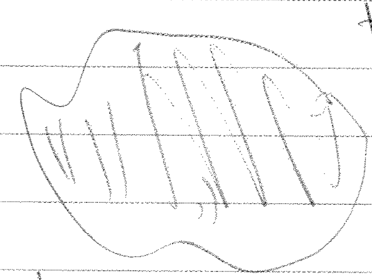
Part I: Fund results using potential theory to obtain results in dynamics.

Part II: • More recent results (higher dimensions)
• results in potential theory using cpx dynamics.

Potential theory in $\mathbb{C}$ (or $\mathbb{R}^2$)

$K \subseteq \mathbb{C}$ compact.



total charge -1

What is the dist of charge (total charge say -1) at equilibrium.

Mathematically: measure μ $\operatorname{supp}\mu \subseteq K$ $\mu(K) = 1$.

Energy functional $E(\mu) = \iint \log \frac{1}{|z-w|} \, d\mu(z) d\mu(w)$ (could be ∞).

Thm (Frostman, 1931) If $\exists$ a prob. measure μ with $E(\mu) < \infty$

then $\exists!$ measure μ_K s.t. $E(\mu_K) = \inf_{\mu} E(\mu)$

μ_K = harmonic measure (for $\mathbb{C} \setminus K$ rel ∞).

and $\operatorname{supp} \mu_K \subseteq \partial K$

Capacity $\quad \mathrm{cap}\, K := e^{-\inf_{\mu} E(\mu)}$

Note: $\mathrm{cap}\, K = 0 \iff E(\mu) = \infty \quad \forall \mu$ on K

$\overset{\text{def}}{\iff}$ K is polar.

$\iff$ $\exists$ subharmonic function $u \not\equiv -\infty$ s.t. $K \subseteq \{u = -\infty\}$

<u>Example:</u>

- $\mathrm{cap}\, D(p,R) = R$
- $\mathrm{cap}\left(\overset{\ell}{\bullet\!\!\longrightarrow\!\!\bullet} \right) = \dfrac{\ell}{4}$

<u>Green's function:</u> (potential function)

- $\mathrm{cap}\, K \neq 0 \Rightarrow \exists!\, G \quad \mathbb{C} \setminus K \to \mathbb{R}$

  s.t.

  - G is harmonic
  - $G(z) = \log|z| + \gamma + o(1)$ as $z \to \infty$
  - $\lim_{z \to \partial K} G(z) = 0$ quasi-everywhere on ∂K

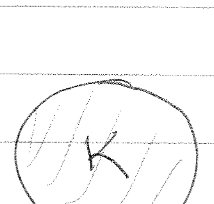
    except pts of capacity 0

Extend G by 0 on K

Then $\frac{1}{2\pi} \Delta G = \mu_K$ in the sense of distributions

$\mathrm{cap}\, K = e^{-\gamma}$

<u>Ex:</u>

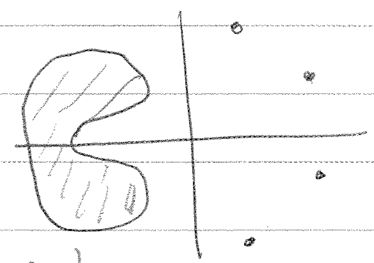


$G = \log^+|z|$

$\mu_K =$ normalized lebesgue measure on S^1.

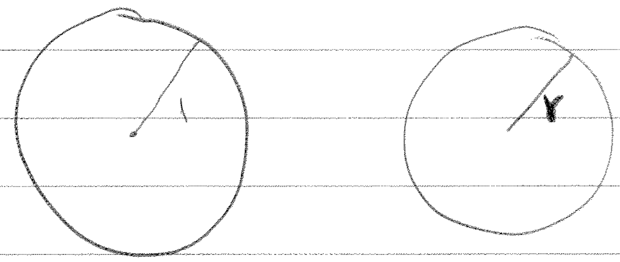
Thm (Fekete–Szegö, 1955) Let $K \subseteq \mathbb{C}$ be compact and symmetric about $\mathbb{R}$

Then $\operatorname{cap} K \geq 1 \iff$ every open neighborhood of K contains inf many complete sets of alg. integers.



[recall: β alg integer $\iff$ β solution to monic poly $= 0$
complete sets of alg integers]

Ex:



$r < 1$ cannot contain inf many since product of roots has to be the coeff $a_0 \in \mathbb{Z}$ in monic poly. So "$\Longleftarrow$" is clear in thm abov

Polynomial dynamics (in $\mathbb{C}$)

$$f(z) = z^2 + c \qquad c \in \mathbb{C}.$$

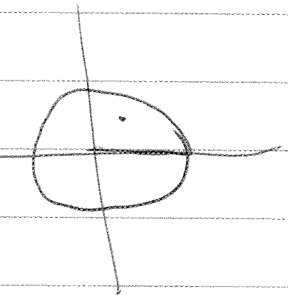
Study $f^n = \underbrace{f \circ \cdots \circ f}_{n \text{ times}}$ as $n \to \infty$.

Filled Julia set $K(f) = \{ z \in \mathbb{C} \mid f^n(z) \not\to \infty \text{ as } n \to \infty \}$.

Julia set: $J(f) = \partial K(f)$ (chaotic locus of $\{f^n\}$).

Ex: For f polynomium $\operatorname{cap} K(f) = |\text{leading coeff}|^{-1/(d-1)}$

<u>Simplest example:</u> $f(z) = z^2$



$K(f) = \overline{\mathbb{D}}$ closure of unit disk

$J(f) = S^1$

Escape-rate function $\quad d = \deg(f)$

$$G_f(z) = \lim_{n \to \infty} \frac{1}{d^n} \log^+ |f^n(z)|$$

$$G_f(z) = 0 \iff z \in K(f)$$

In fact $G_f(z) =$ Greens function for $K(f)$.

$\mu_f = \frac{1}{2\pi} \Delta G_f$ harmonic measure on $J(f)$

invariant for f (i.e. $f_* \mu_f = \mu_f$) Brolin 1965

= Unique measure of max entropy for f. (entropy will be $\log d$)

(invariance property $\frac{1}{d} f^* \mu_f = \mu_f$) Lyubich, 198

where pullback f^* defined as $\int \varphi (f^* \mu) = \int \sum_{f(x)=y} \varphi(x) \, d\mu(y)$

Lyapunov exponent $L(f, \mu_f) = \int \log |f'(z)| \, d\mu_f$

$$\underset{\text{Mañé}}{=} \frac{\log d}{\dim \mu_f}$$

What happens with $L(f, \mu_f)$ when we vary f ??

Ex: family: $f(z) = z^2 + c \quad c \in \mathbb{C}$.

then

Thm $L(f_c, \mu_c)$ is continuous + subharmonic on $\mathbb{C}$ and it is harmonic near $c_0 \iff$
f_{c_0} is stable $\iff \partial M$

Mandelbrot set.

— Break —

$$L(f, \mu_f) = \int \log|f'(z)|\, d\mu_f$$

$$= \sum_{f'(c_j)=0} G_f(c_j) + \log d \qquad \left(\begin{matrix}\text{Manning}\\ \text{Przytycki}\end{matrix}\right)$$

Integration by parts

Douady-Hubbard, Sibony: The Green function for the Mandelbrot set $\mathcal{M} = \{c \in \mathbb{C} \mid J(f_c) \text{ is connected}\}$ is $2 G_{f_c}(0)$ ← critical pts for f_c

$\Rightarrow c \mapsto L(f_c, \mu_c)$ is subharmonic
$$\| $$
$$\frac{1}{2}\left(G_c(0) + \log 2\right)$$

$\Rightarrow \Delta L(f_c, \mu_c)$ is a measure with support $=$
$\partial\mathcal{M}$ = bifurcation locus for $\{f_c\}$.

Riemann sphere

More general family $\mathcal{H}ol(\hat{\mathbb{C}}) = \{ f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ holomorphic

$\cup\!\!\mid$
Poly

1. course in cpx functions $\Rightarrow f(z) = \frac{P(z)}{Q(z)}$

$d = \deg f = \max\{\deg P, \deg Q\} \geq 2$ ← assume, since case of Mobius trans easy (?).

μ_f = Unique measure of max entropy (Lyubich, Mané).

$\operatorname{supp} \mu_f = J(f)$ = complement of region where $\{f^n\}$ is a normal family.

NB: no obvious potential-theoretic understanding of μ_f (on $\hat{\mathbb{C}}$).

Hubbard-Papadopol + Fornaess-Sibony } Lift to $\mathbb{C}^2$ ~~then~~ we get dynamics.

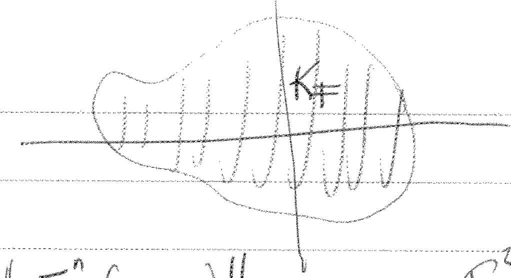
Write $F(z,w) = \left(P(z/w)\, w^d,\ Q(z/w)\, w^d \right)$

polynomial map $F : \mathbb{C}^2 \to \mathbb{C}^2$.

$\pi : \mathbb{C}^2 - 0 \to \hat{\mathbb{C}}$ $\qquad \pi \circ F = f \circ \pi$
$(z,w) \longmapsto \frac{z}{w}$

On $\mathbb{C}^2$ dealing with polys, so can use potential theory.

K_F = filled Julia set = $\{ (z,w) \in \mathbb{C}^2 \mid F^n(z,w) \not\to \infty$ as $n \to \infty \}$



Escape rate function

$$G_F(z,w) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ \| F^n(z,w) \| \qquad \mathbb{C}^2$$

$= $ pluricomplex Green's function for K_F.

Eq measure: $\mu_F = \underbrace{(dd^c G_F) \wedge (dd^c G_F)}_{\substack{\text{potential measure} \\ \text{support} \subseteq \partial K}}$

$$\left[\begin{array}{c} \operatorname{supp} \mu_F = \partial_S K \\ \text{Shilov boundary (?)} \end{array} \right]$$

$=$ Unique measure of max entropy for F

$\pi_* \mu_F = \mu_f$ on $\hat{\mathbb{C}}$

f rational $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$

$$L(f, \mu_f) = \int_{\hat{\mathbb{C}}} \log \|Df\| \, d\mu \qquad \left(= \frac{\log d}{\dim \mu_f} \right)$$

$$= \underbrace{\int_{\mathbb{C}^2} \log |\det DF| \, d\mu_F}_{L(F)} - \log d$$

Have formula as before

$$L(F) = \sum G_F(c_j) + \log(\operatorname{cap} K_F)$$

points chosen according to normalization.

$f = P/Q$

$\operatorname{cap} K_F = |\operatorname{Res}(P,Q)|^{*}$ $\quad * = -1/d(d-1)$

Bifurcation locus Bif For family $\{f_\lambda \mid \lambda \in X$ cpx manifold$\}$ of rational functions.

one says that λ_0 is stable if $\lambda \mapsto J(f_\lambda)$ is continuous near λ_0.

$\iff f_{\lambda_0}|_{J_{\lambda_0}} \overset{\text{conjugate}}{\sim} f_\lambda|_{J_\lambda}$ $\forall \lambda$ in neighborhood of λ_0

Ex: $\{z^2+c\}$ Bif $= \partial \underbrace{M}_{\text{mandelbrot set}}$

Potential theory in $\mathbb{C}^N$ — generalizations of Capacity — transfinite diameter.

Def of transfinite diam:

$N=1$ $K \subseteq \mathbb{C}$ $\quad d_\infty(K) = \lim_{n\to\infty}\Big(\sup_{\zeta_1,\dots,\zeta_n\in K} \underbrace{\prod_{i<j}|\zeta_i-\zeta_j|}_{\text{Vandermonde det}}\Big)^{1/\binom{n}{2}}$

$\overset{\text{Fekete}}{=} \operatorname{cap} K.$

$$\begin{vmatrix} 1 & \zeta_1 & \cdots & \zeta_1^{n-1} \\ \vdots & & & \vdots \\ 1 & \zeta_n & \cdots & \zeta_n^{n-1} \end{vmatrix}$$

$K \subset \mathbb{C}^N$ List the monomials in N var of $\deg \le n$

$e_1, \dots, e_{M(n)}$

$$d_\infty(K) = \lim_{n\to\infty}\left(\sup_{\zeta_1,\dots,\zeta_{M(n)}\in K} |\det(e_i(\zeta_j))|^{1/D(n)}\right)$$

exists by Zaharij - 1975

Fekete-Leja transfinite diam

(D, Rumely)

<u>Thm</u> If $F : \mathbb{C}^N \to \mathbb{C}^N$ is a regular polynomial map of degree d $(\mathrm{Res}\, F_d \neq 0)$

$$d_\infty(F^{-1}K) = |\mathrm{Res}\, F_d|^{\cancel{1/d^N}\ 1/d}\, d_\infty(K)$$

1 variable formula classical
1930's Fekete $f : \mathbb{C} \to \mathbb{C}$ poly $f(z) = a_d z^d + \dots + a_0$

$$\mathrm{Cap}(f^{-1}K) = |a_d|^{1/d}\, (\mathrm{Cap}\, K)^{1/d} \qquad a_d \neq 0.$$

<u>Idea of pf:</u> show 1) $\dfrac{d_\infty(F^{-1}K)}{d_\infty(K)^{1/d}}$ independent of K.

arithmetic! proof

2) choose $K = K_F$ $\quad F^{-1}K_F = K_F$

uses arithmetic intersection theory + dynamics

3) $d_\infty(K_F) = ? = |\mathrm{Res}\, F_d|^*$