Homotopy colimits in model categories

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1 Introduction

In [1], Dwyer and Spalinski construct the so-called homotopy pushout functor, motivated by the following observation. In the category **Top** of topological spaces, one can construct the n-dimensional sphere S^n by glueing together two n-disks D^n along their boundaries S^{n-1}, i.e. by the pushout of

$$D^n \xleftarrow{\ i\ } S^{n-1} \xrightarrow{\ i\ } D^n ,$$

where i denotes the inclusion. Let $*$ be the one point space. Observe, that one has a commutative diagram

$$\begin{array}{ccccc} D^n & \xleftarrow{\ i\ } & S^{n-1} & \xrightarrow{\ i\ } & D^n , \\ \downarrow & & \downarrow{\scriptstyle \mathrm{id}_{S^{n-1}}} & & \downarrow \\ * & \longleftarrow & S^{n-1} & \longrightarrow & * \end{array}$$

where all vertical maps are homotopy equivalences, but the pushout of the bottom row is the one-point space $*$ and therefore not homotopy equivalent to S^n. One probably prefers the homotopy type of S^n. Having this idea of calculating the prefered homotopy type in mind, they equip the functor category $\mathbf{C}^{\mathbf{D}}$, where $\mathbf{C}$ is a model category and $\mathbf{D} = a \leftarrow b \rightarrow c$ the category consisting out of three objects a, b, c and two non-identity morphisms as indicated, with a suitable model category structure. This enables them to construct out of the pushout functor $\mathrm{colim} : \mathbf{C}^{\mathbf{D}} \rightarrow \mathbf{C}$ its so-called total left derived functor $\mathbf{L}\mathrm{colim} : \mathrm{Ho}(\mathbf{C}^{\mathbf{D}}) \rightarrow \mathrm{Ho}(\mathbf{C})$ between the corresponding homotopy categories, which defines the homotopy pushout functor.

Dwyer and Spalinski further indicate how to generalize this construction to define the so-called homotopy colimit functor as the total left derived functor for the colimit functor $\mathrm{colim} : \mathbf{C}^{\mathbf{D}} \rightarrow \mathbf{C}$ in the case, where $\mathbf{D}$ is a so-called very small category. The goal of this paper is to give a proof of this generalization, since in [1] it is omitted to the reader. The main work lies in proving our Theorem 2, which equips $\mathbf{C}^{\mathbf{D}}$ with the suitable model category structure. For the existence of the total left derived functor for colim, we will use a result from [1].

The paper contains four sections. In section 2 and 3 we recall some definitions and results related to colimits and model categories, respectively. The introduced terminology will be used in section 4, where we construct the homotopy colimit functor.

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2 Colimits

In this section let $\mathbf{C}$ be a category, let $\mathbf{D}$ be a small category and $F : \mathbf{D} \to \mathbf{C}$ a functor. Mainly to fix notations, we recall some definitions and results related to colimits.

Definition 2.1. The *functor category* $\mathbf{C}^{\mathbf{D}}$, also called the *category of diagrams in $\mathbf{C}$ with the shape of $\mathbf{D}$*, is the category, where the objects are functors $\mathbf{D} \to \mathbf{C}$ and the morphisms are natural transformations.

Example 2.2. If $\mathbf{D}$ is the category $a \leftarrow b \to c$ with three objects a, b, c and two non-identity morphisms $a \leftarrow b$, $b \to c$, then an object X of $\mathbf{C}^{a \leftarrow b \to c}$ is just a diagram $X(a) \leftarrow X(b) \to X(c)$ in $\mathbf{C}$ and a morphism from X to Y in $\mathbf{C}^{a \leftarrow b \to c}$ is given by a triple (s_a, s_b, s_c) of morphisms in $\mathbf{C}$ making

$$
\begin{array}{ccccc}
X(a) & \longleftarrow & X(b) & \longrightarrow & X(c) \\
\downarrow{\scriptstyle s_a} & & \downarrow{\scriptstyle s_b} & & \downarrow{\scriptstyle s_c} \\
Y(a) & \longleftarrow & Y(b) & \longrightarrow & Y(c)
\end{array}
$$

commute.

Example 2.3. If $\mathbf{D}$ is the category $a \to b$ with two objects a, b and one non-identity morphism $a \to b$, then an object of $\mathbf{C}^{a \to b}$ is a morphism $f : X(a) \to X(b)$ in $\mathbf{C}$ and a morphism from $f : X(a) \to X(b)$ to $g : Y(a) \to Y(b)$ in $\mathbf{C}^{a \to b}$ is just a pair of morphisms (s_a, s_b) in $\mathbf{C}$ making

$$
\begin{array}{ccc}
X(a) & \xrightarrow{f} & X(b) \\
\downarrow{\scriptstyle s_a} & & \downarrow{\scriptstyle s_b} \\
Y(a) & \xrightarrow{g} & Y(b)
\end{array}
$$

commute. We call $\mathbf{C}^{a \to b}$ the *category of morphisms in $\mathbf{C}$* and denote it by $\mathbf{Mor}(\mathbf{C})$.

Example 2.4. If $\mathbf{D}$ is the category $\mathbf{1}$ consisting only out of one object 1 and one morphism, then $\mathbf{C}^{\mathbf{D}}$ is isomorphic to $\mathbf{C}$ via the functor given by $X \mapsto X(1)$ on objects and $f \mapsto f_1$ on morphisms.

Definition 2.5. The *constant diagram functor* $\Delta = \Delta_{\mathbf{D}} : \mathbf{C} \to \mathbf{C}^{\mathbf{D}}$ is the functor, which sends an object C of $\mathbf{C}$ to the functor $\Delta(C)$ given by

$$d \mapsto C \text{ on objects and } g \mapsto \mathrm{id}_C \text{ on morphisms,}$$

and which sends a morphism f of $\mathbf{C}$ to the natural transformation $\Delta(f)$ given by $\Delta(f)_d = f$ in an object d of $\mathbf{D}$.

Note that a functor $j : \mathbf{D}' \to \mathbf{D}$ from a small category $\mathbf{D}'$ to $\mathbf{D}$ induces a functor $(\cdot)|_{\mathbf{D}',j} = (\cdot)|_{\mathbf{D}'} = (\cdot)|_j : \mathbf{C}^{\mathbf{D}} \to \mathbf{C}^{\mathbf{D}'}$, which sends an object X of $\mathbf{C}^{\mathbf{D}}$ to $X \circ j$ and a morphism $f : X \to Y$ in $\mathbf{C}^{\mathbf{D}}$ to the natural transformation $f|_{\mathbf{D}'} : X|_{\mathbf{D}'} \to Y|_{\mathbf{D}'}$ given by $f_{j(d')}$ in an object d' of $\mathbf{D}'$. Furthermore, if $j' : \mathbf{D}'' \to \mathbf{D}'$ is a functor from a small category $\mathbf{D}''$ to $\mathbf{D}'$, then

$$(\cdot)|_{j \circ j'} = (\cdot)|_{j'} \circ (\cdot)|_j . \tag{2.1}$$

Example 2.6. The functor $\mathbf{D} \to \mathbf{1}$ induces the functor $(\cdot)|_{\mathbf{1}} : \mathbf{C}^{\mathbf{D}} \to \mathbf{C}^{\mathbf{D}}$, which composed with the isomorphism $\mathbf{C} \cong \mathbf{C}^{\mathbf{1}}$ yields $\Delta_{\mathbf{D}}$. By (2.1), it follows that $\Delta_{\mathbf{D}'} = (\cdot)|_j \Delta_{\mathbf{D}}$ for any functor j from a small category $\mathbf{D}'$ to $\mathbf{D}$.

Definition 2.7. A *colimit* $C = (C, t)$ for $F : \mathbf{D} \to \mathbf{C}$ is an object C of $\mathbf{C}$ together with a natural transformation $t : F \to \Delta(C)$ such that for any object X of $\mathbf{C}$ and any natural transformation $s : F \to \Delta(X)$ there exists a unique morphism $s' : C \to X$ in $\mathbf{C}$ satisfying $\Delta(s')t = s$.

Example 2.8. If $\mathbf{D} = a \leftarrow b \to c$, then a colimit for an object X of $\mathbf{C}^{a \leftarrow b \to c}$ is just a pushout of the diagram $X(a) \leftarrow X(b) \to X(c)$.

Remark 2.9. If a colimit for F exists, we will sometimes speak of the colimit for F for the following reason. If (C, t) and (C', t') are colimits for F, then the unique morphism $h : C \to C'$ in $\mathbf{C}$ such that $\Delta(h)t = t'$ holds, is an isomorphism, which will be called the canonical isomorphism. Given furthermore an object X of $\mathbf{C}$ and a natural transformation $s : F \to \Delta(X)$, let $s'' : C' \to X$ be the unique morphism in $\mathbf{C}$ satisfying $\Delta(s'')t' = s$. Then the unique morphism $s' : C \to X$ satisfying $\Delta(s')t = s$ is given by $s''h$.

Remark 2.10. Assume that for any functor $G : \mathbf{D} \to \mathbf{C}$ the colimit $(\mathrm{colim}(G), t_G)$ exists. Then the chosen colimits yield a functor

$$\mathrm{colim} : \mathbf{C}^{\mathbf{D}} \to \mathbf{C},$$

called *colimit functor*, which maps a morphism $s : G \to G'$ in $\mathbf{C}^{\mathbf{D}}$ to the unique morphism $s' : \mathrm{colim}(G) \to \mathrm{colim}(G')$ in $\mathbf{C}$ such that $\Delta(s')t_G = t_{G'}s$. Furthermore, we have an adjunction $(\mathrm{colim}, \Delta, \alpha)$, where the natural equivalence α is given in an object (G, X) of $(\mathbf{C}^{\mathbf{D}})^{\mathrm{op}} \times \mathbf{C}$ by the bijection

$$\begin{aligned} \alpha = \alpha_{G,X} : \mathrm{Hom}_{\mathbf{C}}(\mathrm{colim}(G), X) &\to \mathrm{Hom}_{\mathbf{C}^{\mathbf{D}}}(G, \Delta(X)), \\ s' &\mapsto \Delta(s')t_G. \end{aligned} \tag{2.2}$$

In particular, any two colimit functors $\mathbf{C}^{\mathbf{D}} \to \mathbf{C}$ are naturally isomorphic and therefore, we will sometimes speak of the colimit functor.

Remark 2.11. Let $\mathbf{D}'$ be a small category such that there exists an isomorphism $J : \mathbf{C}^{\mathbf{D}} \to \mathbf{C}^{\mathbf{D}'}$ with $J \circ \Delta_{\mathbf{D}} = \Delta_{\mathbf{D}'}$. Assume there exists a colimit functor $\operatorname{colim} : \mathbf{C}^{\mathbf{D}'} \to \mathbf{C}$. Then the composition $\operatorname{colim} \circ J : \mathbf{C}^{\mathbf{D}} \to \mathbf{C}$ is a colimit functor. Furthermore, for any object X of $\mathbf{C}$ and any natural transformation $s : F \to \Delta_{\mathbf{D}}(X)$ the induced morphism from the colimit of F to X is given by the morphism $\operatorname{colim}(J(F)) \to X$ induced by $J(s) : J(F) \to \Delta_{\mathbf{D}'}(X)$.

In the definition of a model category we will use both, the notion of colimit and limit.

Definition 2.12. A *limit* $L = (L, t)$ for $F : \mathbf{D} \to \mathbf{C}$ is an object L of $\mathbf{C}$ together with a natural transformation $t : \Delta(L) \to F$ such that for any object X of $\mathbf{C}$ and any natural transformation $s : \Delta(X) \to F$ there exists a unique morphism $s' : X \to L$ in $\mathbf{C}$ satisfying $t\Delta(s') = s$.

The following result is proved on page 115 in [3].

Proposition 2.13. *Let $\boldsymbol{D}'$ be a small category and $G : \boldsymbol{D}' \to \boldsymbol{C}^{\boldsymbol{D}}$ a functor. Denote by $\mathrm{ev}_d : \boldsymbol{C}^{\boldsymbol{D}} \to \boldsymbol{C}$ the evaluation functor in the object d of $\boldsymbol{D}$. Assume that for all objects d of $\boldsymbol{D}$ a limit for the composition $\mathrm{ev}_d \circ G : \boldsymbol{D}' \to \boldsymbol{C}$ exists. Then there exists a limit for G.*

The notion of colimit is dual to the notion of limit:

Remark 2.14. A colimit for F is the same as a limit for the dual functor $F^{\mathrm{op}} : \mathbf{D}^{\mathrm{op}} \to \mathbf{C}^{\mathrm{op}}$. More precisely, sending a colimit (C, t) for F to (C, t'), where the natural transformation $t' : \Delta_{\mathbf{D}^{\mathrm{op}}}(C) \to F^{\mathrm{op}}$ is given by $(t_d)^{\mathrm{op}}$ in an object d of $\mathbf{D}^{\mathrm{op}}$, defines a one-to-one correspondence between colimits for F and limits for F^{op}.

We conclude this section with a result about pushouts.

Proposition 2.15. *Assume that*

$$\begin{array}{ccc} X & \xrightarrow{i} & W \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j'} \\ Y & \xrightarrow{i'} & P \end{array} \quad \textit{and} \quad \begin{array}{ccc} Y & \xrightarrow{i'} & P \\ \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle k'} \\ Z & \xrightarrow{i''} & Q \end{array}$$

are pushout squares in $\boldsymbol{C}$. Then so is

$$\begin{array}{ccc} X & \xrightarrow{i} & W \\ \downarrow{\scriptstyle kj} & & \downarrow{\scriptstyle k'j'} \\ Z & \xrightarrow{i''} & Q \end{array} \quad .$$

Proof. From the commutativity of the first two squares one concludes that $k'j'i = i''kj$, thus the third one commutes too. Given now any commutative square

$$\begin{array}{ccc} X & \xrightarrow{i} & W \\ \downarrow{\scriptstyle kj} & & \downarrow{\scriptstyle r} \\ Z & \xrightarrow{s} & V \end{array},$$

one has to show, that there is a unique morphism $t : Q \to V$ such that $tk'j' = r$ and $ti'' = s$. Using that P is a pushout, let $t' : P \to V$ be the unique morphism such that $t'j' = r$ and $t'i' = sk$. Using that also Q is a pushout, define $t : Q \to V$ to be the unique morphism such that $tk' = t'$ and $ti'' = s$. This gives the desired morphism t. To show the uniqueness, let $t'' : Q \to V$ be a morphism with $t''k'j' = r$ and $t''i'' = s$. By the universal property of P, one concludes $t''k' = t'$. Hence, by the universal property of Q follows $t'' = t$. □

3 Model categories and homotopy categories

3.1 Model categories

The following terms will be used in the definition of a model category.

Definition 3.1. A morphism $f : X \to X'$ of a category $\mathbf{C}$ is called a *retract* of a morphism $g : Y \to Y'$ of $\mathbf{C}$, if there exists a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{i} & Y & \xrightarrow{r} & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{i'} & Y' & \xrightarrow{r'} & X' \end{array},$$

such that $ri = \mathrm{id}_X$ and $r'i' = \mathrm{id}_{X'}$.

Definition 3.2. Let $\mathbf{C}$ be a category.

i) Given a commutative diagram in $\mathbf{C}$ of the form

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{g} & Y \end{array}, \qquad (3.1)$$

a *lift* in the diagram is a morphism $h : B \to X$ such that $hi = f$ and $ph = g$.

ii) Let $i : A \to B$, $p : X \to Y$ be morphisms in $\mathbf{C}$. We say that i has the *left lifting property* (LLP) with respect to p and that p has the *right lifting property* (RLP) with respect to i, if there exists a lift in any commutative diagram of the form (3.1).

Now we are ready for the definition of a model category.

Definition 3.3. A *model category* is a category $\mathbf{C}$ together with three classes of morphisms of $\mathbf{C}$, the class of *weak equivalences* $W = W(\mathbf{C})$, of *fibrations* $Fib = Fib(\mathbf{C})$ and of *cofibrations* $Cof = Cof(\mathbf{C})$, each of which is closed under composition and contains all identity morphisms of $\mathbf{C}$, such that the following five conditions hold:

MC1: Every functor from a finite category to $\mathbf{C}$ has a limit and a colimit.

MC2: If f and g are morphisms of $\mathbf{C}$ such that gf is defined and if two out of the three morphisms f, g and gf are weak equivalences, then so is the third.

MC3: If f is a retract of a morphism g of $\mathbf{C}$ and g is a weak equivalence, a fibration or a cofibration, then so is f.

MC4:

i) Every cofibration has the LLP with respect to all $p \in W \cap Fib$.

ii) Every fibration has the RLP with respect to all $i \in W \cap Cof$.

MC5: Any morphism f of $\mathbf{C}$ can be factored as

i) $f = pi$, where $i \in Cof$, $p \in W \cap Fib$, and as

ii) $f = pi$, where $i \in Cof \cap W$, $p \in Fib$.

By a model category structure for a category $\mathbf{C}$ we mean a choice of three classes of morphisms of $\mathbf{C}$ such that $\mathbf{C}$ together with these classes is a model category. Let $\mathbf{C}$ be a model category until the end of this subsection. The following two remarks follow immediately from the definition of a model category.

Remark 3.4. Since any isomorphism $f : X \to Y$ of $\mathbf{C}$ is a retract of id_Y and since $\mathrm{id}_Y \in W \cap Fib \cap Cof$, it follows by **MC3** that also $f \in W \cap Fib \cap Cof$.

Remark 3.5. The opposite category $\mathbf{C}^{\mathrm{op}}$ is a model category by defining a morphism f^{op} in $\mathbf{C}^{\mathrm{op}}$ to be in $W(\mathbf{C}^{\mathrm{op}})$ if f is in $W(\mathbf{C})$, to be in $Fib(\mathbf{C}^{\mathrm{op}})$ if f is in $Cof(\mathbf{C})$ and to be in $Cof(\mathbf{C}^{\mathrm{op}})$ if f is in $Fib(\mathbf{C})$.

The proofs of the following two propositions can be found on page 87 and 88 of [1].

Proposition 3.6. *A morphism i of $\boldsymbol{C}$ is*

i) in Cof, if and only if it has the LLP with respect to all $p \in W \cap Fib$,

ii) in $W \cap Cof$, if and only if it has the LLP with respect to all $p \in Fib$.

Proposition 3.7. *Let* $\begin{array}{ccc} X & \xrightarrow{j} & Z \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\ Y & \xrightarrow{j'} & P \end{array}$ *be a pushout square in* $\boldsymbol{C}$*.*

i) If i *is in* $Cof(\boldsymbol{C})$*, then so is* i'*.*

ii) If i *is in* $Cof(\boldsymbol{C}) \cap W(\boldsymbol{C})$*, then so is* i'*.*

The next two results are concerned with cofibrations and pushouts and will be used in the proof of Theorem 2.

Lemma 3.8. *Given a commutative square in* $\boldsymbol{C}$ *of the form*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array} \quad . \tag{3.2}$$

Let P*,* Q *be pushouts of* $C \leftarrow A \rightarrow B$ *and let* $i_P : P \rightarrow D$*,* $i_Q : Q \rightarrow D$ *be the morphisms induced by* (3.2)*. Then* i_P *is a cofibration [resp. weak equivalence] if and only if* i_Q *is a cofibration [resp. weak equivalence].*

Proof. Let $j : P \rightarrow Q$ be the canonical isomorphism, then $i_P = i_Q j$ by Remark 2.9. Now, since $Cof(\mathbf{C})$ [resp. $W(\mathbf{C})$] is closed under composition, Lemma 3.8 follows from Remark 3.4. □

Proposition 3.9. *Given a commutative cube in* $\boldsymbol{C}$ *of the form*

$$\begin{array}{ccccc} A & & \longrightarrow & & B & & \\ & \searrow & & & & \searrow{\scriptstyle i_B} & \\ \downarrow & & A' & \longrightarrow & \downarrow & & B' \\ & & \downarrow & & & & \downarrow \\ C & & \longrightarrow & & D & & \downarrow \\ & \searrow & \downarrow & & & \searrow{\scriptstyle i_D} & \\ & & C' & & \longrightarrow & & D' \end{array} \quad , \tag{3.3}$$

where the back face and the front face are pushout squares. Let P *denote the pushout of the diagram* $C \leftarrow A \rightarrow A'$ *and let* $i_P : P \rightarrow C'$ *be the morphism induced by the left-hand face of* (3.3)*. If* i_B *and* i_P *are cofibrations, then so is* i_D*.*

Proof. By Proposition 3.6i), it's enough to find a lift in any given commutative diagram

$$\begin{array}{ccc} D & \longrightarrow & X \\ {\scriptstyle i_D}\downarrow & & \downarrow{\scriptstyle p} \\ D' & \longrightarrow & Y \end{array} \quad ,$$

where p is in $W \cap Fib$. Since i_B is a cofibration, there exists a lift $h_{B'} : B' \to X$ in

$$\begin{array}{ccccc} B & \longrightarrow & D & \longrightarrow & X \\ \downarrow{\scriptstyle i_B} & & & & \downarrow{\scriptstyle p} \\ B' & \longrightarrow & D' & \longrightarrow & Y \end{array} .$$

Defining $h_{A'}$ as the composition $A' \to B' \overset{h_{B'}}{\to} X$ yields a commutative diagram

$$\begin{array}{ccccc} A & & \longrightarrow & & A' \\ \downarrow & & & & \downarrow{\scriptstyle h_{A'}} \\ C & \longrightarrow & D & \longrightarrow & X \end{array}$$

and hence an induced morphism $P \to X$. This morphism makes the diagram

$$\begin{array}{ccccc} P & & \longrightarrow & & X \\ \downarrow{\scriptstyle i_P} & & & & \downarrow{\scriptstyle p} \\ C' & \longrightarrow & D' & \longrightarrow & Y \end{array} \qquad (3.4)$$

commute, as one checks using the universal property of the pushout P. Since i_P is a cofibration by assumption, there exists a lift $h_{C'} : C' \to X$ in (3.4). It makes the square

$$\begin{array}{ccc} A' & \longrightarrow & B' \\ \downarrow & & \downarrow{\scriptstyle h_{B'}} \\ C' & \xrightarrow{h_{C'}} & X \end{array}$$

commute. This square induces a morphism h from the pushout D' to X. One checks that h is the desired lift, using the universal property of the pushouts D and D'. □

3.2 The homotopy category of a model category

Definition 3.10. Let $\mathbf{C}$ be a category and W a class of morphisms of $\mathbf{C}$. A *localization of $\mathbf{C}$ with respect to W* is a category $\mathbf{D}$ together with a functor $F : \mathbf{C} \to \mathbf{D}$ such that the following two conditions hold:

i) $F(f)$ is an isomorphism for every f in W.

ii) If G is a functor from $\mathbf{C}$ to a category $\mathbf{D'}$, such that $G(f)$ is an isomorphism for every f in W, then there exists a unique functor $G' : \mathbf{D} \to \mathbf{D'}$ with $G'F = G$.

Remark 3.11. Let $\mathbf{C}$ be a category and W a class of morphisms of $\mathbf{C}$. If $(\mathbf{D}, F)$ and $(\mathbf{D}', F')$ are localizations of $\mathbf{C}$ with respect to W, then the unique functor G' such that $G'F = F'$ is an isomorphism. Therefore, if a localization exists, we will sometimes speak of the localization.

Let $\mathbf{C}$ be a model category until the end of this subsection. The localization $(\mathrm{Ho}(\mathbf{C}), \gamma_{\mathbf{C}})$ of $\mathbf{C}$ with respect to the class of weak equivalences W exists by Theorem 6.2 of [1]. This fact makes the following definition possible.

Definition 3.12. The *homotopy category* of the model category $\mathbf{C}$ is the localization of $\mathbf{C}$ with respect to W.

Until the end of this subsection let F be a functor from $\mathbf{C}$ to another model category $\mathbf{D}$. The homotopy colimit functor will be defined as a total left derived functor, which is defined as follows.

Definition 3.13. A *total left derived functor* $\mathbf{L}F$ for the functor F is a functor $\mathbf{L}F : \mathrm{Ho}(\mathbf{C}) \to \mathrm{Ho}(\mathbf{D})$ together with a natural transformation $t : (\mathbf{L}F)\gamma_{\mathbf{C}} \to \gamma_{\mathbf{D}}F$ such that for any pair (G, s) of a functor $G : \mathrm{Ho}(\mathbf{C}) \to \mathrm{Ho}(\mathbf{D})$ and a natural transformation $s : G\gamma_{\mathbf{C}} \to \gamma_{\mathbf{D}}F$, there exists a unique natural transformation $s' : G \to \mathbf{L}F$ satisfying

$$t \circ s'|_{\gamma_{\mathbf{C}}} = s,$$

where the natural transformation $s'|_{\gamma_{\mathbf{C}}} : G\gamma_{\mathbf{C}} \to (\mathbf{L}F)\gamma_{\mathbf{C}}$ is given by $s'_{\gamma_{\mathbf{C}}(X)}$ in an object X of $\mathbf{C}$.

Remark 3.14. Assume that $(\mathbf{L}F, t)$ and $(\mathbf{L}'F, t')$ are total left derived functors for F. Then the unique natural transformation $s' : \mathbf{L}'F \to \mathbf{L}F$ such that $t \circ s'|_{\gamma_{\mathbf{C}}} = t'$ is a natural equivalence. Therefore, if a total left derived functor for F exists, we will sometimes speak of the left derived functor.

The proof of the following theorem is given on page 114 in [1].

Theorem 1. *Assume $G : \boldsymbol{D} \to \boldsymbol{C}$ is a functor, which is right adjoint to F and which carries morphisms of $Fib(\boldsymbol{D})$ to $Fib(\boldsymbol{C})$ and morphisms of $Fib(\boldsymbol{D}) \cap W(\boldsymbol{D})$ to $Fib(\boldsymbol{C}) \cap W(\boldsymbol{C})$. Then the total left derived functor $\boldsymbol{L}F : \mathrm{Ho}(\boldsymbol{C}) \to \mathrm{Ho}(\boldsymbol{D})$ for F exists.*

4 Homotopy colimits

In this section let $\mathbf{C}$ be a model category. We want to define the homotopy colimit functor as the total left derived functor for the functor $\mathrm{colim} : \mathbf{C}^{\mathbf{D}} \to \mathbf{C}$. Therefore, we have to equip the functor category $\mathbf{C}^{\mathbf{D}}$ with a suitable model category structure, which can be done under the assumption that $\mathbf{D}$ is as in the next definition.

Definition 4.1. A non-empty, finite category $\mathbf{D}$ is said to be *very small* if there exists an integer $N \geq 1$ such that for any composition $f_N ... f_2 f_1$ of morphisms $(f_i)_{1 \leq i \leq N}$ in $\mathbf{D}$, at least one morphism f_i is an identity morphism.

More geometrically, note that a non-empty, finite category $\mathbf{D}$ is very small if and only if it has no cycles, i.e. given any integer $n \geq 1$ and any composition $f_n...f_2 f_1 : d \to d$ of morphisms $(f_i)_{1\leq i\leq n}$ in $\mathbf{D}$, then each morphism f_i is the identity morphism id_d. The advantage of a very small category is that it enables us to do induction involving the degree, which is defined as follows.

Definition 4.2. Let $\mathbf{D}$ be a very small category and d any object of $\mathbf{D}$. The *degree* $\deg(d)$ of d is defined by

$$\begin{aligned}\deg(d) := \max(\{0\} \cup \{n \geq 1;\ &\text{there exists a composition } f_n...f_2 f_1 : e \to d \text{ of} \\ &\text{morphisms } (f_i \neq \mathrm{id}_d)_{1\leq i\leq n} \text{ in } \mathbf{D}\}),\end{aligned}$$

the *total degree* $\deg(\mathbf{D})$ of $\mathbf{D}$ by $\deg(\mathbf{D}) := \max(\{\deg(d); d \text{ an object of } \mathbf{D}\})$.

Remark 4.3. Let e, d be objects of a very small category $\mathbf{D}$ and assume, there exists a non-identity morphism $e \to d$. Then $\deg(e) < \deg(d)$.

4.1 A model category structure for $\mathbf{C}^{\mathbf{D}}$

Let $\mathbf{D}$ be a very small category. The functor category $\mathbf{C}^{\mathbf{D}}$ can be given a model category structure in the following way.

Let d be an object of $\mathbf{D}$. Recall that an object m of the over category $\mathbf{D}\downarrow d$ is given by a morphism $m : e \to d$ in $\mathbf{D}$ and that a morphism k in $\mathbf{D}\downarrow d$ from $m : e \to d$ to $m' : e' \to d$ is a morphism $k : e \to e'$ in $\mathbf{D}$ such that $m'k = m$. Denote by ∂d the full subcategory of $\mathbf{D}\downarrow d$ which contains all objects of $\mathbf{D}\downarrow d$ except id_d. Let the functor $j_d : \partial d \to \mathbf{D}$ be given by $(m : e \to d) \mapsto e$ on objects and $k \mapsto k$ on morphisms. Composing the induced functor $(\cdot)|_{\partial d} : \mathbf{C}^{\mathbf{D}} \to \mathbf{C}^{\partial d}$ with $\mathrm{colim} : \mathbf{C}^{\partial d} \to \mathbf{C}$ gives a functor

$$\partial_d := \mathrm{colim} \circ (\cdot)|_{\partial d} : \mathbf{C}^{\mathbf{D}} \to \mathbf{C}.$$

Let X be an object of $\mathbf{C}^{\mathbf{D}}$. The natural transformation $s^X_d : X|_{\partial d} \to \Delta(X(d))$ given in an object $m : e \to d$ of ∂d by $X(m)$, induces the morphism

$$\alpha^{-1}(s^X_d) : \partial_d(X) \longrightarrow X(d),$$

where the bijection α comes from the adjunction $(\mathrm{colim}, \Delta_{\partial d}, \alpha)$. This induced morphism is natural. Indeed, given any morphism $f : X \to Y$ in $\mathbf{C}^{\mathbf{D}}$, we show that the diagram

$$\begin{array}{ccc} \partial_d(X) & \longrightarrow & X(d) \\ \big\downarrow{\scriptstyle \partial_d(f)} & & \big\downarrow{\scriptstyle f_d} \\ \partial_d(Y) & \longrightarrow & Y(d) \end{array} \tag{4.1}$$

commutes or equivalently, that

$$\alpha(\alpha^{-1}(s_d^Y)\partial_d(f)) = \alpha(f_d \alpha^{-1}(s_d^X)) \tag{4.2}$$

holds. Using the naturality of α, one concludes that the left-hand side of (4.2) equals $s_d^Y \circ f|_{\partial d}$, which is $Y(m) \circ f_e$ in an object $m : e \to d$, and that the right-hand side equals $\Delta(f_d) \circ s_d^X$, which is $f_d \circ X(m)$ in $m : e \to d$. Finally, the equation $Y(m) \circ f_e = f_d \circ X(m)$ holds, since f is a natural transformation by assumption. Define the functor δ_d as the composition

$$\delta_d : \mathbf{Mor}(\mathbf{C}^{\mathbf{D}}) \longrightarrow \mathbf{C}^{a \leftarrow b \rightarrow c} \overset{\text{colim}}{\longrightarrow} \mathbf{C},$$

where the first functor is given by

$$(f : X \to Y) \mapsto (\partial_d(Y) \overset{\partial_d(f)}{\longleftarrow} \partial_d(X) \longrightarrow X(d)) \text{ on objects,}$$
$$(s, s') \mapsto (\partial_d(s'), \partial_d(s), s_d) \text{ on morphisms.}$$

Given any morphism $f : X \to Y$ in $\mathbf{C}^{\mathbf{D}}$, consider that $\delta_d(f)$ is a pushout by definition and that therefore the commutative square (4.1) induces a morphism

$$i_d(f) : \delta_d(f) \to Y(d),$$

which is natural. Indeed, given any morphism (s, s') in $\mathbf{Mor}(\mathbf{C}^{\mathbf{D}})$ from $f : X \to X'$ to $g : Y \to Y'$, one checks that

$$\begin{array}{ccc} \delta_d(f) & \xrightarrow{i_d(f)} & X'(d) \\ {\scriptstyle \delta_d(s,s')}\downarrow & & \downarrow{\scriptstyle s'_d} \\ \delta_d(g) & \xrightarrow{i_d(g)} & Y'(d) \end{array}$$

commutes using the universal property of pushouts, the equation $s'_d f_d = g_d s_d$ which holds by assumption and the naturality of the morphism $\partial_d(X') \to X'(d)$. Using the above constructed morphisms $(i_d(f))_d$ for a morphism f in $\mathbf{C}^{\mathbf{D}}$, we give $\mathbf{C}^{\mathbf{D}}$ a model category structure.

Theorem 2. *Define a morphism f of $\boldsymbol{C}^{\boldsymbol{D}}$ to be in*

i) $W(\boldsymbol{C}^{\boldsymbol{D}})$, if f_d is in $W(\boldsymbol{C})$ for all objects d of $\boldsymbol{D}$,

ii) $Fib(\boldsymbol{C}^{\boldsymbol{D}})$, if f_d is in $Fib(\boldsymbol{C})$ for all objects d of $\boldsymbol{D}$, and to be in

iii) $Cof(\boldsymbol{C}^{\boldsymbol{D}})$, if $i_d(f)$ is in $Cof(\boldsymbol{C})$ for all objects d of $\boldsymbol{D}$.

Then $\boldsymbol{C}^{\boldsymbol{D}}$ together with these three classes is a model category.

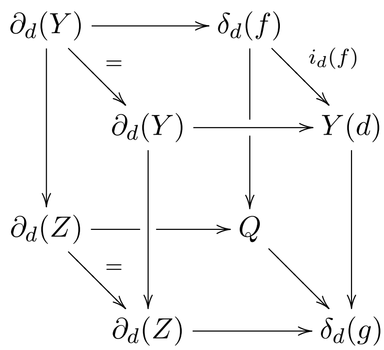
Remark 4.4. One can check directly, that the property of a morphism f in $\mathbf{C}^{\mathbf{D}}$ to be in $Cof(\mathbf{C}^{\mathbf{D}})$ doesn't depend on the choices of colimits involved in the construction of the morphisms $(i_d(f))_d$. However, this fact follows from Proposition 3.6i) after having proved Theorem 2.

Proof of Theorem 2. Since $W(\mathbf{C})$ and $Fib(\mathbf{C})$ are closed under composition and contain all identity morphisms, it follows immediately that $W(\mathbf{C}^{\mathbf{D}})$ and $Fib(\mathbf{C}^{\mathbf{D}})$ share the same properties.

To show that $Cof(\mathbf{C}^{\mathbf{D}})$ is closed under composition, let two morphisms $f : X \to Y$, $g : Y \to Z$ in $Cof(\mathbf{C}^{\mathbf{D}})$ be given. We have to prove, that $i_d(gf)$ is in $Cof(\mathbf{C})$ for all objects d of $\mathbf{D}$. By Lemma 3.8, it's enough to show that for any pushout Q of $\partial_d(Z) \leftarrow \partial_d(X) \to X(d)$, the morphism $i : Q \to Z(d)$ induced by the diagram

$$\begin{array}{ccc} \partial_d(X) & \longrightarrow & X(d) \\ {\scriptstyle \partial(g)\partial(f)=\partial(gf)}\downarrow & & \downarrow{\scriptstyle (gf)_d=g_d f_d} \\ \partial_d(Z) & \longrightarrow & Z(d) \end{array}$$

is a cofibration in $\mathbf{C}$. Using Proposition 2.15, define such a Q as the pushout of $\partial_d(Z) \leftarrow \partial_d(Y) \to \delta_d(f)$. To show that the induced morphism $i : Q \to Z(d)$ is in $Cof(\mathbf{C})$, let $Q \to \delta_d(g)$ be the unique morphism such that



(4.3)

commutes. The pushout P of $\partial_d(Z) \leftarrow \partial_d(Y) \overset{=}{\to} \partial_d(Y)$ is just $\partial_d(Z)$ and the morphism $P \to \partial_d(Z)$ induced by the left-hand face of (4.3) is $\mathrm{id}_{\partial_d(Z)}$ and therefore in $Cof(\mathbf{C})$. Applying Proposition 3.9, we get that $Q \to \delta_d(g)$ is in $Cof(\mathbf{C})$. Finally, using the universal property of the pushout Q of $\partial_d(Z) \leftarrow \partial_d(X) \to X(d)$, one checks that i equals the composition

$$Q \to \delta_d(g) \overset{i_d(g)}{\to} Z(d)$$

and therefore is in $Cof(\mathbf{C})$ as a composition of two cofibrations in $\mathbf{C}$.

To prove that $Cof(\mathbf{C}^{\mathbf{D}})$ contains all identity morphisms of $\mathbf{C}^{\mathbf{D}}$, one has to show the following claim. For any object d of $\mathbf{D}$ and any object X of $\mathbf{C}^{\mathbf{D}}$, the morphism $i_d(\mathrm{id}_X) : \delta_d(\mathrm{id}_X) \to X(d)$ is a cofibration in $\mathbf{C}$. Since $\partial_d(\mathrm{id}_X) = \mathrm{id}_{\partial_d(X)}$, it follows that $X(d)$ is a pushout of $\partial_d(X) \leftarrow \partial_d(X) \to X(d)$. The claim is now a consequence of Lemma 3.8.

4.1.1 Proof of MC1-MC3

By **MC1** in $\mathbf{C}$ and Proposition 2.13, it follows that any functor F from a finite category to $\mathbf{C}^{\mathbf{D}}$ has a limit. We want to show that F has a colimit or equivalently by Remark 2.14 that F^{op} has a limit. Note that $(\mathbf{C}^{\mathbf{D}})^{\mathrm{op}}$ is isomorphic to $(\mathbf{C}^{\mathrm{op}})^{\mathbf{D}^{\mathrm{op}}}$ and conclude from Proposition 2.13 and **MC1** in $\mathbf{C}^{\mathrm{op}}$, that F^{op} has a limit. Hence, **MC1** holds in $\mathbf{C}^{\mathbf{D}}$.

From **MC2** for $\mathbf{C}$, we will deduce **MC2** for $\mathbf{C}^{\mathbf{D}}$. Let f, g be morphisms in $\mathbf{C}^{\mathbf{D}}$ such that gf is defined and such that two of the three morphisms f, g, gf are in $W(\mathbf{C}^{\mathbf{D}})$. Then for all objects d of $\mathbf{D}$, two of the three morphisms f_d, g_d, $g_d f_d$ are in $W(\mathbf{C})$. Thus by **MC2** for $\mathbf{C}$ and the equality $g_d f_d = (gf)_d$, all three morphisms f_d, g_d, $(gf)_d$ are in $W(\mathbf{C})$ and hence f, g, gf are in $W(\mathbf{C}^{\mathbf{D}})$.

To show **MC3**, let $f : X \to X'$, $g : Y \to Y'$ be morphisms in $\mathbf{C}^{\mathbf{D}}$ such that f is a retract of g, i.e. there exists a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{i} & Y & \xrightarrow{r} & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{i'} & Y' & \xrightarrow{r'} & X' \end{array},$$

such that $ri = \mathrm{id}_X$ and $r'i' = \mathrm{id}_{X'}$. Note that therefore f_d is a retract of g_d for all objects d of $\mathbf{D}$. Thus the part of **MC3** dealing with fibrations [resp. weak equivalences] is a direct consequence of **MC3** for $\mathbf{C}$. Assume that g is a cofibration, i.e. for all objects d of $\mathbf{D}$ the morphism $i_d(g)$ is in $Cof(\mathbf{C})$. We will deduce that f is a cofibration by showing that $i_d(f)$ is a retract of $i_d(g)$ and hence is in $Cof(\mathbf{C})$ by **MC3** for $\mathbf{C}$. Consider the diagram

$$\begin{array}{ccccc} \delta_d(f) & \xrightarrow{\delta_d(i,i')} & \delta_d(g) & \xrightarrow{\delta_d(r,r')} & \delta_d(f) \\ \downarrow{\scriptstyle i_d(f)} & & \downarrow{\scriptstyle i_d(g)} & & \downarrow{\scriptstyle i_d(f)} \\ X'(d) & \xrightarrow{i'_d} & Y'(d) & \xrightarrow{r'_d} & X'(d) \end{array}.$$

The morphism $i_d(f)$ is natural as already shown, so the diagram commutes. By functoriality and since $ri = \mathrm{id}_X$, $r'i' = \mathrm{id}_{X'}$, it follows that the composition in the top row is the identity morphism. The equation $r'i' = \mathrm{id}_{X'}$ implies that the composition in the bottom row equals $\mathrm{id}_{X'(d)}$. Hence, $i_d(f)$ is a retract of $i_d(g)$.

4.1.2 Proof of MC4 and MC5

We will use induction over the total degree $\deg(\mathbf{D})$ of $\mathbf{D}$ to prove **MC4**, **MC5** and the following proposition.

Proposition 4.5. *Let $i : A \to B$ be a morphism in $\boldsymbol{C}^{\boldsymbol{D}}$. Then i is in $Cof(\boldsymbol{C}^{\boldsymbol{D}}) \cap W(\boldsymbol{C}^{\boldsymbol{D}})$, if and only if $i_d(i)$ is in $Cof(\boldsymbol{C}) \cap W(\boldsymbol{C})$ for all objects d of $\boldsymbol{D}$.*

Remark 4.6. Note that to prove any direction of Proposition 4.5, it's enough to show that $\partial_d(i)$ is in $Cof(\mathbf{C}) \cap W(\mathbf{C})$ for all objects d of $\mathbf{D}$ by Proposition 3.7ii) and **MC2** in **C**.

For the proof of the initial case $\deg(\mathbf{D}) = 0$, we will use the following lemma.

Lemma 4.7. *Assume* $\deg(\boldsymbol{D}) = 0$. *Let d be any object of $\boldsymbol{D}$ and $i : A \to B$ a morphism in $\boldsymbol{C}^{\boldsymbol{D}}$. Then $\partial_d(i)$ is an isomorphism in $\boldsymbol{C}$. Furthermore, the morphism $i_d(i)$ is in $Cof(\boldsymbol{C})$, if and only if i_d is in $Cof(\boldsymbol{C})$.*

Proof of Lemma 4.7. Since $\deg(\mathbf{D}) = 0$ implies $\deg(d) = 0$, it follows by Remark 4.3 that ∂d is the empty category. Hence, the colimits $\partial_d(A)$ and $\partial_d(B)$ are initial objects and $\partial_d(i)$ is an isomorphism. A pushout of $\partial_d(B) \leftarrow \partial_d(A) \to A(d)$ is given by $A(d)$ and the morphism $A(d) \to B(d)$ induced by the commutative square

$$\begin{array}{ccc} \partial_d(A) & \longrightarrow & A(d) \\ \downarrow & & \downarrow{\scriptstyle i_d} \\ \partial_d(B) & \longrightarrow & B(d) \end{array}$$

is i_d. Hence, the second statement of Lemma 4.7 follows from Lemma 3.8. □

We show Proposition 4.5, **MC4** and **MC5** in the initial case. By Remark 4.6, Lemma 4.7 implies Proposition 4.5, since any isomorphism in **C** is in $Cof(\mathbf{C}) \cap W(\mathbf{C})$ by Remark 3.4.

To prove **MC4**, let a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{g} & Y \end{array}$$

in $\mathbf{C}^{\mathbf{D}}$ be given, such that i is in $Cof(\mathbf{C}^{\mathbf{D}})$ and p is in $Fib(\mathbf{C}^{\mathbf{D}})$. We have to find a lift $h : B \to X$, whenever p is in $W(\mathbf{C}^{\mathbf{D}})$ [resp. i is in $W(\mathbf{C}^{\mathbf{D}})$]. By Lemma 4.7, it follows that i_d is in $Cof(\mathbf{C})$ for all objects d of **D**. If p is in $W(\mathbf{C}^{\mathbf{D}})$ [resp. if i is in $W(\mathbf{C}^{\mathbf{D}})$], then applying **MC4**i) [resp. **MC4**ii)] in **C** yields an objectwise lift $h_d : B(d) \to X(d)$. These objectwise lifts $(h_d)_d$ fit together to give the desired lift $h : B \to X$.

To prove **MC5**i) [resp. **MC5**ii)], let $f : A \to B$ be a morphism in $\mathbf{C}^{\mathbf{D}}$. Using **MC5**i) [resp. **MC5**ii)] in **C**, factor for every object d of **D** the morphism f_d as $f_d = p_d i_d$, where i_d is in $Cof(\mathbf{C})$ [resp. $Cof(\mathbf{C}) \cap W(\mathbf{C})$] and p_d is in $W(\mathbf{C}) \cap Fib(\mathbf{C})$ [resp. $Fib(\mathbf{C})$]. By construction and Lemma 4.7, the morphisms $(i_d)_d$ fit together to give a morphism i in $Cof(\mathbf{C}^{\mathbf{D}})$

[resp. $Cof(\mathbf{C}^{\mathbf{D}}) \cap W(\mathbf{C}^{\mathbf{D}})$] and the morphisms $(p_d)_d$ define a morphism p in $W(\mathbf{C}^{\mathbf{D}}) \cap Fib(\mathbf{C}^{\mathbf{D}})$ [resp. $Fib(\mathbf{C}^{\mathbf{D}})$]. The factorization $f = pi$ shows that **MC5**i) [resp. **MC5**ii)] holds.

To show the induction step, assume that $n := \deg(\mathbf{D}) \geq 1$. We prove Proposition 4.5. Let a morphism $i : A \to B$ in $\mathbf{C}^{\mathbf{D}}$ and any object d of $\mathbf{D}$ be given. For the proof of any direction, it's enough to show that $\partial_d(i)$ is in $Cof(\mathbf{C}) \cap W(\mathbf{C})$ or equivalently by Proposition 3.6ii), to find a lift in any commutative diagram

$$\begin{array}{ccc} \partial_d(A) & \xrightarrow{f} & C \\ \downarrow{\scriptstyle \partial_d(i)} & & \downarrow{\scriptstyle p} \\ \partial_d(B) & \xrightarrow{g} & D \end{array} \tag{4.4}$$

in $\mathbf{C}$, where p is in $Fib(\mathbf{C})$. From the commutativity of the diagram (4.4) and the definition of $\partial_d(i)$, it follows that also the diagram

$$\begin{array}{ccccc} A|_{\partial d} & \longrightarrow & \Delta(\partial_d(A)) & \xrightarrow{\Delta(f)} & \Delta(C) \\ \downarrow{\scriptstyle i|_{\partial d}} & & \downarrow{\scriptstyle \Delta(\partial_d(i))} & & \downarrow{\scriptstyle \Delta(p)} \\ B|_{\partial d} & \longrightarrow & \Delta(\partial_d(B)) & \xrightarrow{\Delta(g)} & \Delta(D) \end{array} \tag{4.5}$$

in $\mathbf{C}^{\partial d}$ commutes. In any of the two directions of Proposition 4.5 we will apply the induction hypothesis to find a lift $h : B|_{\partial d} \to \Delta(C)$ in (4.5), which will induce the desired lift $\partial_d(B) \to C$ in (4.4). Indeed, assume h is a lift in (4.5) and let $h' : \partial_d(B) \to C$ be the induced morphism, i.e. $h' = \alpha^{-1}(h)$, where the bijection α comes from the adjunction $(\mathrm{colim}, \Delta_{\partial d}, \alpha)$. Note that by the naturality of α the equations $\alpha(h'\partial_d(i)) = \alpha(h')i|_{\partial d}$ and $\alpha(ph') = \Delta(p)\alpha(h')$ hold. Since $h = \alpha(h')$ is a lift, deduce that $\alpha(f) = \alpha(h'\partial_d(i))$ and $\alpha(g) = \alpha(ph')$. This shows that h' is a lift in (4.4). To find a lift h in (4.5), note that the category ∂d is very small with $\deg(\partial d) < n$ and that $\Delta(p)$ is in $Fib(\mathbf{C}^{\partial d})$. Hence, using the induction hypothesis to apply **MC4**ii) in $\mathbf{C}^{\partial d}$, it's enough to show that $i|_{\partial d}$ is in $Cof(\mathbf{C}^{\partial d}) \cap W(\mathbf{C}^{\partial d})$, that is by definition that $i_m(i|_{\partial d})$ is in $Cof(\mathbf{C})$ and that $(i|_{\partial d})_m$ is in $W(\mathbf{C})$ for every object $m : e \to d$ of ∂d. We calculate $i_m(i|_{\partial d})$. Recall that an object of the subcategory ∂m of $\partial d{\downarrow} m$ is given by a non-identity morphism $k : (m' : e' \to d) \to m$ in ∂d, which by definition of ∂d is a non-identity morphism $k : e' \to e$ in $\mathbf{D}$ such that $mk = m'$. Furthermore, a morphism $g : k \to l$ in ∂m from $k : (m' : e' \to d) \to m$ to $l : (m'' : e'' \to d) \to m$ is a morphism $g : m' \to m''$ in ∂d such that $lg = k$, which is a morphism $g : e' \to e''$ in $\mathbf{D}$ with $m''g = m'$ and $lg = k : e' \to e$. Hence, we can define a functor $j' : \partial e \to \partial m$ by

$(k : e' \to e) \mapsto (k : mk \to m)$ on objects and
$(g : k \to l) \mapsto (g : j'(k) \to j'(l))$ on morphisms,

which is an isomorphism. Using (2.1), one concludes that the induced functor $(\cdot)|_{j'} : \mathbf{C}^{\partial m} \to \mathbf{C}^{\partial e}$ is an isomorphism. Since $\Delta_{\partial e} = (\cdot)|_{j'} \Delta_{\partial m}$ by Example 2.6, it follows by Remark 2.11 that the composition $\mathrm{colim} \circ (\cdot)|_{j'}$ of $\mathrm{colim} : \mathbf{C}^{\partial e} \to \mathbf{C}$ with $(\cdot)|_{j'}$ is the colimit functor $\mathbf{C}^{\partial m} \to \mathbf{C}$. Hence, the functor $\partial_m : \mathbf{C}^{\partial d} \to \mathbf{C}$ is the composition $\mathrm{colim} \circ (\cdot)|_{j'} \circ (\cdot)|_{\partial m}$. The composition $j_d \circ j_m \circ j'$ of the functors $j' : \partial e \to \partial m$, $j_m : \partial m \to \partial d$, $j_d : \partial d \to \mathbf{D}$ equals j_e. It follows by (2.1) that $(\cdot)|_{j'} (\cdot)|_{\partial m} (\cdot)|_{\partial d} = (\cdot)|_{\partial e}$ and hence $\partial_m(A|_{\partial_d}) = \mathrm{colim}(A|_{\partial_e}) = \partial_e(A)$, $\partial_m(B|_{\partial_d}) = \partial_e(B)$ and $\partial_m(i|_{\partial d}) = \partial_e(i)$. Note that $i|_{\partial d} : A|_{\partial d} \to B|_{\partial d}$ is just i_e in the object $m : e \to d$ of ∂d. It follows that the diagram

$$\begin{array}{ccc} \partial_m(A|_{\partial d}) & \longrightarrow & (A|_{\partial d})(m) \\ \downarrow{\scriptstyle \partial_m(i|_{\partial d})} & & \downarrow{\scriptstyle (i|_{\partial d})_m} \\ \partial_m(B|_{\partial d}) & \longrightarrow & (B|_{\partial d})(m) \end{array}$$

is just

$$\begin{array}{ccc} \partial_e(A) & \longrightarrow & A(e) \\ \downarrow{\scriptstyle \partial_e(i)} & & \downarrow{\scriptstyle i_e} \\ \partial_e(B) & \longrightarrow & B(e) \end{array}$$

by Remark 2.11, since for any Y in $\mathbf{C}^{\mathbf{D}}$, the natural transformation

$$s_m^{Y|_{\partial d}} : (Y|_{\partial d})|_{\partial m} \to \Delta_{\partial m}(Y|_{\partial d}(m))$$

is in an object $k : m' \to m$ of ∂m given by $Y|_{\partial d}(k) = Y(j_d(k)) = Y(k)$ and hence the equation $(s_m^{Y|_{\partial d}})|_{j'} = s_e^Y$ holds. It follows that $i_m(i|_{\partial d})$ equals $i_e(i)$. Assuming now first, that i is in $Cof(\mathbf{C}^{\mathbf{D}}) \cap W(\mathbf{C}^{\mathbf{D}})$, one concludes that for every object $m : e \to d$ of ∂d the morphism $i_m(i|_{\partial d})$ is in $Cof(\mathbf{C})$ and that $(i|_{\partial d})_m = i_e$ is in $W(\mathbf{C})$. This shows one direction of Proposition 4.5. For the other direction, assume that $i_{d'}(i)$ is in $Cof(\mathbf{C}) \cap W(\mathbf{C})$ for all objects d' of $\mathbf{D}$. It follows that for any object $m : e \to d$ in ∂d the morphism $i_m(i|_{\partial d})$ is in $Cof(\mathbf{C}) \cap W(\mathbf{C})$. Using the induction hypothesis to apply Proposition 4.5 one deduces that $i|_{\partial d}$ is in $Cof(\mathbf{C}^{\partial d}) \cap W(\mathbf{C}^{\partial d})$, which completes the proof of Proposition 4.5.

In the proof of **MC4** and **MC5** we will use the following notation. Let $\mathbf{D}^{n-1}$ be the full subcategory of $\mathbf{D}$ which contains precisely the objects e of $\mathbf{D}$ with $\deg(e) \leq n-1$. It is very small and has total degree $n-1$. The inclusion functor $j : \mathbf{D}^{n-1} \to \mathbf{D}$ induces the functor $(\cdot)|_{\mathbf{D}^{n-1}} : \mathbf{C}^{\mathbf{D}} \to \mathbf{C}^{\mathbf{D}^{n-1}}$, which carries weak equivalences [resp. fibrations] in $\mathbf{C}^{\mathbf{D}}$ to weak equivalences [resp. fibrations] in $\mathbf{C}^{\mathbf{D}^{n-1}}$. Note that $\mathbf{D}^{n-1} \downarrow e = \mathbf{D} \downarrow e$ for any object e of $\mathbf{D}$ with $\deg(e) < n$. One concludes that for any morphism f of $\mathbf{C}^{\mathbf{D}}$ the equation

$$i_e(f|_{\mathbf{D}^{n-1}}) = i_e(f) \qquad (4.6)$$

holds and in particular that $(\cdot)|_{\mathbf{D}^{n-1}}$ carries cofibrations in $\mathbf{C}^{\mathbf{D}}$ to cofibrations in $\mathbf{C}^{\mathbf{D}^{n-1}}$.

To show **MC4**i) [resp. **MC4**ii)], let a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{g} & Y \end{array}$$

in $\mathbf{C}^{\mathbf{D}}$ be given, such that i is in $Cof(\mathbf{C}^{\mathbf{D}})$ and p is in $Fib(\mathbf{C}^{\mathbf{D}})$. Assuming that p is in $W(\mathbf{C}^{\mathbf{D}})$ [resp. i is in $W(\mathbf{C}^{\mathbf{D}})$], one has to find a lift $h : B \to X$. Use the induction hypothesis to apply **MC4**i) [resp. **MC4**ii)] to find a lift $h^{n-1} : B|_{\mathbf{D}^{n-1}} \to X|_{\mathbf{D}^{n-1}}$ in the commutative square

$$\begin{array}{ccc} A|_{\mathbf{D}^{n-1}} & \xrightarrow{f|_{\mathbf{D}^{n-1}}} & X|_{\mathbf{D}^{n-1}} \\ {\scriptstyle i|_{\mathbf{D}^{n-1}}}\downarrow & & \downarrow{\scriptstyle p|_{\mathbf{D}^{n-1}}} \\ B|_{\mathbf{D}^{n-1}} & \xrightarrow{g|_{\mathbf{D}^{n-1}}} & Y|_{\mathbf{D}^{n-1}} \end{array} \quad .$$

Now, the strategy is to find for each object d of degree n of $\mathbf{D}$ an objectwise lift $B(d) \to X(d)$, such that these lifts and h^{n-1} fit together to give the desired lift $h : B \to X$. Let the natural transformation

$$B|_{\partial d} \to \Delta_{\partial d}(X(d))$$

be given by the composition $B(e) \overset{(h^{n-1})_e}{\longrightarrow} X(e) \overset{X(m)}{\longrightarrow} X(d)$ in an object $m : e \to d$ of ∂d. The induced morphism $\partial_d(B) \to X(d)$ makes the diagram

$$\begin{array}{ccc} \partial_d(A) & \xrightarrow{\alpha^{-1}(s_d^A)} & A(d) \\ {\scriptstyle \partial_d(i)}\downarrow & & \downarrow{\scriptstyle f_d} \\ \partial_d(B) & \longrightarrow & X(d) \end{array} \tag{4.7}$$

commute. Indeed, by the naturality of α, one conludes that $\alpha(f_d\alpha^{-1}(s_d^A)) = \Delta(f_d)s_d^A$, which is $f_dA(m)$ in an object $m : e \to d$ of ∂d, and that α of the composition $\partial_d(A) \to \partial_d(B) \to X(d)$ equals the composition $A|_{\partial_d} \to B|_{\partial_d} \to \Delta(X(d))$, which is $X(m)(h^{n-1})_e i_e$ in $m : e \to d$. Finally, the equation $f_dA(m) = X(m)(h^{n-1})_e i_e$ holds, since $h^{n-1}i|_{\mathbf{D}^{n-1}} = f|_{\mathbf{D}^{n-1}}$ and since f is a natural transformation by assumption. Let $\delta_d(i) \to X(d)$ be the morphism induced by the commutative diagram (4.7). Using the universal property of pushouts, one checks that it makes

$$\begin{array}{ccc} \delta_d(i) & \longrightarrow & X(d) \\ {\scriptstyle i_d(i)}\downarrow & & \downarrow{\scriptstyle p_d} \\ B(d) & \xrightarrow{g_d} & Y(d) \end{array} \tag{4.8}$$

commute. Apply **MC4**i) in **C** [resp. apply **MC4**ii) in **C** and Proposition 4.5] to find a lift $h_d : B(d) \to X(d)$ in (4.8). One checks that any lift $B(d) \to X(d)$ in (4.8) is also a lift in

$$\begin{array}{ccc} A(d) & \xrightarrow{f_d} & X(d) \\ {\scriptstyle i_d}\downarrow & & \downarrow{\scriptstyle p_d} \\ B(d) & \xrightarrow{g_d} & Y(d) \end{array} .$$

The desired lift $h : B \to X$ can now be defined as $(h^{n-1})_e$ in an object e of $\mathbf{D}$ with $\deg(e) < n$ and as the constructed lift h_d of (4.8) in an object d of degree n of $\mathbf{D}$. To show that h is a natural transformation, note that by Remark 4.3 and since h^{n-1} is a natural transformation, one only has to consider morphisms $m : e \to d$ in $\mathbf{D}$, where $\deg(e) < n$, $\deg(d) = n$, and to check that $X(m)(h^{n-1})_e = h_d B(m)$ holds. Denoting $\operatorname{colim}(B|_{\partial_d}) = (\partial d(B), t)$, this is done by the following sequence of equations of compositions of morphisms:

$$\begin{aligned} B(e) \xrightarrow{B(m)} B(d) \xrightarrow{h_d} X(d) &= B(e) \xrightarrow{t_m} \partial_d(B) \longrightarrow B(d) \xrightarrow{h_d} X(d) \\ &= B(e) \xrightarrow{t_m} \partial_d(B) \longrightarrow \delta_d(i) \xrightarrow{i_d(i)} B(d) \xrightarrow{h_d} X(d) \\ &= B(e) \xrightarrow{t_m} \partial_d(B) \longrightarrow \delta_d(i) \longrightarrow X(d) \\ &= B(e) \xrightarrow{t_m} \partial_d(B) \longrightarrow X(d) \\ &= B(e) \xrightarrow{(h^{n-1})_e} X(e) \xrightarrow{X(m)} X(d). \end{aligned}$$

To prove **MC5**, we have to factor a given morphism $f : A \to B$ in $\mathbf{C}^{\mathbf{D}}$ in the two ways i) and ii). Use the induction hypothesis to factor $f|_{\mathbf{D}^{n-1}}$ as

$$A|_{\mathbf{D}^{n-1}} \xrightarrow{h^{n-1}} X^{n-1} \xrightarrow{g^{n-1}} B|_{\mathbf{D}^{n-1}} \tag{4.9}$$

as in **MC5**i) [resp. **MC5**ii)]. Let d be an object of degree n of $\mathbf{D}$. Note that the functor $j_d : \partial d \to \mathbf{D}$ carries any object m of ∂d to an object $j_d(m)$ with $\deg(j_d(m)) < n$. It therefore induces a functor $j'_d : \partial d \to \mathbf{D}^{n-1}$, which composed with the inclusion functor $j : \mathbf{D}^{n-1} \to \mathbf{D}$ equals j_d. For the induced functors follows

$$(\cdot)|_{\partial d} = (\cdot)|_{j'_d} \circ (\cdot)|_{\mathbf{D}^{n-1}} \tag{4.10}$$

by (2.1). Hence, applying the composition $\operatorname{colim} \circ (\cdot)|_{j'_d}$ of $\operatorname{colim} : \mathbf{C}^{\mathbf{D}} \to \mathbf{C}$ with $(\cdot)|_{j'_d}$ to (4.9) yields a factorization

$$\partial_d(A) \to \operatorname{colim}(X^{n-1}|_{j'_d}) \to \partial_d(B) \tag{4.11}$$

of $\partial_d(f)$. Now define $X(d)$ and h_d through the pushout square

$$\begin{array}{ccc} \partial_d(A) & \longrightarrow & A(d) \\ \downarrow & & \downarrow{\scriptstyle h_d} \\ \operatorname{colim}(X^{n-1}|_{j'_d}) & \longrightarrow & X(d) \end{array} \quad . \tag{4.12}$$

Denote $\operatorname{colim}(X^{n-1}|_{j'_d}) = (\operatorname{colim}(X^{n-1}|_{j'_d}), t^d)$ and for a non-identity morphism $m : e \to d$ define the morphism $X(m) : X^{n-1}(e) \to X(d)$ as the composition

$$X^{n-1}(e) \overset{(t^d)_m}{\to} \operatorname{colim}(X^{n-1}|_{j'_d}) \to X(d). \tag{4.13}$$

Furthermore, set $X(\mathrm{id}_d) = \mathrm{id}_{X(d)}$. Then these choices, X^{n-1} and h^{n-1} fit together to give a functor $X : \mathbf{D} \to \mathbf{C}$ and a natural transformation $h : A \to X$. Indeed, let $m : e \to d$ be a non-identity morphism in $\mathbf{D}$ with $\deg(d) = n$. If $m' : e' \to e$ is another non-identity morphism in $\mathbf{D}$, then one deduces by the naturality of t^d that $(t^d)_m X^{n-1}(m') = (t^d)_{mm'}$ holds and therefore $X(mm') = X(m)X(m')$. To prove the naturality of h, check that

$$\begin{array}{ccc} A(e) & \xrightarrow[(h^{n-1})_e]{} & X(e) \\ {\scriptstyle A(m)}\downarrow & & \downarrow{\scriptstyle X(m)} \\ A(d) & \xrightarrow{h_d} & X(d) \end{array}$$

commutes by noting that $(h^{n-1})_e = (h^{n-1}|_{j'_d})_m$ and by using the commutativity of the diagram (4.12) and of the diagram

$$\begin{array}{ccc} A(e) & \xrightarrow{(h^{n-1}|_{j'_d})_m} & (X^{n-1}|_{j'_d})(m) \\ \downarrow & & \downarrow{\scriptstyle (t^d)_m} \\ \partial_d(A) & \xrightarrow[\operatorname{colim}(h^{n-1}|_{j'_d})]{} & \operatorname{colim}(X^{n-1}|_{j'_d}) \end{array} \quad .$$

We want to show that h is in $Cof(\mathbf{C}^{\mathbf{D}})$ [resp. h is in $Cof(\mathbf{C}^{\mathbf{D}}) \cap W(\mathbf{C}^{\mathbf{D}})$]. Note that $X|_{\mathbf{D}^{n-1}} = X^{n-1}$ and $h|_{\mathbf{D}^{n-1}} = h^{n-1}$ by definition. In case i), since by construction h^{n-1} is in $Cof(\mathbf{C}^{\mathbf{D}^{n-1}})$, it follows by (4.6) that $i_e(h)$ is in $Cof(\mathbf{C})$ for every object e of $\mathbf{D}$ with $\deg(e) < n$. Similarly, in case ii), since by construction h^{n-1} is in $Cof(\mathbf{C}^{\mathbf{D}^{n-1}}) \cap W(\mathbf{C}^{\mathbf{D}^{n-1}})$, it follows by Proposition 4.5 and by (4.6) that $i_e(h)$ is in $Cof(\mathbf{C}) \cap W(\mathbf{C})$ for every object e of $\mathbf{D}$ with $\deg(e) < n$. Let d be an object of degree n of $\mathbf{D}$. By (4.10), deduce the equations $X|_{\partial d} = X^{n-1}|_{j'_d}$, $h|_{\partial d} = h^{n-1}|_{j'_d}$ and therefore $\partial_d(X) = \operatorname{colim}(X^{n-1}|_{j'_d})$, $\partial_d(h) = \operatorname{colim}(h^{n-1}|_{j'_d})$. Recall that the morphism

$\partial_d(X) \to X(d)$ is induced by the natural transformation s_d^X which is given in an object $m : e \to d$ of ∂d by $X(m)$. Since $X(m)$ is the composition (4.13), it follows that $\partial_d(X) \to X(d)$ is just $\operatorname{colim}(X^{n-1}|_{j'_d}) \to X(d)$. Hence, $\delta_d(h) = X(d)$ and $i_d(h)$ equals $\mathrm{id}_{X(d)}$ and in particular is in $Cof(\mathbf{C}) \cap W(\mathbf{C})$. Thus in case i) we have shown that h is in $Cof(\mathbf{C}^{\mathbf{D}})$ and by Proposition 4.5, it follows in case ii) that h is in $Cof(\mathbf{C}^{\mathbf{D}}) \cap W(\mathbf{C}^{\mathbf{D}})$.

Using g^{n-1}, we want to define a natural transformation $g : X \to B$ such that $f = gh$. Let d be an object of degree n of $\mathbf{D}$. The natural transformation $X|_{\partial d} \to \Delta(B(d))$ given by $B(m)(g^{n-1})_e$ in an object $m : e \to d$ of ∂d, induces a morphism $\partial_d(X) \to B(d)$ in $\mathbf{C}$. One checks that this morphism makes the square

$$\begin{array}{ccc} \partial_d(A) & \longrightarrow & A(d) \\ {\scriptstyle \partial_d(h)}\downarrow & & \downarrow{\scriptstyle f_d} \\ \partial_d(X) & \longrightarrow & B(d) \end{array}$$

commute. This gives an induced morphism g_d from the pushout $X(d)$ to $B(d)$ such that $f_d = g_d h_d$. The constructed morphisms $(g_d)_d$ and g^{n-1} fit together to give a natural transformation $g : X \to B$. Indeed, if $m : e \to d$ is a non-identity morphism in $\mathbf{D}$ with $\deg(d) = n$, then $B(m)g_e = g_d X(m)$ holds, as is shown by the following sequence of equations of compositions of morphisms:

$$\begin{aligned} X(e) \overset{(g^{n-1})_e}{\to} B(e) \overset{B(m)}{\to} &= X(e) \overset{(t^d)_m}{\to} \partial_d(X) \to B(d) \\ &= X(e) \overset{(t^d)_m}{\to} \partial_d(X) \to X(d) \overset{g_d}{\to} B(d) \\ &= X(e) \overset{X(m)}{\to} X(d) \overset{g_d}{\to} B(d). \end{aligned}$$

We have factored f as gh, such that h is in $Cof(\mathbf{C}^{\mathbf{D}})$ [resp. $Cof(\mathbf{C}^{\mathbf{D}}) \cap W(\mathbf{C}^{\mathbf{D}})$]. Thus to prove **MC5**i) and **MC5**ii), it's enough to factor g as pi in $\mathbf{C}^{\mathbf{D}}$ such that respectively, $i \in Cof(\mathbf{C}^{\mathbf{D}})$, $p \in W(\mathbf{C}^{\mathbf{D}}) \cap Fib(\mathbf{C}^{\mathbf{D}})$ and $i \in Cof(\mathbf{C}^{\mathbf{D}}) \cap W(\mathbf{C}^{\mathbf{D}})$, $p \in Fib(\mathbf{C}^{\mathbf{D}})$. Set $Z^{n-1} := X|_{\mathbf{D}^{n-1}} : \mathbf{D}^{n-1} \to \mathbf{C}$, let $i^{n-1} : X|_{\mathbf{D}^{n-1}} \to Z^{n-1}$ be the identity morphism and define the natural transformation $p^{n-1} := g^{n-1} : Z^{n-1} \to B|_{\mathbf{D}^{n-1}}$. Let d be any object of degree n of $\mathbf{D}$. Using **MC5** in $\mathbf{C}$, factor g_d as

$$X(d) \overset{i_d}{\to} Z(d) \overset{p_d}{\to} B(d),$$

as in **MC5**i) [resp. **MC5**ii)]. For any non-identity morphism $m : e \to d$ in $\mathbf{D}$ set $Z(m) := i_d X(m) : Z(e) \to Z(d)$ and let the morphism $Z(\mathrm{id}_d)$ in $\mathbf{C}$ be given by $\mathrm{id}_{Z(d)}$. Then these choices and Z^{n-1} fit together to give a functor $Z : \mathbf{D} \to \mathbf{C}$. Indeed, if $m' : e' \to e$, $m : e \to d$ are non-identity morphisms in $\mathbf{D}$ such that $\deg(d) = n$, then

$$Z(mm') = i_d X(mm') = i_d X(m) X(m') = Z(m) Z(m').$$

The constructed morphisms $(i_d)_d$ and i^{n-1} fit together to give a natural transformation $i : X \to Z$. Indeed, given a non-identity morphism $m : e \to d$ in $\mathbf{D}$ with $\deg(d) = n$, one has $i_e = \mathrm{id}_{Z(e)}$ and thus $Z(m)i_e = i_d X(m)$. The morphisms $(p_d)_d$ and p^{n-1} yield a natural transformation $p : Z \to B$. Indeed, for a non-identity morphism $m : e \to d$ in $\mathbf{D}$ with $\deg(d) = n$ holds

$$B(m)p_e = B(m)g_e = g_d X(m) = p_d i_d X(m) = p_d Z(m).$$

Note that for an object d of arbitrary degree of $\mathbf{D}$, the functor $X|_{\partial d}$ equals $Z|_{\partial d}$ and that $i|_{\partial d} = \mathrm{id}_{X|_{\partial d}}$. Hence a pushout of $\partial_d(Z) \xleftarrow{=} \partial_d(X) \to X(d)$ is given by $X(d)$ and the morphism $X(d) \to Z(d)$ induced by the commutative square

$$\begin{array}{ccc} \partial_d(X) & \longrightarrow & X(d) \\ \big\downarrow & & \big\downarrow{\scriptstyle i_d} \\ \partial_d(Z) & \longrightarrow & Z(d) \end{array}$$

is just i_d. By Lemma 3.8, it follows that $i_d(i)$ is a cofibration [resp. weak equivalence] if and only if i_d is a cofibration [resp. weak equivalence]. Furthermore, note that if $\deg(d) < n$, then $i_d = \mathrm{id}_{Z(d)}$ and hence is in $Cof(\mathbf{C}) \cap W(\mathbf{C})$.

In case i), since by construction for any object d of degree n of $\mathbf{D}$ the morphism i_d is in $Cof(\mathbf{C})$, it follows that i is in $Cof(\mathbf{C}^{\mathbf{D}})$. One checks that by construction p is in $Fib(\mathbf{C}^{\mathbf{D}}) \cap W(\mathbf{C}^{\mathbf{D}})$. Similarly, in case ii), the morphism i_d is in $Cof(\mathbf{C}) \cap W(\mathbf{C})$ for every object d of degree n of $\mathbf{D}$. By Proposition 4.5, it follows that $i \in Cof(\mathbf{C}^{\mathbf{D}}) \cap W(\mathbf{C}^{\mathbf{D}})$. One checks that $p \in Fib(\mathbf{C}^{\mathbf{D}})$. This completes the proof of **MC5** and hence the induction step.

We have shown Theorem 2. □

4.2 The homotopy colimit functor

Let $\mathbf{D}$ be a very small category. Let $\mathbf{C}^{\mathbf{D}}$ be the model category of Theorem 2. Recall that by Remark 2.10 the functor $\Delta : \mathbf{C} \to \mathbf{C}^{\mathbf{D}}$ is right adjoint to $\mathrm{colim} : \mathbf{C}^{\mathbf{D}} \to \mathbf{C}$ and note that it carries morphisms of $Fib(\mathbf{C})$ and morphisms of $Fib(\mathbf{C}) \cap W(\mathbf{C})$ to $Fib(\mathbf{C}^{\mathbf{D}})$ and $Fib(\mathbf{C}^{\mathbf{D}}) \cap W(\mathbf{C}^{\mathbf{D}})$ respectively. By Theorem 1 we are finally in position to define homotopy colimits or more precisely, the homotopy colimit functor.

Definition 4.8. The *homotopy colimit functor* is the total left derived functor $\mathbf{L}\mathrm{colim} : \mathrm{Ho}(\mathbf{C}^{\mathbf{D}}) \to \mathrm{Ho}(\mathbf{C})$ for the functor $\mathrm{colim} : \mathbf{C}^{\mathbf{D}} \to \mathbf{C}$.

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