Interactions between complex dynamics and potential theory

Part I: Find results using potential theory to obtain results in dynamics.

Part II: More recent results (higher dimension).

Results in potential theory using C^p dynamics.

Potential theory in C (or R^2)

\[ K \subset \mathbb{C} \text{ compact.} \]

What is the drift of change
(total change \( \sum_{-1}^0 \)) at equilibrium.

Mathematically: measure \( \mu \) s.t. \( supp \mu \subseteq K, \mu(K) = 1 \).

Energy functional \( E(\mu) = \int \int \frac{1}{|z-w|} \mu(\{z\})d\mu(w) \) (could be \( \infty \)).

Thm (Frostman, 1931) \( \exists \) a measure \( \mu \) with \( E(\mu) < \infty \) then \( \exists \) measure \( \mu_K \) s.t. \( E(\mu_K) = \inf_{\mu} E(\mu) \)

\( \mu_K \) = harmonic measure (for \( \partial K \) rel \( \partial \Omega ) \) and \( supp \mu_K \subseteq \partial K \).
Capacity \[ \text{cap } K := e^{-\int E(\mu)} \]

Note:
- \[ \text{cap } K = 0 \iff E(\mu) = \infty \text{ } \forall \mu \text{ on } K \]
- \( K \) is polar
- \( \exists \) subharmonic function \( \mu \neq -\infty \)
  such that \( K \leq \mu = -\infty \)

Example:
- \( \text{cap } D(p, R) = R \)
- \( \text{cap } (\text{line}) = \frac{2}{\pi} \)

Green's function: (potential function)

- \( \text{cap } K = 0 \implies \exists G: \mathbb{C} \setminus K \to \mathbb{R} \)
- \( G \) is harmonic
- \( G(z) = \log|z| + \gamma + o(1) \text{ as } z \to \infty \)
- \( \lim_{z \to \partial K} G(z) = 0 \text{ quasi-everywhere on } \partial K \)

Extend \( G \) by \( 0 \) on \( K \)

Then \( \frac{1}{2\pi} \Delta G = \mu_K \text{ in the sense of distributions} \)

\( \text{cap } K = e^0 = 1 \)

Example:
- \( G = \log^+ |z| \)
- \( \mu_K = \text{normalized Lebesgue measure on } S^1 \).
Thm (Fekete–Szegő, 1955) Let $K \subseteq \mathbb{C}$ be compact and symmetric about $\mathbb{R}$.

Then $\operatorname{cap} K \geq 1$ if every open neighborhood

$\text{of } K \text{ contains infinitely many complete sets of algebraic integers.}$

[recall: $\mathbb{P}$ a prime & $p$ solution to monic poly $= 0$]

[complete set of algebraic integers]

Example:

Polynomial dynamics (in $\mathbb{C}$)

$$ f(z) = z^2 + c, \quad c \in \mathbb{C}, $$

Study $f^n = f \circ \cdots \circ f \quad (n \text{ times})$ as $n \to \infty$.

Filled Julia set $K(f) = \{ z \in \mathbb{C} \mid f^n(z) \to \infty \text{ as } n \to \infty \}$

Julia set: $J(f) = \partial K(f)$ (chaotic locus of $f^n$).

Example: $\operatorname{cap} K(f) = 1$ leading coefficient...
Simplest example: \( f(z) = z^2 \)

\[
K(f) = \overline{D} \quad \text{closure of unit disk}
\]

\[
J(f) = S
\]

Escape-rate function \( d = \deg(f) \)

\[
G_f(z) = \lim_{n \to \infty} \frac{1}{d^n} \log f^n(z)
\]

\[
G_f(z) = 0 \iff z \in K(f)
\]

In fact \( G_f(z) = \text{Green's function for } K(f) \).

\[
\mu_f = \frac{1}{2\pi} \Delta G_f \quad \text{harmonic measure on } J(f)
\]

invariant for \( f \) (i.e. \( f_*\mu_f = \mu_f \)) Broli, 1965

= unique measure of maximal entropy for \( f \) (entropy will be \( \log d \))

\[
\frac{1}{d} f^* \mu_f = \mu_f
\]

Lyubich, 198

where pullback \( f^* \) defined as \( \int f^*(y) = \int y \mu(y) f^{(*)}(y) \)

Lyapunov exponent \( L(\mu_f) = \int \log |f'(z)| \, d\mu_f \)

\[
= \log d \quad \text{min over } \dim \mu_f
\]
What happens with $L(f, p_c)$ when we vary $f$?

**Example:**

Family: $f(z) = z^2 + c$, $c \in \mathbb{C}$.

Then

**Theorem:** $L(f_c, p_c)$ is continuous + subharmonic on $\mathbb{C}$ and it is harmonic near $c_0$ if:

- $f_{c_0}$ is stable $\iff$ $c_0 \in \mathcal{M}$

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**Proof:**

$$L(f, p_c) = \int \log |f'(z)| \, dp_c$$

$$= \sum_{j=0} G_f(c_j) + \log 2 \quad \text{(Manning–Przybicki)}$$

- Integration by parts
- $f'(c_j) = 0$

**Douady–Hubbard.** The Green function $G_f$ for the Sierpinski Mandelbrot set $M = \{ f \in \mathbb{C} \mid J(f) \text{ is connected} \}$ is:

- $2 G_f(0)$ at critical points $f_0$ for $f_c$

$\Rightarrow c \iff L(f_c, p_c)$ is subharmonic

$$\frac{1}{2} \left( G_f(0) + \log 2 \right)$$

$\Rightarrow \Delta L(f_c, p_c)$ is a measure with support $\mathcal{M} = \text{bifurcation locus for } f_c$. 

$\mathcal{M}$
More general family $\ Hol(\hat{C}) = \{ f: \hat{C} \to \hat{C} \text{ holomorphic} \}$

1. course: cpx functions $\Rightarrow f(z) = \frac{P(z)}{Q(z)}$

$\deg f = \max \{ \deg P, \deg Q \} \geq 2$

$\mu_f = \text{unique measure of max entropy (Lyubich, Mane)}$

$\text{supp } \mu_f = J(f) = \text{complement of region where } |f^n| \text{ is a normal family}$

$\text{no obvious potential-theoretic understanding of } \mu_f \text{ (on } \hat{C})$

Hubbard-Papadopoulou + Fornaess-Sibony

"Lift to $C^2$ then we get dynamics + moduli"
Escape rate function

\[ G_f(z, w) = \lim_{n \to \infty} \frac{1}{d^n} \log^+ \| F^n(z, w) \| \quad \text{supp} \leq \partial K \]

= pluricomplex Green function for \( K_f \).

By measure:

\[ \mu_f = (dd^c G_f) \wedge (dd^c G_f) \]

= Unique measure of max entropy for \( F \)

\[ \text{supp} \mu_f = \partial S K \]

\[ \text{shift boundary} \ (3) \]

= Unique measure of max entropy for \( F \)

\[ T^* \mu_f = \mu_f \quad \text{on} \ \hat{C} \]

\[ L(f, \mu_f) = \int_C \log \| DF \| \, d\mu_f \left( \frac{\log \lambda}{\dim \mu_f} \right) \]

= \int_C \log | \det DF | \, d\mu_f - \log \lambda

= \int_C \frac{\log \| DF \| \, d\mu_f}{\dim \mu_f}

L(F)

Have formula as before

\[ L(F) = \sum G_f(s_j) + \log \left( \cap \mu F_s \right) \]

\( s_j \) points chosen according to normalization.
Cap \( K \) = \text{Re}(p, q)\) \* \* = \( -\frac{1}{d(d-1)} \)

**Bifurcation Locus** bif For family \( \{ f_\lambda \} \) of rational functions,

one says that \( \lambda_0 \) is stable if \( \lambda \to J(f_\lambda) \) is

\[ \Rightarrow \{ f_{\lambda_0} \} \to J_{\lambda_0} \Rightarrow f_{\lambda_0}|J_{\lambda_0} \to f_{\lambda}|J_{\lambda} \text{ for } \lambda \text{ in neighborhood of } \lambda_0 \]

Ex. 12 \( \alpha < 5 \) bif = \( \partial M \) - shell but not.

Potential theory in \( \mathbb{C}^N \) generalizations of capacity transfinite diameter.

Def of transfinite diam: \( n = 1 \) \( K \subset \mathbb{C} \)

\[ d_\infty(K) = \lim_{n \to \infty} \left( \sup_{\xi_j \in K} \left| \frac{1}{\sum_{1 \leq j \leq n} |\xi_j|} \right| \right) \]

\[ = \text{Cap} \ K \text{.} \]

Fekete - Vandersilde det

\[ \begin{vmatrix} 1 & 3^{-1} \\ 3 & 3^{-1} \\ \vdots & \vdots \end{vmatrix} \]

\( K \subset \mathbb{C}^N \) list the monomials in \( N \) var. of deg \( \leq n \)

\[ e_1, \ldots, e_{\text{min}(n)} \]

\[ d_\infty(K) = \lim_{n \to \infty} \left( \sup_{\xi_j \in K} \left| \frac{1}{\det \{ e_i(\xi_j) \}^{1/\text{dim}(n)} } \right| \right) \]

Fekete-Leja - transfinite diam

Zaharji 1975
Thm. If $F: \mathbb{C}^n \to \mathbb{C}^n$ is a regular polynomial map of degree $d$, \( \text{Res} \, F_d \neq 0 \),

$$d_{\infty}(F^{-1} K) = \left| \text{Res} \, F_d \right| d_{\infty}(K)^{1/d}$$

In 1 variable: formula classical 1930's Fekete $f: \mathbb{C} \to \mathbb{C}$ poly $f(z) = a_d z^d + \ldots + a_0$

$$\text{cap}(f^{-1} K) = \left| a_d \right| \left( \text{cap} K \right)^{1/d}$$

Idea of pf: show 1) $d_{\infty}(F^{-1} K) = d_{\infty}(K)^{1/d}$ independent of $K$.

- arithmetic
- proof

use arithmetic + intersection theory + dynamics

2) choose $K = K_F$, $F^{-1} K_F = K_F$

3) $d_{\infty}(K_F) = ? = \left| \text{Res} \, F_d \right|$