Morse theory I

$\boxed{\begin{array}{c}\text{4 lectures}\\ \text{21 pages notes}\end{array}}$

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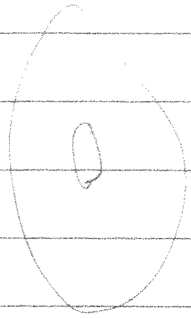
M^n C^∞ closed.

$f: M \to \mathbb{R}$ Morse all crit pts nondeg.

Classically



$t \in \mathbb{R}$ $M^t = \{x \in M \mid f(x) \leq t\}$.

t_0 crit value. $a \in \text{Crit}(f)$

$f(a) = t_0$ only one.

$$\text{index}(a) = \lambda(a) = i$$

$$M^{t_0+\varepsilon} \simeq M^{t_0-\varepsilon} \cup_\alpha D^i \times D^{n-i}$$

$$\alpha: \underset{\substack{\text{S1}\\ S^{i-1}}}{\partial D^i} \times D^{n-i} \to M^{t_0-\varepsilon}$$

$$M^{t_0+\varepsilon} \simeq M^{t_0-\varepsilon} \cup D^i$$

$M \simeq$ CW cpx with 1 cell of dim i for each crit pt.

Cellular chain cpx C^f_*

$$\to \cdots \to C^f_q \to C^f_{q-1} \to \cdots$$

C^f_q = free ab gp gen by crit pts of index q.

Conseq: $\underset{\substack{\uparrow\\ \text{alg. top.}}}{\text{rank}(H_q(M))} \leq \underset{\text{geom/analytic.}}{\underbrace{\# \text{ crit pts of index } q}}$.

Given M a Riemannian metric.

$x \in M \qquad \langle \, , \, \rangle : T_xM \times T_xM \to \mathbb{R}$

Gradient vector field $\qquad \nabla f(x) \in T_x(M)$.

$$\langle \nabla f(x), v \rangle = df(x)(v).$$

Existence + uniqueness of solutions to ODE's.

For every $x \in M \quad \exists! \ \gamma_x : \mathbb{R} \to M$ s.t.

1) $\dfrac{d\gamma_x}{dt}(t) = -\nabla f(x) \qquad$ <u>flow equation</u>.

2) $\gamma_x(0) = x$

γ_x "flow line"

$$\lim_{t \to -\infty} \gamma_x(t) \qquad \lim_{t \to +\infty} \gamma_x(t) \quad \text{crit pts.}$$

$$\begin{array}{cc} \| & \| \\ s(\gamma_x) & e(\gamma_x) \\ \text{"start"} & \text{"end"} \end{array}$$

a crit pt. $\quad \lambda(a) = i$

Unstable manifold of $a \qquad W^u(a) = \{x \in M \mid s(\gamma_x) = a\}$.

Stable $\qquad W^s(a) = \{x \in M \mid e(\gamma_x) = a\}$

<u>Thm</u> $\quad W^u(a) \cong D^{\lambda(a)}, \qquad W^s(a) \cong D^{n-\lambda(a)}$

Morse

$T_a W^u \subseteq T_a M$

$= \text{negative eigenspace of } \mathrm{Hess}(f(a))$

$f, \langle\,\rangle$ is Morse-Smale
if $\forall a, b \in \mathrm{Crit}(f)$

$W^u(a) \pitchfork W^s(b)$ transversally

<u>Thm</u> (Smale) Morse-Smale is generic in both f and $\langle,\rangle$

$f, \langle\,\rangle$ morse smale.

$W^u(a) \cap W^s(b) = W(a,b)$

$= \{x \in M \mid s(\gamma_x) = a \quad e(\gamma_x) = b\}$

is a manifold of dim $\lambda(a) - \lambda(b) = \lambda(a,b)$ "relative index"

$W(a,b)$ has a free action of $\mathbb{R}$
given by

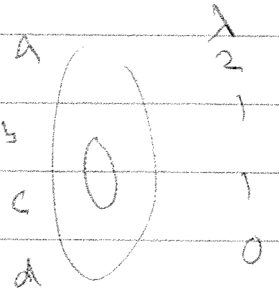
$s * x = \gamma_x(s)$

$\mathcal{M}(a,b) = W(a,b)/\mathbb{R}$ is again a manifold.

= space of gradient flow lines connecting a to b.

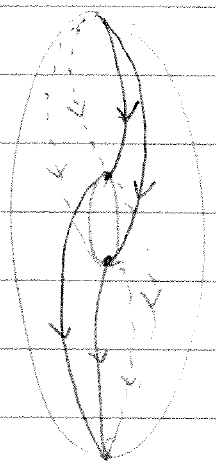
of dim $\lambda(a) - \lambda(b) - 1$

<u>Note:</u>



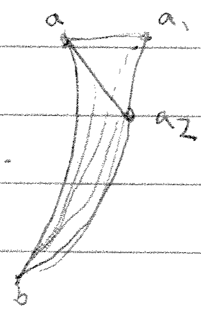
$\lambda(b) = \lambda(c) = 1$ so
$\mathcal{M}(b,c)$ should be -1 dim
ie $\emptyset$.
So std morse function is
<u>not</u> morse-Smale.

Tilt:



i.e. 8 discrete flow lines.

Compactness of $\mathcal{M}(a,b)$



Could have broken flow lines.

open 2-simplex

Smale: $\overline{W(a,b)} \subset M$

← closure.

$$\overline{W(a,b)} = \bigcup_{a \geq \alpha \geq \beta \geq b} W(\alpha,\beta)$$

where $\alpha \geq \beta$ if $\mathcal{M}(\alpha,\beta) \neq \emptyset$.

Compactify $\mathcal{M}(a,b)$

$$\overline{\mathcal{M}(a,b)} = \bigcup_{a \geq a_1 \geq \dots \geq a_k \geq b} \mathcal{M}(a,a_1) \times \mathcal{M}(a_1,a_2) \times \dots \times \mathcal{M}(a_k$$

How to topologize?

$\gamma \in \mathcal{M}(a,b)$ $\quad \mathbb{R} \xrightarrow{\gamma} M \xrightarrow{f} \mathbb{R}$

$f \circ \gamma : \mathbb{R} \xrightarrow{\cong} (f(b), f(a))$

Reparametrize.

$$\tilde{\gamma} = \gamma \circ (f \circ \gamma)^{-1} : (f(b), f(a)) \longrightarrow M$$

$$\frac{d\tilde{\gamma}(t)}{dt} + \frac{\nabla f(\gamma(t))}{\|\nabla f(\gamma(t))\|^2} = 0$$

Extend: $\tilde{\gamma} : [f(b), f(a)] \to M$

$$f(b) \longmapsto b$$
$$f(a) \longmapsto a$$

Define: $\overline{\mathcal{M}}(a,b) = \{ w : [f(b), f(a)] \to M$ cont piecewise smooth satisfy
$$\frac{dw}{dt} + \frac{\nabla f(w(t))}{\|\nabla f(w(t))\|^2} = 0$$
$$w(f(b)) = b$$
$$w(f(a)) = a \}.$$

$\mathcal{M}(a,b) \subseteq \overline{\mathcal{M}}(a,b)$ as an open dense subset

$\overline{\mathcal{M}}(a,b)$ is compact.

Also note $\overline{\mathcal{M}(a,b)} \times \overline{\mathcal{M}(b,c)} \subseteq \overline{\mathcal{M}(a,c)}$

Flow category $\mathcal{C}_f$ obj crit points of f

$$\mathrm{Mor}(a,b) = \overline{\mathcal{M}(a,b)}$$

Morse theory II

$f: M \to \mathbb{R}$ Morse $\quad \langle , \rangle$ metric on TM

$$\overline{\mathcal{M}}(a,b) = \bigcup_{a>a_1>\dots a_k>b} \mathcal{M}(a,a_1)\times \dots \times \mathcal{M}(a_k,b)$$

$$\mathcal{M}(a,b) \subseteq \Omega^f_{b,a}M = \left\{\alpha:[f(b),f(a)]\to M \;\middle|\; \begin{array}{l}\alpha(f(b))=b\\ \alpha(f(a))=a\end{array}\right\}$$

$$\simeq \Omega M = \{\alpha: S^1\to M \mid \alpha(0)=x_0\in M\}.$$

$\dim \mathcal{M}(a,b) = \lambda(a)-\lambda(b)-1.$

$$\begin{tikzcd}
\overline{\mathcal{M}}(a,b)\times\overline{\mathcal{M}}(b,c) \arrow[r, hook] \arrow[d] & \overline{\mathcal{M}}(a,c) \\
\Omega_{b,a}M \times \Omega_{c,b}M \arrow[r] & \Omega_{c,a}M
\end{tikzcd}$$

embeds as part of ∂, codim 1.

$f, \langle -,- \rangle$ Morse-Smale.

$\overline{\mathcal{M}}(a,b)$ is a C^∞ framed manifold with corners.

Flow category

$\mathcal{C}_f$ obj = crit points, topological category

$\mathrm{Mor}(a,b) = \overline{\mathcal{M}}(a,b)$

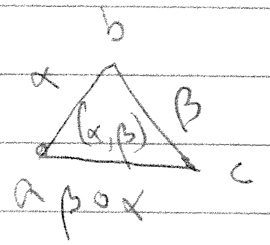
Thm (C-Jones Segal) The geometric realization

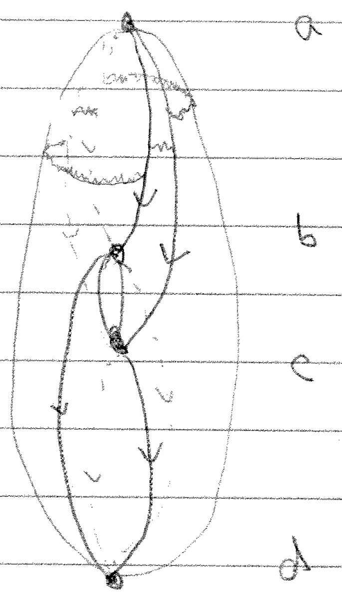
$|\mathcal{C}_f| \; (B\mathcal{C}_f) \underset{\text{homeo}}{\cong} M$ if f is Morse-Smale.

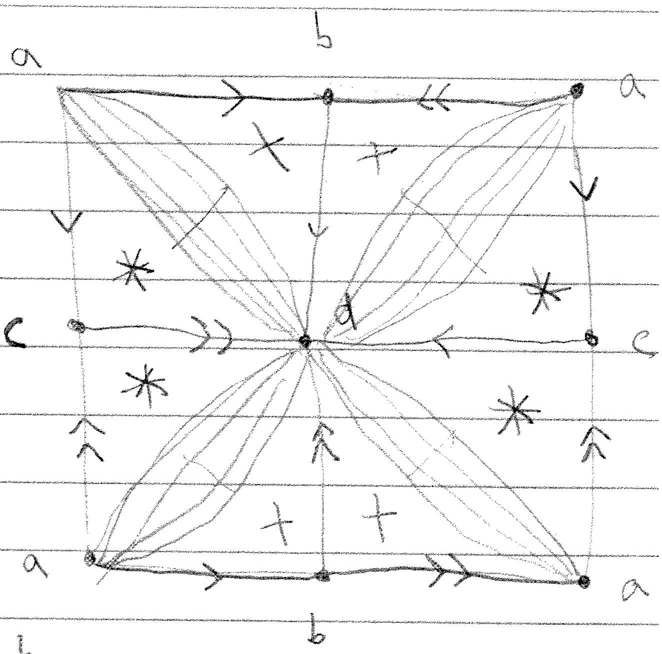
$\simeq M$ if f is Morse.

$$|\mathcal{C}| = \bigcup_k \Delta^k \times Mor(a_1, a_2) \times Mor(a_2, a_3) \cdots \times Mor(a_k, a_{k+1}) \Big/ \sim$$

$a_1 \quad a_{k+1}$







$m(a,b) = 2$ pts

$m(a,c) = 2$ pts

$m(b,d) = 2$ pts.

$m(c,d) = 2$ pts.

$m(a,d) = \coprod\,]0,1[$

$\overline{m}(a,d) = \coprod_4 [0,1]$, marked by $\times$

need Δ^2 for every pair in $M(a,b) \times M(b,d)$ 4 pts

$m(a,c) \times m(c,d)$ 4 pts

(marked by $*$ in discs).

Given any X, define path category $\mathcal{P}(X)$

$$\text{Obj} = \{x \in X\}$$
$$\text{Mor}(x,y) = \{\alpha : [0,r] \to X \mid \alpha(0) = x \quad \alpha(r) = y\}.$$

<u>Exercise:</u> $|\mathcal{P}(X)| \simeq X$.

$$\mathcal{C}_f \xhookrightarrow{i} \mathcal{P}(M)$$

Reformulation of part 2 of thm

$|i| : |\mathcal{C}_f| \to |\mathcal{P}(M)|$ is a homotopy eq.

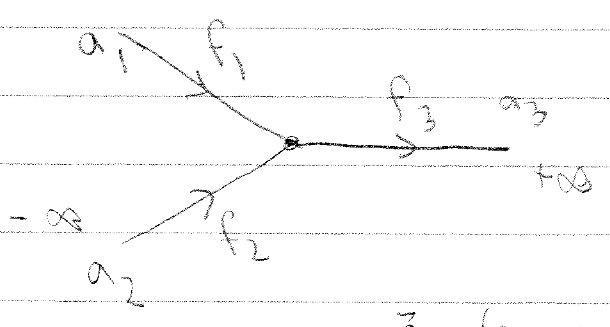
Consider diag $\underset{a}{\bullet} \xrightarrow{\quad f \quad} \underset{v_0}{\bullet}$ $\quad f : M \to \mathbb{R}$ Morse, $a \in \text{Crit}(f)$.

$$\mathcal{M}(R, M, a) = \left\{ \gamma : R \to M \;\middle|\; \begin{array}{l} \frac{d\gamma}{dt} + \nabla f = 0 \\ \lim_{t \to -\infty} \gamma(t) = a \end{array} \right\}$$

$R = (-\infty, 0]$

<u>Obs:</u> $\mathcal{M}(R, M, a) \xrightarrow{\cong} W^u(a)$ by existence + uniqueness
$\gamma \longmapsto \gamma(v_0)$

dim $= \lambda(a)$.



$f_i : M \to \mathbb{R}$ Morse
$a_i \in \text{Crit}(f_i)$

$\vec{a} = (a_1, a_2, a_3)$

$\mathcal{M}(\Gamma, M, \vec{a}) = \{ \gamma : \Gamma \to M$ cont, smooth on open edges

$\frac{d\gamma_i}{dt} + \nabla f_i = 0$

$\lim_{t \to \pm\infty} \gamma_i(t) = a_i$

$$\mathcal{M}(\Gamma, M, \vec{a}) \longrightarrow M$$

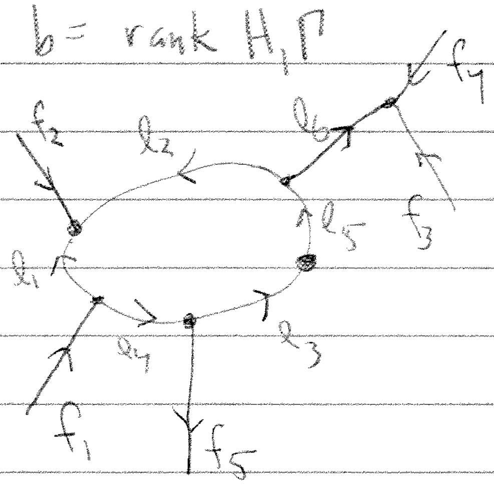
$$\gamma \longmapsto \gamma(v_0)$$

diffeo onto $W^u_{f_1}(a_1) \cap W^u_{f_2}(a_2) \cap W^s_{f_3}(a_3)$

$D^{\lambda(a_1)} \cap D^{\lambda(a_2)} \cap D^{n-\lambda(a_3)}$

If intersect transversally manifold of dim $= \lambda(a_1) + \lambda(a_2) - \lambda(a_3) - n$

Let $\Gamma_{b,p+q}$ ~~a~~ ft. graph $b = \text{rank } H_1\Gamma$



$p+q$ non-compact edges (external)

edges oriented

p incoming external
q outgoing

M Morse labeling.

Label each edge by function $f_i : M \to \mathbb{R}$ distinct.

Label each internal edge by number $l_i \in$

$\vec{a} = (a_1, \ldots, a_{p+q})$ $\qquad a_i \in \mathrm{Crit}(f_i)$ external edges

<u>Define:</u> $\mathcal{M}^\sigma(\Gamma, M, \vec{a})$ space of graph flows.

<u>Thm</u> (Betz, Fukaya) For generic choice of σ, $\langle\ \rangle$ on

$\mathcal{M}^\sigma(\Gamma, M, \vec{a})$ is smooth of dim

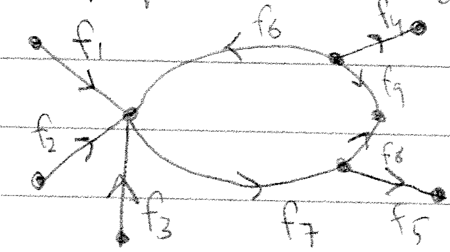
$$= \underbrace{\sum_{i=1}^{p} \lambda(a_i)}_{\text{incoming}} - \underbrace{\sum_{p=1}^{p+q} \lambda(a_j)}_{\text{out}}$$

$$- np + n(\chi(\Gamma))$$

Morse Theory III

$\Gamma_{b,p+q}$ $\quad b = \operatorname{rank} H_1(\Gamma)$

$p+q$ "leaves" (univalent vertex



$b = 1$

$p =$ incoming leaves $= 3$

$q =$ out $=$ $= 2$

M-Morse labeling: • Label edges by distinct functions $f_i : M \to \mathbb{R}$

• internal edges by $l_i > 0$

$E_1, \ldots, E_p$ incoming external

$E_{p+1}, \ldots, E_{p+q}$ outgoing

Define parametrizations on internal edges.

$$[0, l_i] \xrightarrow[\varphi_i]{} E_i$$

External $\mathring{E}_j = E_j$ - leaf.

$$(-\infty, 0] \to \mathring{E}_j \quad \text{incoming}$$
$$[0, +\infty] \to \mathring{E}_j \quad \text{out.}$$

Let $\vec{a} = (a_1, \ldots, a_{p+q})$ $\quad a_i \in \operatorname{Crit}(f_i)$

$\mathcal{M}^\sigma(\Gamma, M, \vec{a})$ = Space of graph flows wrt σ boundary conditions $\vec{a}$

$$= \left\{ \gamma : \Gamma \to M,\ \gamma_i : E_i \to M \quad \frac{d\gamma_i}{dt} + \nabla f_i = 0 \quad \forall i \right.$$

$$\lim_{t \to -\infty} \gamma_i(t) = a_i \quad i = 1, \ldots, p$$

$$\lim_{t \to +\infty} \gamma_j(t) = a_j \quad j = p+1, \ldots, p+q$$

<u>Thm</u> (M. Betz) For generic choice of σ,

1) $\mathcal{M}^\sigma(\Gamma, M, \vec{a})$ is smooth of dim

$$\sum_{i=1}^{p} \lambda(a_i) - \sum_{j=p+1}^{q} \lambda(a_j) - n \cdot p + n(\chi(\Gamma)$$

2) An orientation of M induces an orientation of $\mathcal{M}(\Gamma, M, \vec{a})$

3) Not compact, but can compactify by allowing "broken legs".

<u>Invariants:</u>

$$q(\Gamma) = \sum_{\dim \mathcal{M}(\Gamma, M, \vec{a}) = 0} \# \mathcal{M}(\Gamma, M, \vec{a}) \, [\vec{a}]$$

$$\in \bigotimes_{i=1}^{p} C^*_{f_i}(M) \otimes \bigotimes_{j=p+1}^{p+q} C_*^{f_j}(M)$$

<u>Prop:</u> $d(q(\Gamma)) = 0$. i.e. $q(M)$ cycle.

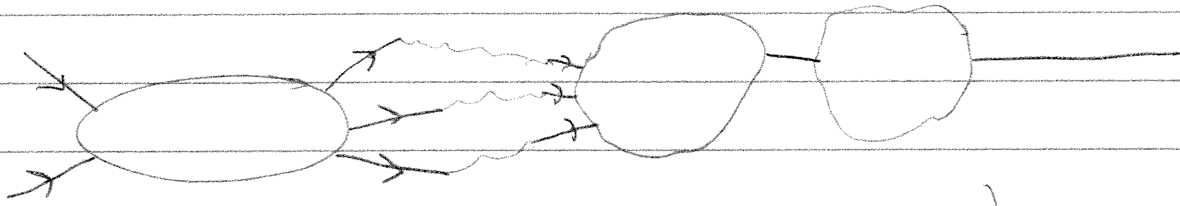
i.e. $q(\Gamma) : \bigotimes_{i=1}^{p} C_*^{f_i} \longrightarrow \bigotimes_{j=p+1}^{p+q} C_*^{f_j}$ is a chain map.

<u>Thm</u> (Betz, Fukaya) 1) $q(\Gamma) : \bigotimes_p H_* M \longrightarrow \bigotimes_q H_* M$

is independent of choice of σ & metric.

only depends on (oriented homotopy type of Γ, rel to leaves)

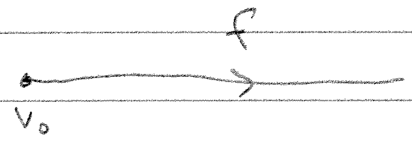
2) Glueing $\Gamma_{b,p+q} \times \Gamma_{b',q+r} \longrightarrow \Gamma_{b+b'+q-1,\,p+r}$



$$q(\Gamma_1 \# \Gamma_2) = q(\Gamma_2) \circ q(\Gamma_2)$$

"TQFT" $\quad q :$ Finite graphs $\to \mathrm{End}_{H_*M}$

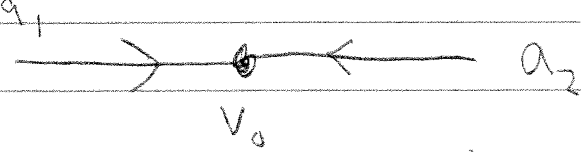
morphism of PROPS.

Example: 1)  Ray $= R$

$$\mathcal{M}(\Gamma, M; a) = W^s(a) \quad \dim \; n - \lambda(a)$$
$= 0$ iff a is max

so $\quad q(\Gamma, M) = \sum_{a \max} [a] \in H_* M$

$= [M]$

2) 

$$\dim \mathcal{M}(\Gamma, M; \vec{a}) = \lambda(a_1) + \lambda(a_2) - n$$

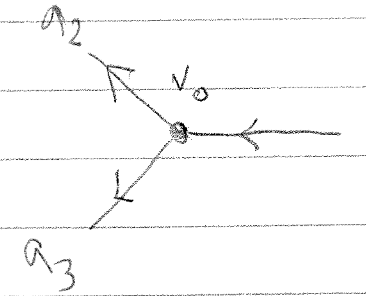
$$\| $$

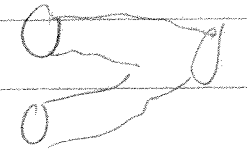
$$W^u_f(a_1) \cap W^u_{f_2}(a_2)$$

$$q(\Gamma, M) \in \bigoplus_q \mathrm{Hom}(H_q M \otimes H_{n-q} M ; k)$$

intersection product

(Poincaré duality)

3) 

(poor man's (thin man's)
pair of pants )

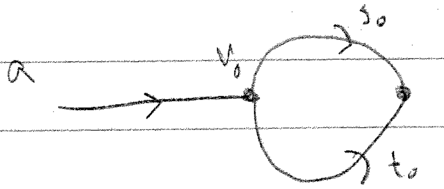
$$\dim \mathfrak{M}(\Gamma, M; \vec{a}) = \lambda(a_1) - \lambda(a_2) - \lambda(a_3)$$

$$= 0 \iff \lambda(a_1) = \lambda(a_2) + \lambda(a_3)$$

$$q(\Gamma, M) \in \bigoplus_{p \geq r} \mathrm{Hom}(H_p, H_r \otimes H_{p-r})$$

$$\Delta_* \; H_* M \to H_* M \otimes H_* M$$

q^* = cup product.



$$\dim \mathfrak{M}(\Gamma, M, a) = \lambda(a) - n$$

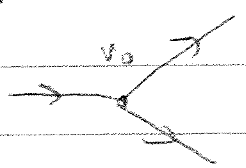
(poor man's 

$= 0$ iff a max.

$$q(\Gamma, M) \in \mathrm{Hom}(H_n(M), k) = H^n(M; k) \cong k$$

$$q(\Gamma, M) = e(M) = \chi(M) \cdot \text{generator}$$

$\tilde{\Gamma} =$



$$\mathfrak{M}(\tilde{\Gamma}, M) \cong M$$
$$\gamma \longmapsto \gamma(v_0)$$

$$M \cong \mathfrak{M}(\tilde{\Gamma}, M) \xrightarrow{\Delta_{s,t}} M \times M$$
$$\gamma \longmapsto (\gamma(s), \gamma(t))$$

$$m(\Gamma, M) = \Delta^{-1}_{s_0, t_0}(\Delta(M))$$

$$= \Delta_{s_0, t_0}(M) \cap \Delta(M)$$

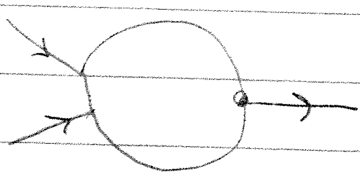
dim 0

$\#$ self intersections of diagonal $= \chi(M)$

(for more detail
see paper by Betz + Cohen in Turkish J Math
Betz + Cohen + Norbury on arXiv.)

Morse Theory IV

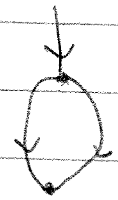
(joint w. P. Norbury), "Morse Field Theory"

$\Gamma_{b,p+q}$ 

Morphisms $\Gamma_1 \xrightarrow{\varphi} \Gamma_2$ maps of graphs (cellular)

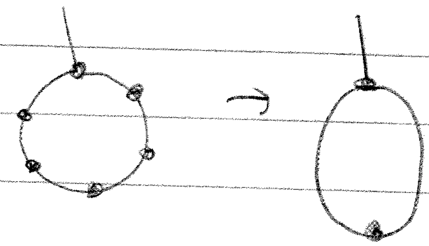
preserve incoming, out leaves

- $\varphi^{-1}(\text{edge}) = \text{edge}$
- $\varphi^{-1}(\text{vertex}) = \text{tree}$ (contractible)

Fix $\Gamma_0 =$ 

$\mathcal{C}_{\Gamma_0}$ = category of graphs over Γ_0

Obj = morphisms to Γ_0



$\Gamma \longrightarrow \Gamma_0$

Mor:
$$\begin{tikzcd} \Gamma_1 \arrow[rr, "\psi"] \arrow[dr] & & \Gamma_2 \arrow[dl] \\ & \Gamma_0 & \end{tikzcd}$$

$|\mathcal{C}_{\Gamma_0}|$ contractible since $\mathcal{C}_{\Gamma_0}$ has terminal obj.

$|\mathcal{C}_{\Gamma_0}|$ = Space of "metrics on subdivisions of Γ_0"

$|\mathcal{C}_{\Gamma_0}| = \bigcup \Delta^k \times \text{Mor} \times \text{Mor} \times \qquad \times \text{Mor}$

$|\mathcal{E}_{\Gamma_0}|$ has free action of $\mathrm{Aut}(\Gamma_0)$

So we have $\overset{\text{univ}}{\vee}$ covering

$$\mathrm{Aut}(\Gamma_0) \to |\mathcal{E}_{\Gamma_0}| \to |\mathcal{E}_{\Gamma_0}| / \mathrm{Aut}(\Gamma_0) \simeq B\mathrm{Aut}(\Gamma_0)$$

$\mathcal{E}_{\Gamma_0} \int S_{\Gamma_0}$ $\qquad$ S_{Γ_0} = spaces of labellings of a graph by distinct functions

grothendieck construction

Since $|\mathcal{E}_{\Gamma_0}|$ and $|S_{\Gamma_0}|$ contractible $|\mathcal{E}_{\Gamma_0} \int S_{\Gamma_0}|$ is as well

essentially this is contractible since $F_k(C^\infty(M;\mathbb{R}))$ is contractible

$\mathcal{E}_{\Gamma_0} \int S_{\Gamma_0}$ = {metric on a subdivision of Γ_0, together with a ~~Morse~~ labeling}

Let $\mathcal{J}_{\Gamma_0} = |\mathcal{E}_{\Gamma_0} \int S_{\Gamma_0}|$

$$\mathcal{M}(\Gamma_0) = \mathcal{J}_{\Gamma_0} / \mathrm{Aut}(\Gamma_0) \simeq B\mathrm{Aut}\,\Gamma_0$$

$$\widetilde{\mathcal{M}}(\Gamma_0, M) = \left\{ (\Gamma, \sigma, \varphi),\ (\Gamma, \sigma) \in \mathcal{J}_{\Gamma_0}\ \ \underset{\substack{\text{graph flow} \\ \text{w.r.t. } \sigma}}{\varphi : \Gamma \to M} \right\}$$

$$\mathcal{M}(\Gamma_0, M) = \widehat{\mathcal{M}}(\Gamma_0, M) \Big/ Aut(\Gamma_0) = \text{Space of } \Gamma_0\text{-flows}$$

$$\begin{tikzcd} \widehat{\mathcal{M}}(\Gamma_0, M) \arrow[r, "\rho_{in}"'] \arrow[d, "\rho_{out}"] & M^p \times \mathcal{J}_{\Gamma_0} \\ M^q & \end{tikzcd}$$

restrict to incoming leaves

outgoing leaves

Use Pontryagin-Thom (more sophisticated way of doing intersection theory

It is $Aut(\Gamma_0)$-equivariant

$$q_\Gamma = \rho_{out*} \circ (\rho_{in})_! : H_*(\mathcal{M}(\Gamma_0)) \otimes H_*^{\Sigma_p}(M^p) \to H_*^{\Sigma_q}(M^q)$$

$$H_*(\mathcal{M}(\Gamma_0)) = H_*(BAut(\Gamma_0))$$

Interlude:

Easier example of Pontryagin-Thom and $(\ )_!$ const.: Given $P \overset{e}{\hookrightarrow} M$ $\alpha \in C_*(M)$ $e_!(\alpha) =$ "$\alpha \cap P$"

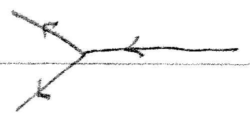
find tubular neighborhood

$$P \subset N(P) \subset M \qquad \text{Get collapse map}$$

$$C_*(M) \longrightarrow C_*(M / M - N(P))$$

$$= C_*\left(Thom\left(\begin{matrix} N P \\ \downarrow \\ P \end{matrix}\right)\right) \underset{\text{Thom iso}}{\cong} C_{*-k}(P)$$

$e_!$

Example:  $= \Gamma_0$ $\quad Aut(\Gamma) = \mathbb{Z}/2 \quad BAut(\Gamma) = \mathbb{RP}^\infty$

Take $\mathbb{Z}/2$-coeff $\quad q_\Gamma : H_*(\mathbb{RP}^\infty) \otimes H_* M \longrightarrow H_*^{\mathbb{Z}/2}(M \times M)$

Steenrod's equivariant diagonal

$$M \xrightarrow{\Delta} M \times M \qquad \mathbb{Z}/2\text{-equiv.}$$

Apply $\quad S^\infty \times_{\mathbb{Z}/2} (-)$

$\quad\quad\quad\quad\quad\quad \| $

$\quad\quad\quad\quad\quad\quad E\mathbb{Z}/2$

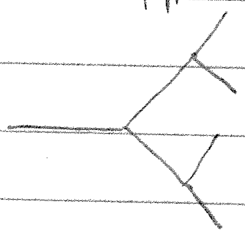
Recall const of Steenrod Sq:

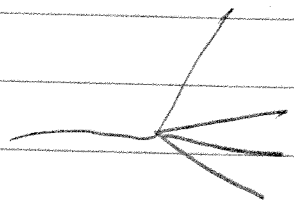
$$B\mathbb{Z}/2 \times M \xrightarrow{\Delta} E\mathbb{Z}/2 \times_{\mathbb{Z}/2} (M \times M)$$

$$\Delta^* = q_p^* : H^*_{\mathbb{Z}/2}(M \times M) \longrightarrow H^*(M) \otimes H^*(\mathbb{R}P^\infty)$$

$$\alpha \in H^q(M) \qquad \alpha \otimes \alpha \xrightarrow{q_p^*} \sum Sq^{q-i}\alpha \otimes a^i$$

Iterate q_p $\quad\quad\quad\quad$ same invariant as



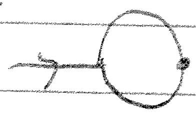


$\mathrm{Aut} = \mathbb{Z}/2 \int \mathbb{Z}/2 \quad\quad\quad\quad \mathrm{Aut} = \Sigma_4$

$\quad\quad = D_8$

From this we get Adem relations + Cartan formula.

~~Exp.~~ Consider



$$\mathrm{Aut}\,\Gamma = \mathbb{Z}/2$$

$$q_p : H_*\mathbb{R}P^\infty \otimes H_*M \longrightarrow \mathbb{Z}/2$$

$$q_p \in H^*\mathbb{R}P^\infty \otimes H^*(M)$$

q_p = equivariant Euler class of M.

$\mathbb{Z}/2$ acts on $T_*M \quad v \mapsto -v$

get bundle $E\mathbb{Z}/2 \times_{\mathbb{Z}/2} TM$

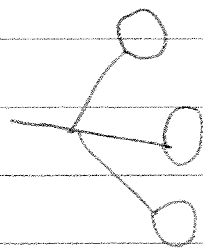
$$\begin{array}{c} E\mathbb{Z}/2 \times_{\mathbb{Z}/2} TM \\ \downarrow \\ \mathbb{R}P^\infty \times M \end{array}$$

$$e^{\mathbb{Z}/2}(M) := e(E\mathbb{Z}/2 \times_{\mathbb{Z}/2} M) \in H^n(\mathbb{R}P^\infty \times M)$$

$$= \sum a^i \otimes w_{n-i}(M)$$

Stiefel-Whitney class

Looking at



gives you Cartan formula for Stiefel-Whitney classes

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References for Lecture Series **Morse theory, graphs, and strings**

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