

PhD thesis

Burnside rings of fusion systems

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PhD thesis submitted to the PhD School of The Faculty of Science, University of Copenhagen, Denmark in April 2014.

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Abstract

In this thesis we study the interactions between saturated fusion systems and group actions of the underlying p -groups. For a saturated fusion system \mathcal{F} on a finite p -group S we construct the Burnside ring of \mathcal{F} in terms of the finite S -sets whose actions respect the structure of the fusion system, and we produce a basis for the Burnside ring that shares properties with the transitive sets for a finite group. We construct a transfer map from the p -local Burnside ring of the underlying p -group S to the p -local Burnside ring of \mathcal{F} . Using such transfer maps, we give a new explicit construction of the characteristic idempotent of \mathcal{F} – the unique idempotent in the p -local double Burnside ring of S satisfying properties of Linckelmann and Webb. We describe this idempotent both in terms of fixed points and as a linear combination of transitive bisets. Additionally, using fixed points we determine the map of Burnside rings given by multiplication with the characteristic idempotent, and we show that this map is the transfer map previously constructed. Applying these results, we show that for every saturated fusion system the ring generated by all (non-idempotent) characteristic elements in the p -local double Burnside ring is isomorphic to the p -local Burnside ring of the fusion system, and we disprove a conjecture by Park-Ragnarsson-Stancu on the composition product of fusion systems.

Resumé

I denne afhandling undersøger vi de gensidige relationer mellem mættede fusions systemer og gruppevirkninger af de underliggende p -grupper. For et mættet fusionssystem \mathcal{F} på en endelig p -gruppe S konstruerer vi Burnside-ringen for \mathcal{F} ud fra de endelige S -mængder hvis gruppevirkninger respekterer strukturen fra fusion systemet, og vi producerer en basis for Burnside-ringen som deler egenskaber med de transitive mængder for en endelig gruppe. Vi konstruerer en induceringsafbildning fra den p -lokale Burnside-ring for S til den p -lokale Burnside-ring for \mathcal{F} . Ved at benytte sådanne induceringsafbildninger giver vi en ny explicit konstruktion af den karakteristiske idempotent for \mathcal{F} – den entydige idempotent i den p -lokale dobbelte Burnside ring for S der tilfredsstiller egenskaber fra Linckelmann og Webb. Vi beskriver både denne idempotent med hensyn til fikspunkter og som en linearkombination af transitive bimængder. Derudover bestemmer vi via fikspunkter den afbildning mellem Burnside-ringe som er givet ved multiplikation med den karakteristiske idempotent, og vi viser at denne afbildning er lig induceringsafbildningen fra tidligere. Ved at anvende sidde resultater viser vi at for ethvert mættet fusion system vil ringen frembragt af alle (ikke-idempotente) karakteristiske elementer i den p -lokale dobbelte Burnside ring er isomorf med den p -lokale Burnside ring for fusions systemet, og vi modbeviser en formodning af Park-Ragnarsson-Stancu angående kompositionsproduktet af fusionsystemer.

Acknowledgements

First and foremost I would like to thank my advisor Jesper Grodal for four excellent years of being his student. I always enjoy our meetings and discussions, and I have received a lot of great advice and feedback through the years no matter whether I have been fighting against a tricky problem, writing papers, or even when figuring out how to teach algebraic topology to high school students.

I thank my fellow PhD students and postdocs: We have had a great time, held interesting seminars, watched good movies, and eaten a lot of excellent cake. It has been a pleasure being webmaster for our cake club, and I apologize strongly for the time when I forgot to bake cake and had to *buy* one instead. A particular thanks goes to Martin Wedel Jacobsen, my fellow fusion system PhD student and friend, for listening whenever I wanted to share a new discovery or a frustrating problem, and for great company through all of our years studying together at the University of Copenhagen.

Thanks go to the Centre for Symmetry and Deformation¹, and to the Department of Mathematical Sciences in general, for a great scientific environment with lots of guest and events, and for making it possible for me to go to conferences and visit fellow researchers throughout the world. My thanks go as well to all those abroad who have given me excellent feedback and suggestions, these include Matthew Gelvin, Ergün Yalçın, Serge Bouc, Radu Stancu, Sejong Park and Haynes Miller.

Thank you to all my fellow organizers in the Danish Youth Association of Science (UNF) and in particular all those who have helped organize the UNF MatCamp throughout the years. It has been great and a fun challenge to teach at MatCamp and to show the high school students how incredible mathematics can be. Keep up the good work all of you!

I thank my fellow organizers of the Georg Mohr math competition, and I thank the Danish IMO² teams of 2009, 2011, 2012, and 2013 – It was a pleasure being your Deputy Leader.

Finally, I thank my family and friends for supporting me. Troels and Jørgen I thank in particular for being great friends for many years so far, and I hope for many more. Last but not least I thank my loving parents, and I apologize for not following in your footsteps and becoming a computer scientist.³

¹Funded by the Danish National Research Foundation. (DNRF92)

²the International Mathematical Olympiad

³At least I write computer programs to do fusion system calculations on examples ;-)

Contents

Abstracts	iii
Acknowledgements	iv
Contents	v
Part I Thesis overview	1
1 Background on fusion systems and Burnside rings	3
2 Main results	6
3 Perspectives	12
Bibliography	15
Part II The papers	17
A The abelian monoid of fusion-stable finite sets is free	19
<i>submitted, 14 pages, also available at arXiv:1302.4628</i>	
1. Introduction	19
2. Fusion systems	21
3. Burnside rings for groups	23
4. Stable sets for a fusion system	24
4.1. Proving theorems A and B	26
References	32
B Transfer and characteristic idempotents for saturated fusion systems	35
<i>submitted, 39 pages, also available at arXiv:1306.4162</i>	
1. Introduction	35
2. Fusion systems	40
3. Burnside rings for groups and fusion systems	41
3.1. The Burnside ring of a saturated fusion system	43
4. The p -localized Burnside ring	44
4.1. Equivalent Burnside rings	53
5. The characteristic idempotent	55
5.1. The category of Burnside modules	56
5.2. A new construction of the characteristic idempotent	56
5.3. The action of the characteristic idempotent	62
6. The Burnside ring embeds in the double Burnside ring	65
7. On the composition product of saturated fusion systems	70
References	73

Part I

Thesis overview

Background on fusion systems and Burnside rings

Part I of this thesis is intended as an overview of the scientific content of the thesis as well as an overview of the previous scientific work that forms the context in which this thesis was made. Part II then consists of the actual papers constituting this PhD thesis.

This first section of part I describes some of the scientific background for the thesis. The section is intended as an overview and motivation, and for this reason many details have been left out and can instead be found in [A], [B] or elsewhere. The main results of the thesis are described in the later section 2.

Fusion systems

Fusion systems are a way of studying the p -structure of a finite group G , by describing which of its elements of p -power order are related by conjugation in G . This is encoded by writing down a Sylow p -subgroup of G along with all conjugations from G between the subgroups of the Sylow.

Such structures were initially axiomatized by Puig in [27] under the name of “Frobenius categories”. However, in the terminology mostly used, these structures go by the name of *(abstract) fusion systems*. An abstract fusion system over a p -group S is a category with the subgroups of S as objects, and the morphisms between subgroups are actual group homomorphism required to behave like conjugation maps. See [A, Section 2] or [B, Section 2] for the precise definition. Of particular interest are the so-called *saturated* fusion systems, which satisfy analogues of Sylow’s theorems. The fusion system generated by any finite group on a Sylow p -subgroup is always saturated.

The algebraic theory of fusion systems were connected to classifying space of finite groups by Broto-Levi-Oliver in their papers [4–7] laying the groundwork for the homotopy theory of saturated fusion systems. They introduced the notion of a centric linking system associated to a saturated fusion system, which were later shown by Chermak [10] to always exist uniquely. Through linking systems Broto-Levi-Oliver defined the classifying space of a fusion system, and showed that if a fusion system \mathcal{F} comes from a finite group G , then the classifying space of \mathcal{F} is equivalent to the p -completion BG_p^\wedge of the classifying space of G . The Martino-Priddy conjecture furthermore states that finite groups have equivalent p -completed classifying spaces if and only if they have isomorphic fusion systems. The Martino-Priddy conjecture has been proven true, but so far the proof as well as the existence and uniqueness of centric linking systems depend upon the classification of finite simple groups.

Burnside rings

In group theory the Burnside ring of a finite group G emerges when we consider actions of G on finite sets, up to isomorphism. The isomorphism classes of finite G -sets form a semiring $A_+(G)$ with disjoint union as addition and cartesian product as multiplication. The Burnside ring of G , denoted $A(G)$, is then the Grothendieck group of $A_+(G)$ where we add formal additive inverses. Basic properties of Burnside rings are listed in [A, Section 3] as well as [B, Section 3]. For later reference we make note of two of these properties:

First of all, a finite G -set is determined up to isomorphism by its “fixed point vector” consisting of the number of fixed points with respect to each subgroup $H \leq G$. This embeds the Burnside ring $A(G)$ as the set of fixed point vectors inside the set of all vectors

$$\tilde{\Omega}(G) := \prod_{\substack{H \leq G \text{ up to} \\ \text{conjugation}}} \mathbb{Z}.$$

The ring $\tilde{\Omega}(G)$ is usually called the *ghost ring* for $A(G)$.

Secondly, the transitive G -sets G/H , $H \leq G$, form a basis for the additive monoid $A_+(G)$ of all finite G -sets. Furthermore, these basis elements are in one-to-one correspondence with the conjugacy classes of subgroups in G .

The first main focus of this thesis is to find a suitable Burnside ring $A(\mathcal{F})$ for a saturated fusion system \mathcal{F} , with properties similar to the Burnside ring of a group. All of [A] as well as part of [B] is dedicated to this pursuit (see section 2 below).

As described in [B, Section 5.1] there is another construction similar to the Burnside ring, but instead of having one group acting on finite sets, we instead consider two finite groups G and H acting simultaneously (G from the right, H from the left) resulting in an additive monoid of *bisets*. The resulting additive groups $A(G, H)$ in many ways interact as if they consist of “maps” from G to H , and we for instance have composition maps

$$\circ: A(H, K) \times A(G, H) \rightarrow A(G, K)$$

for all triples G, H, K . The ring of endomorphisms $A(G, G)$ is called the *double Burnside ring* of G .

A further reason to view the elements of $A(G, H)$ as maps comes from the Segal conjecture [8, 22], where we know for finite groups G, H that the collection of all stable maps $BG_+ \rightarrow BH_+$, up to homotopy, forms a completion of the double Burnside module $A(G, H)$ as long as we restrict our attention to bisets on which H acts freely. This enables us to study stable maps between classifying spaces through purely algebraic means.

Characteristic elements for fusion systems

Results of Linckelmann-Webb and later Broto-Levi-Oliver make use of the relation between stable maps $BG_+ \rightarrow BH_+$ and $A(G, H)$ to construct a classifying spectrum for each saturated fusion system. An important ingredient in this construction is a so-called *characteristic biset* for the fusion system: If a fusion system \mathcal{F} on S is realized by a finite group G , then instead of letting G act on the subgroups of S , we let S act on G to give us a biset ${}_S G_S \in A(S, S)$. The core properties of the biset ${}_S G_S$ in relation to \mathcal{F} were recognized by Linckelmann-Webb, and any (S, S) -biset or element in $A(S, S)$ is said to be \mathcal{F} -characteristic if it satisfies these properties (see [B, Definition 5.4]).

The construction of a classifying spectrum for a saturated fusion system \mathcal{F} on S was made more precise by Ragnarsson-Stancu in [28, 29] where they proved that each \mathcal{F} has a unique *characteristic idempotent* inside the p -localization $A(S, S)_{(p)}$, and via this idempotent they construct the classifying spectrum. Ragnarsson-Stancu furthermore showed how to reconstruct \mathcal{F} from its characteristic idempotent, thus giving a one-to-one correspondence between saturated fusion systems and characteristic idempotents. In particular for a finite group G , the corresponding $\mathcal{F}_S(G)$ -characteristic idempotent contains all the p -local information of BG_p^\wedge up to stable homotopy.

The second focus of this thesis is to give a new explicit construction of the characteristic idempotents, and to describe in detail their properties and how each characteristic idempotent interacts with the Burnside ring and double Burnside ring of the associated fusion system and its underlying p -group. This is the content of the second part of [B] (see section 2 below).

Main results

This section gives an overview of the main results of the thesis in the order that they appear in the two papers [A] and [B].

Paper A: The abelian monoid of fusion-stable finite sets is free

If \mathcal{F} is a saturated fusion system on a p -group S , we consider all the finite S -sets that additionally respects the conjugations coming from \mathcal{F} . These S -sets are called \mathcal{F} -stable and they form a subsemiring $A_+(\mathcal{F})$ inside the semiring of all finite S -sets (see [A, Definition 4.2]). The goal of paper [A] is to classify all the \mathcal{F} -stable sets and use them to construct a Burnside ring for each saturated fusion system \mathcal{F} .

For the semiring $A_+(S)$ of finite S -sets we know that each set decomposes uniquely as a sum of transitive S -sets, and the transitive sets correspond to the S -conjugacy classes of subgroups in S . The first and main theorem of [A] states that a similar result holds for the \mathcal{F} -stable sets:

Theorem 2.1 ([A, Theorem A]). *Let \mathcal{F} be a saturated fusion system on the underlying finite p -group S .*

Every \mathcal{F} -stable S -set splits uniquely (up to S -isomorphism) as a disjoint union of irreducible \mathcal{F} -stable sets, and there is a one-to-one correspondence between the irreducible \mathcal{F} -stable sets and conjugacy classes of subgroups in the fusion system \mathcal{F} .

Hence the semiring $A_+(\mathcal{F})$ of \mathcal{F} -stable sets is additively a free commutative monoid with rank equal to the number of conjugacy classes of subgroups in \mathcal{F} .

This theorem was originally conjectured by Matthew Gelvin, and was previously unknown even in the group case where we consider G -stable S -sets for $S \in \text{Syl}_p(G)$. As part of the proof for this theorem, [A] gives an algorithm constructing the irreducible \mathcal{F} -stable set α_P corresponding to each $P \leq S$ up to \mathcal{F} -conjugacy, but a closed formula for the orbits or fixed points of each α_P is not immediately evident (see section 3 on further research perspectives).

We define the Burnside ring of \mathcal{F} to be the Grothendieck group of $A_+(\mathcal{F})$. Because $A_+(\mathcal{F})$ is a free commutative monoid, it follows that the irreducible \mathcal{F} -stable sets α_P form an additive basis for the Burnside ring $A(\mathcal{F})$.

In the same way that elements of $A(S)$ are completely determined by the number of fixed points with respect to the subgroup $Q \leq S$ up to S -conjugation, the second theorem of [A] describes how the elements of $A(\mathcal{F})$ are determined by the number of Q fixed points for $Q \leq S$ up to \mathcal{F} -conjugation. In addition, this describes $A(\mathcal{F})$ as a finite index subring

of fixed point vectors inside a suitable ghost ring

$$\tilde{\Omega}(\mathcal{F}) := \prod_{\substack{Q \leq S \text{ up to} \\ \mathcal{F}\text{-conjugation}}} \mathbb{Z},$$

and we show that the resulting *mark homomorphism* $\Phi: A(\mathcal{F}) \rightarrow \tilde{\Omega}(\mathcal{F})$ has properties similar to the mark homomorphism for groups:

Theorem 2.2 ([A, Theorem B]). *Let \mathcal{F} be a saturated fusion system over a p -group S , and let $A(\mathcal{F})$ be the Burnside ring of \mathcal{F} – i.e. the subring consisting of the \mathcal{F} -stable elements in the Burnside ring of S .*

Then there is a ring homomorphism Φ and a group homomorphism Ψ that fit together in the following short-exact sequence:

$$0 \rightarrow A(\mathcal{F}) \xrightarrow{\Phi} \tilde{\Omega}(\mathcal{F}) \xrightarrow{\Psi} \prod_{\substack{P \leq S \text{ up to} \\ \mathcal{F}\text{-conjugation,} \\ P \text{ fully } \mathcal{F}\text{-normalized}}} \mathbb{Z}/|W_S P|\mathbb{Z} \rightarrow 0,$$

where $W_S P := N_S P/P$.

Φ comes from restricting the mark homomorphism of $A(S)$ to $A(\mathcal{F})$, and Ψ is given by the coordinate functions¹

$$\Psi_P(f) := \sum_{\bar{s} \in W_S P} f_{\langle s \rangle P} \pmod{|W_S P|}$$

when P is a fully normalized representative of its conjugacy class in \mathcal{F} . Here $\Psi_P = \Psi_{P'}$ if $P \sim_{\mathcal{F}} P'$ are both fully normalized.

The theorem provides a way of checking for each vector $f \in \tilde{\Omega}(\mathcal{F})$ whether f is the fixed point vector of an element in the Burnside ring of \mathcal{F} . This generalizes previous results by Dress and others (see [15], [14, Section 1] or [31]) concerning the mark homomorphism and congruence relations for Burnside rings of finite groups.

Paper B: Transfer and characteristic idempotents for saturated fusion systems

Paper [B] concerns the p -localization of the Burnside ring $A(\mathcal{F})_{(p)}$ as well as the characteristic idempotent for \mathcal{F} as defined by Ragnarsson in [28]. Paper [B] introduces a transfer map $\pi: A(S)_{(p)} \rightarrow A(\mathcal{F})_{(p)}$ between the Burnside rings for a saturated fusion system \mathcal{F} and its underlying p -group S . By using this transfer map [B] gives a new explicit construction of the characteristic idempotent $\omega_{\mathcal{F}}$ for a saturated fusion system \mathcal{F} . This enables us to calculate the fixed points and coefficients of $\omega_{\mathcal{F}}$ and give a precise description of the products $\omega_{\mathcal{F}} \circ X$ and $X \circ \omega_{\mathcal{F}}$ for any element X of the double Burnside ring of S . Finally, [B] gives an application of these results to a conjecture by Park-Ragnarsson-Stancu on the composition product of saturated fusion systems.

¹For a vector $f \in \tilde{\Omega}(\mathcal{F})$ we use $f_{\langle s \rangle P}$ to denote the coordinate of f corresponding to the subgroup $\langle s \rangle P \leq S$.

In more detail, we first consider the transfer map for Burnside rings of fusion systems: In the case of groups whenever we have an inclusion $H \leq G$ we get both a restriction map $A(G) \rightarrow A(H)$ restricting the action of each G -set to H , as well as a transfer map $A(H) \rightarrow A(G)$ that takes an H -set X to the G -set $G \times_H X$. The restriction map is a ring homomorphism while the transfer map is a homomorphism of $A(G)$ -modules, where $A(H)$ becomes an $A(G)$ -module through the restriction map. In the case of a saturated fusion system \mathcal{F} on S , we have $A(\mathcal{F}) \leq A(S)$ by construction, so the inclusion plays the role of a restriction map in this context. The following theorem shows that when we p -localize, we receive a transfer map as well:

Theorem 2.3 ([B, Theorem A]). *Let \mathcal{F} be a saturated fusion system on a finite p -group S . We let $A(\mathcal{F})_{(p)}$ denote the p -localized Burnside ring of \mathcal{F} as a subring of the p -localized Burnside ring $A(S)_{(p)}$ for S . Then there is a transfer map $\pi: A(S)_{(p)} \rightarrow A(\mathcal{F})_{(p)}$, which is a homomorphism of $A(\mathcal{F})_{(p)}$ -modules and which restricts to the identity on $A(\mathcal{F})_{(p)}$. In terms of (virtual) fixed points $|X^P|$ for elements $X \in A(S)_{(p)}$ the transfer map π satisfies*

$$|\pi(X)^Q| = \frac{1}{|[Q]_{\mathcal{F}}|} \sum_{Q' \in [Q]_{\mathcal{F}}} |X^{Q'}|,$$

where $[Q]_{\mathcal{F}}$ is the conjugacy class of $Q \leq S$ in \mathcal{F} .

If we apply the π to the transitive S -sets S/P for $P \leq S$, we get elements $\beta_P := \pi(S/P)$ in $A(\mathcal{F})_{(p)}$. These form a $\mathbb{Z}_{(p)}$ -basis for the p -localized Burnside ring $A(\mathcal{F})_{(p)}$, and we have $\beta_P = \beta_Q$ if and only if P and Q are conjugate in \mathcal{F} . The basis formed by the β_P is in many ways a preferable alternative to the basis consisting of irreducible \mathcal{F} -stable sets α_P ; however the β_P only exist once we p -localize the Burnside ring. When \mathcal{F} arises from a finite group G with Sylow p -subgroup S , then the basis elements β_P are closely related to the transitive G -sets G/P for $P \leq S$, and the p -localized Burnside ring $A(\mathcal{F})_{(p)}$ is in fact isomorphic to the part of $A(G)_{(p)}$ where all stabilizers are p -subgroups (see [B, Proposition 4.7]).

Next in paper [B] is a new and explicit proof that every saturated fusion system \mathcal{F} on S has an associated element $\omega_{\mathcal{F}} \in A(S, S)_{(p)}$ that is both \mathcal{F} -characteristic and idempotent. To construct $\omega_{\mathcal{F}}$ we consider the product fusion system $\mathcal{F} \times \mathcal{F}$ on $S \times S$ and apply the transfer map of [B, Theorem A] to $S \times S/\Delta(S, id)$, which is S itself considered as a biset. The resulting element $\beta_{\Delta(S, id)}$ then turns out to be both \mathcal{F} -characteristic and idempotent when considered as an element of the double Burnside ring $A(S, S)_{(p)}$. This new construction of the characteristic idempotent for instance enables us to calculate the (virtual) fixed points of $\omega_{\mathcal{F}}$:

Theorem 2.4 ([B, Theorem B]). *Let \mathcal{F} be a saturated fusion system on a finite p -group S . Then there exists a unique \mathcal{F} -characteristic idempotent $\omega_{\mathcal{F}} \in A(S, S)_{(p)}$, and it satisfies:*

For all graphs $\Delta(P, \varphi) \leq S \times S$ with $\varphi \in \mathcal{F}(P, S)$, we have

$$\left| (\omega_{\mathcal{F}})^{\Delta(P, \varphi)} \right| = \frac{|S|}{|\mathcal{F}(P, S)|};$$

and $|(\omega_{\mathcal{F}})^D| = 0$ for all other subgroups $D \leq S \times S$. Consequently, if we write $\omega_{\mathcal{F}}$ in the basis of $A(S, S)_{(p)}$, we get the expression

$$\omega_{\mathcal{F}} = \sum_{\substack{[\Delta(P, \varphi)]_{S \times S} \\ \text{with } \varphi \in \mathcal{F}(P, S)}} \frac{|S|}{|([P, \varphi]_S^S)^{\Delta(P, \varphi)}|} \left(\sum_{P \leq Q \leq S} \frac{|\{\psi \in \mathcal{F}(Q, S) \mid \psi|_P = \varphi\}|}{|\mathcal{F}(Q, S)|} \cdot \mu(P, Q) \right) [P, \varphi]_S^S,$$

where the outer sum is taken over $(S \times S)$ -conjugacy classes of subgroups, and where μ is the Möbius function for the poset of subgroups in S .

A closer look at the way [B, Theorem A] is applied to construct $\omega_{\mathcal{F}}$ reveals an even closer relationship between the transfer map and the characteristic idempotent, and we get a precise description of what happens when other elements are multiplied by $\omega_{\mathcal{F}}$:

Theorem 2.5 ([B, Theorem C]). *Let \mathcal{F}_1 and \mathcal{F}_2 be saturated fusion systems on finite p -groups S_1 and S_2 respectively, and let $\omega_1 \in A(S_1, S_1)_{(p)}$ and $\omega_2 \in A(S_2, S_2)_{(p)}$ be the characteristic idempotents.*

For every element of the Burnside module $X \in A(S_1, S_2)_{(p)}$, the product $\omega_2 \circ X \circ \omega_1$ is right \mathcal{F}_1 -stable and left \mathcal{F}_2 -stable, and satisfies

$$|(\omega_2 \circ X \circ \omega_1)^D| = \frac{1}{|[D]_{\mathcal{F}_2 \times \mathcal{F}_1}|} \sum_{D' \in [D]_{\mathcal{F}_2 \times \mathcal{F}_1}} |X^{D'}|,$$

for all subgroups $D \leq S_2 \times S_1$, where $[D]_{\mathcal{F}_2 \times \mathcal{F}_1}$ is the isomorphism class of D in the product fusion system $\mathcal{F}_2 \times \mathcal{F}_1$ on $S_2 \times S_1$.

Let $A(\mathcal{F}_1, \mathcal{F}_2)_{(p)}$ denote the right \mathcal{F}_1 -stable and left \mathcal{F}_2 -stable elements of $A(S_1, S_2)_{(p)}$. Then the characteristic idempotents ω_1 and ω_2 act trivially on $A(\mathcal{F}_1, \mathcal{F}_2)_{(p)}$, and [B, Theorem C] gives a transfer homomorphism of modules over the double Burnside rings $A(\mathcal{F}_1, \mathcal{F}_1)_{(p)}$ and $A(\mathcal{F}_2, \mathcal{F}_2)_{(p)}$ – see [B, Proposition 5.10]. In particular, we recover the transfer map of [B, Theorem A] as multiplication by $\omega_{\mathcal{F}}$ on the Burnside module $A(1, S)_{(p)}$ generated by finite left S -sets.

For a saturated fusion system \mathcal{F} on S , we define the double Burnside ring $A(\mathcal{F}, \mathcal{F})_{(p)}$ to be the subring of $A(S, S)_{(p)}$ consisting of all the elements that are both left and right \mathcal{F} -stable. An even smaller subring is the ring generated by all \mathcal{F} -characteristic elements – in the terminology of [B] this subring consists of all the elements that are \mathcal{F} -generated as well as \mathcal{F} -stable. We denote the subring $A^{\text{char}}(\mathcal{F})_{(p)}$ since a generic \mathcal{F} -generated, \mathcal{F} -stable element is actually \mathcal{F} -characteristic. $A^{\text{char}}(\mathcal{F})_{(p)}$ has a $\mathbb{Z}_{(p)}$ -basis consisting of elements $\beta_{\Delta(P, id)}$, which only depends on $P \leq S$ up to \mathcal{F} -conjugation (see [B, Proposition 6.3]), and each element X of $A^{\text{char}}(\mathcal{F})_{(p)}$, written

$$X = \sum_{\substack{P \leq S \text{ up to} \\ \mathcal{F}\text{-conjugation}}} c_{\Delta(P, id)} \beta_{\Delta(P, id)},$$

is \mathcal{F} -characteristic if and only if the coefficient $c_{\Delta(S, id)}$ is invertible in $\mathbb{Z}_{(p)}$.

Hence we have a sequence of inclusions of subrings

$$A^{\text{char}}(\mathcal{F})_{(p)} \subseteq A(\mathcal{F}, \mathcal{F})_{(p)} \subseteq A(S, S)_{(p)}.$$

The last inclusion is not unital since $\omega_{\mathcal{F}}$ is the multiplicative identity of the first two rings, and $S = [S, id]_S^S$ is the identity of $A(S, S)_{(p)}$.

For every (S, S) -biset X , we can quotient out the right S -action in order to get X/S as a left S -set. Quotienting out the right S -action preserves disjoint union and extends to a collapse map $q: A(S, S)_{(p)} \rightarrow A(S)_{(p)}$, and by restriction to subrings we get maps

$$\begin{array}{ccccc} A^{\text{char}}(\mathcal{F})_{(p)} & \subseteq & A(\mathcal{F}, \mathcal{F})_{(p)} & \subseteq & A(S, S)_{(p)} \\ & \searrow & \downarrow & & \downarrow \\ & & A(\mathcal{F})_{(p)} & \subseteq & A(S)_{(p)} \end{array}$$

where \mathcal{F} -stable bisets are collapsed to \mathcal{F} -stable sets. In general the collapse map does not respect the multiplication of the double Burnside ring, but combining the techniques of [B, Theorems A and C] we show that on $A^{\text{char}}(\mathcal{F})_{(p)}$ the collapse map is not only a ring homomorphism but actually an isomorphism of rings!

Theorem 2.6 ([B, Theorem D]). *Let \mathcal{F} be a saturated fusion system defined over a finite p -group S .*

Then the collapse map $q: A^{\text{char}}(\mathcal{F})_{(p)} \rightarrow A(\mathcal{F})_{(p)}$, which quotients out the right S -action, is an isomorphism of rings, and it sends the basis element $\beta_{\Delta(P, id)}$ of $A(\mathcal{F}, \mathcal{F})_{(p)}$ to the basis element β_P of $A(\mathcal{F})_{(p)}$.

This generalizes a similar result for groups where the Burnside ring $A(S)$ embeds in the double Burnside ring $A(S, S)$ with the transitive S -set S/P corresponding to the transitive biset $[P, id]_S^S$. As an immediate consequence of [B, Theorem D] we get an alternative proof that the characteristic idempotent $\omega_{\mathcal{F}}$ is unique: In [B, Corollary 6.6] it is shown that $\beta_{\Delta(S, id)} = \omega_{\mathcal{F}}$ is the only non-zero idempotent of $A^{\text{char}}(\mathcal{F})_{(p)}$ by proving that 0 and S/S are the only idempotents of $A(\mathcal{F})_{(p)}$.

The final section of B applies [B, Theorem C] to disprove a conjecture by Park-Ragnarsson-Stancu, [26], on the composition product of fusion systems. Let \mathcal{F} be a saturated fusion system on a p -group S , and let \mathcal{H}, \mathcal{K} be saturated fusion subsystems on subgroups $R, T \leq S$ respectively. In the terminology of Park-Ragnarsson-Stancu, we then say that \mathcal{F} is the composition product of \mathcal{H} and \mathcal{K} , written $\mathcal{F} = \mathcal{H}\mathcal{K}$, if $S = RT$ and for all subgroups $P \leq T$ it holds that every morphism $\varphi \in \mathcal{F}(P, R)$ can be written as a composition $\varphi = \psi\rho$ where ψ is a morphism of \mathcal{H} and ρ is a morphism of \mathcal{K} .

Park-Ragnarsson-Stancu conjectured that $\mathcal{F} = \mathcal{H}\mathcal{K}$ is equivalent to the following equation of characteristic idempotents:

$$[R, id]_S^R \circ \omega_{\mathcal{F}} \circ [T, id]_T^S = \omega_{\mathcal{H}} \circ [R \cap T, id]_T^R \circ \omega_{\mathcal{K}} \quad (2.1)$$

A special case of the conjecture was proven in [26], in the case where $R = S$ and \mathcal{K} is weakly normal in \mathcal{F} , and the general conjecture was inspired by the group case, where $H, K \leq G$ satisfy $G = HK$ if and only if there is an isomorphism of (K, H) -bisets $G \cong H \times_{H \cap K} K$. Direct calculation via [B, Theorem C] now characterizes all cases where (2.1) holds:

Theorem 2.7 ([B, Theorem E]). *Let \mathcal{F} be a saturated fusion system on a p -group S , and suppose that \mathcal{H}, \mathcal{K} are saturated fusion subsystems of \mathcal{F} on subgroups $R, T \leq S$ respectively. Then the characteristic idempotents satisfy*

$$[R, id]_S^R \circ \omega_{\mathcal{F}} \circ [T, id]_T^S = \omega_{\mathcal{H}} \circ [R \cap T, id]_T^R \circ \omega_{\mathcal{K}} \quad (2.2)$$

if and only if $\mathcal{F} = \mathcal{H}\mathcal{K}$ and for all $Q \leq R \cap T$ we have

$$|\mathcal{F}(Q, S)| = \frac{|\mathcal{H}(Q, R)| \cdot |\mathcal{K}(Q, T)|}{|\text{Hom}_{\mathcal{H} \cap \mathcal{K}}(Q, R \cap T)|}. \quad (2.3)$$

In particular (2.2) always implies $\mathcal{F} = \mathcal{H}\mathcal{K}$, but the converse is not true in general. In [B, Example 7.1], the alternating group A_6 gives rise to a composition product $\mathcal{F} = \mathcal{H}\mathcal{K}$ where (2.3) fails – hence we get a counter-example to the general conjecture of Park-Ragnarsson-Stancu.

At the same time, [B, Proposition 7.2] proves a special case of the conjecture where \mathcal{K} is weakly normal in \mathcal{F} , which is a generalization of the case proved by Park-Ragnarsson-Stancu.

— 3 —

Perspectives

The results of the thesis leads to new questions for possible further research, and the following section lists several of these. Some of the questions are still only ideas, and some of them have already been worked on.

Describing the structure of the irreducible stable sets

In the proof of [A, Theorem A] we have an algorithm for constructing each irreducible \mathcal{F} -set when given a saturated fusion system \mathcal{F} , but the proof does not yield a closed formula for the orbits of α_P nor for the number of fixed points. Furthermore, in order to completely describe the Burnside ring $A(\mathcal{F})$, it would be good to have a formula for each product $\alpha_Q \cdot \alpha_P$ of basis elements in the same vein as the double coset formula for groups describing the products of transitive G -sets.

Last year Ergün Yalçın pointed out a way to approach the basis elements α_P using Möbius functions. This has recently resulted in a collaboration by Ergün Yalçın, Matthew Gelvin, and myself where we succeed in writing down a formula for the orbits and fixed points of α_P in terms of Möbius-like functions involving the fusion system, or equivalently by counting (with signs) suitable chains of subgroups in S mixed with \mathcal{F} -conjugations. These results can be found in [19].

A multiplication formula for $\alpha_Q \cdot \alpha_P$ is however still missing.

A category of \mathcal{F} -sets

Is there a good notion of a category of finite \mathcal{F} -stable sets or “ \mathcal{F} -sets” when \mathcal{F} is a saturated fusion system? Such a category should at least have finite coproducts and products (and preferably pullbacks as well), which would give rise to a semiring structure $A'_+(\mathcal{F})$ on the set of isomorphism classes of finite \mathcal{F} -sets. The Grothendieck group $A'(\mathcal{F})$ would then be an obvious choice for the Burnside ring of \mathcal{F} , and while it seems unlikely that $A'(\mathcal{F})$ would be isomorphic to $A(\mathcal{F})$ directly, the two rings should at least become isomorphic after p -localization.

One attempt in this direction was made by Matthew Gelvin in his PhD thesis [17], and further attempts has followed during discussions between Matthew and myself, but so far no good construction satisfying all these requirements has been found.

Characteristic bisets

When applied to a product fusion system $\mathcal{F} \times \mathcal{F}$ on $S \times S$, [A, Theorem A] can be used to give a classification of all the characteristic bisets for \mathcal{F} , and in particular there turns out to be a unique minimal of these bisets contained in all the others. Studying genuine bisets instead of general elements in $A(S, S)_{(p)}$ has interest for several reasons, one of them being Park’s proof in [25] that each characteristic biset gives rise to a group G containing S and where $\mathcal{F} = \mathcal{F}_S(G)$ – however S is rarely a Sylow p -subgroup of G in this construction.

Studying the unique minimal \mathcal{F} -characteristic biset Ω_{\min} has been a joint project by Matthew Gelvin and myself, and our results were recently written down in [18]. It turns out that there is a very close relation between Ω_{\min} and the centric linking system for \mathcal{F} , and if \mathcal{F} is constrained, then the model for \mathcal{F} is isomorphic to Ω_{\min} as bisets. We also show for each fully normalized subgroup $P \leq S$ how Ω_{\min} induces a characteristic biset for each of the normalizer and centralizer subsystems $N_{\mathcal{F}}(P)$ and $C_{\mathcal{F}}(P)$ respectively. When P is \mathcal{F} -centric, the induced characteristic bisets for $N_{\mathcal{F}}(P)$ and $C_{\mathcal{F}}(P)$ will even be minimal.

Related questions that are still open include: Given Ω_{\min} , how can we find the minimal characteristic biset for $N_{\mathcal{F}}(P)$ and $C_{\mathcal{F}}(P)$ even when P is not \mathcal{F} -centric? How about minimal characteristic bisets for quotient fusion systems? If a group structure can be defined on Ω_{\min} such that it fits with the biset structure and the preexisting multiplication on S , is \mathcal{F} then necessarily constrained?

Fusion systems in terms of characteristic idempotents

We know from Ragnarsson-Stancu, [29], that saturated fusion systems on a finite p -group S are in one-to-one correspondence with those symmetric idempotents of $A(S, S)_{(p)}$ that satisfy Frobenius reciprocity. How much of the theory of saturated fusion systems is it possible to translate via this correspondence into a theory of characteristic idempotents and vice versa? The Park-Ragnarsson-Stancu conjecture on composition products of fusion systems, [26], is one such translation (see [B, Section 7]).

In relation to the previous paragraph on characteristic bisets, we can also ask whether it is possible given the characteristic idempotent for \mathcal{F} to construct the characteristic idempotent for each normalizer system $N_{\mathcal{F}}(P)$ where P is fully normalized. The construction that worked for the minimal characteristic biset does not immediately apply the characteristic idempotents, for instance because the idempotent is not actually a biset.

Mackey functors for fusion systems

If \mathcal{E} is a saturated fusion subsystem of \mathcal{F} , then we both have a restriction map $A(\mathcal{F})_{(p)} \rightarrow A(\mathcal{E})_{(p)}$ and a transfer map $A(\mathcal{E})_{(p)} \rightarrow A(\mathcal{F})_{(p)}$. It seems that this should make $A(-)$ into a Mackey functor on \mathcal{F} , or even a global Mackey functor for fusion systems over finite p -groups, but what does it even mean to be a Mackey functor on \mathcal{F} ?

One possible definition of a Mackey functor on \mathcal{F} , with trivial centralizer action, is as a pair of a co- and a contravariant functors (M_*, M^*) with certain relations and defined on the “orbit category” $\mathcal{O}(\mathcal{F})$ where we mod out all the inner automorphisms. Equivalently we can ask for a functor defined on the category with morphism sets $A_{\mathcal{F}}(Q, P)_{(p)}$ of \mathcal{F} -generated elements for $Q, P \leq S$. This definition is equivalent to the one investigated by Diaz-Park in [13].

In the case where \mathcal{F} is realized by a group G , with $S \in \text{Syl}_p(G)$, we know that the contravariant part of any Mackey functor satisfies

$$\lim_{\mathcal{O}(\mathcal{F}^c)}^i M^* = 0 \text{ for } i > 0,$$

where the higher limit is taken over the \mathcal{F} -centric part of the orbit category. But it is so far still unknown whether the same is true for all saturated fusion systems.

The methods presented in paper B at least seem sufficient to show that

$$\lim_{\mathcal{O}(\mathcal{F})}^1 M^* = 0$$

for each Mackey functor on \mathcal{F} , but here we take \lim^1 over the entire $\mathcal{O}(\mathcal{F})$ and not the centric part as we would really want.

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Part II

The papers

THE ABELIAN MONOID OF FUSION-STABLE FINITE SETS IS FREE

SUNE PRECHT REEH

ABSTRACT. For a finite group G with a Sylow p -subgroup S , we say that a finite set with an action of S is G -stable if the action is unchanged up to isomorphism when we act through conjugation maps in G . We show that the abelian monoid of isomorphism classes of G -stable S -sets is free, and we give an explicit construction of the basis, whose elements are in one-to-one correspondence with G -conjugacy classes of subgroups in S . As a main tool for proving freeness, we describe the Burnside ring of a saturated fusion system, and its embedding into a suitable associated ghost ring.

1. INTRODUCTION

Given a finite group G acting on a finite set X , we can restrict the action to a Sylow p -subgroup S of G . The resulting S -set has the property that it stays the same (up to S -isomorphism) whenever we change the action via a conjugation map from G . More precisely, if $P \leq S$ is a subgroup and $\varphi: P \rightarrow S$ is a homomorphism given by conjugation with some element of G , we can turn X into a P -set by using φ to define the action $p.x := \varphi(p)x$. We denote the resulting P -set by ${}_{P,\varphi}X$. In particular when $\text{incl}: P \rightarrow S$ is the inclusion map, ${}_{P,\text{incl}}X$ has the usual restriction of the S -action to P . When a finite S -set X is the restriction of a G -set, then X has the property

$$(1.1) \quad {}_{P,\varphi}X \text{ is isomorphic to } {}_{P,\text{incl}}X \text{ as } P\text{-sets, for all } P \leq S \text{ and homomorphisms } \varphi: P \rightarrow S \text{ induced by } G\text{-conjugation.}$$

Any S -set with property (1.1) is called G -stable. Whenever we restrict a G -set to S , the resulting S -set is G -stable; however there are G -stable S -sets whose S -actions do not extend to actions of G .

The isomorphism classes of finite S -sets form a semiring $A_+(S)$ with disjoint union as addition and cartesian product as multiplication. The collection of G -stable S -sets is closed under addition and multiplication, hence G -stable sets form a subsemiring.

Theorem A (for finite groups). *Let G be a finite group with Sylow p -group S .*

Every G -stable S -set splits uniquely (up to S -isomorphism) as a disjoint union of irreducible G -stable sets, and there is a one-to-one correspondence between the irreducible G -stable sets and G -conjugacy classes of subgroups in S .

Hence the semiring of G -stable sets is additively a free commutative monoid with rank equal to the number of G -conjugacy classes of subgroups in S .

As part of the proof proposition 4.8 gives an explicit construction of the irreducible G -stable sets.

It is a well-known fact that any finite S -set splits uniquely into orbits/transitive S -sets; and the isomorphism type of a transitive set S/P depends only on the subgroup P up to

2010 *Mathematics Subject Classification*. 20D20 (20J15, 19A22).

Supported by the Danish National Research Foundation through the Centre for Symmetry and Deformation (DNRF92).

S -conjugation. Theorem A then states that this fact generalizes nicely to G -stable S -sets, which is less obvious than it might first appear.

If we consider G -sets and restrict their actions to S , then two non-isomorphic G -sets might very well become isomorphic as S -sets. Therefore even though finite G -sets decompose uniquely into orbits, we have no guarantee that this decomposition remains unique when we restrict the actions to the Sylow subgroup S . In fact, uniqueness of decompositions fails in general when we consider restrictions of G -sets to S , as demonstrated in example 4.3 for the symmetric group S_5 and its Sylow 2-subgroup.

It then comes as a surprise that if we consider *all* G -stable S -sets, and not just the restrictions of actual G -sets, we can once more write stable sets as a disjoint union of irreducibles in a unique way.

The proof of theorem A relies only on the way G acts on the subgroups of S by conjugation. We therefore state and prove the theorem in general for abstract saturated fusion systems, which model the action of a group on a Sylow subgroup.

If \mathcal{F} is a fusion system over a p -group S , we say that an S -set X is \mathcal{F} -stable if it satisfies

$$(1.2) \quad \begin{array}{l} P, \varphi X \text{ is isomorphic to } P, \text{incl} X \text{ as } P\text{-sets, for all } P \leq S \text{ and homomorphisms} \\ \varphi: P \rightarrow S \text{ in } \mathcal{F}. \end{array}$$

The \mathcal{F} -stable S -sets form a semiring $A_+(\mathcal{F})$ since the disjoint union and cartesian product of \mathcal{F} -stable sets is again \mathcal{F} -stable; and theorem A then generalizes to

Theorem A (for fusion systems). *Let \mathcal{F} be a saturated fusion system over a p -group S .*

Every \mathcal{F} -stable S -set splits uniquely (up to S -isomorphism) as a disjoint union of irreducible \mathcal{F} -stable sets, and there is a one-to-one correspondence between the irreducible \mathcal{F} -stable sets and conjugacy/isomorphism classes of subgroups in the fusion system \mathcal{F} .

Hence the semiring $A_+(\mathcal{F})$ of \mathcal{F} -stable sets is additively a free commutative monoid with rank equal to the number of conjugacy classes of subgroups in \mathcal{F} .

In homotopy theory classifying spaces for groups and maps between them play an important role. For finite groups G, H , or in general discrete groups, the homotopy classes of unbased maps $[BG, BH]$ is in bijection with $\text{Rep}(G, H) = H \backslash \text{Hom}(G, H)$, where H acts on $\text{Hom}(G, H)$ by post-conjugation. Hence $[BG, B\Sigma_n]$ corresponds to the different way G can act on a set with n elements up to G -isomorphism. This implies that for a finite group G we have $[BG, \coprod_n B\Sigma_n] \cong A_+(G)$ as monoids.

A argument of Mislin [10, Formula 4] (building on work of Dwyer-Zabrodsky, Lannes, Miller and Carlsson [3, 6, 8, 9]) implies that every finite p -group S has $[BS, \coprod_n (B\Sigma_n)_p^\wedge] \simeq A_+(S)$. For a general finite group G , the set $[BG, \coprod_n (B\Sigma_n)_p^\wedge]$ is highly interesting but still mysterious. Restriction along the inclusion $\iota: S \rightarrow G$ of a Sylow p -subgroup induces a map

$$\iota^*: [BG, \coprod_n (B\Sigma_n)_p^\wedge] \rightarrow [BS, \coprod_n (B\Sigma_n)_p^\wedge] \simeq A_+(S),$$

and the image must necessarily be contained in the collection of G -stable sets $A_+(\mathcal{F}_S(G))$, where $\mathcal{F}_S(G)$ is the fusion system over S generated by G . It is currently unknown whether the map $\iota^*: [BG, \coprod_n (B\Sigma_n)_p^\wedge] \rightarrow A_+(\mathcal{F}_S(G))$ is injective or surjective. However, theorem A shows that the algebraic approximation $A_+(\mathcal{F}_S(G))$ has a very regular structure.

An important tool in proving theorem A, is the subring consisting of \mathcal{F} -stable elements inside the Burnside ring $A(S)$ of S , where the \mathcal{F} -stable elements satisfy a property similar to (1.2). This subring will be the Grothendieck group of the semiring of \mathcal{F} -stable sets, and we call it *the Burnside ring of \mathcal{F}* denoted by $A(\mathcal{F})$.

By restriction, the Burnside ring of \mathcal{F} inherits the homomorphism of marks from $A(S)$, embedding $A(\mathcal{F})$ into a product of a suitable number of copies of \mathbb{Z} . As a main step in proving theorem A, we show that this mark homomorphism has properties analogous the mark homomorphism for groups:

Theorem B. *Let \mathcal{F} be a saturated fusion system over a p -group S , and let $A(\mathcal{F})$ be the Burnside ring of \mathcal{F} – i.e. the subring consisting of the \mathcal{F} -stable elements in the Burnside ring of S .*

Then there is a ring homomorphism Φ and a group homomorphism Ψ that fit together in the following short-exact sequence:

$$0 \rightarrow A(\mathcal{F}) \xrightarrow{\Phi} \prod_{\substack{\text{conj. classes} \\ \text{in } \mathcal{F}}} \mathbb{Z} \xrightarrow{\Psi} \prod_{\substack{[P]_{\mathcal{F}} \text{ conj. class in } \mathcal{F}, \\ P \text{ fully } \mathcal{F}\text{-normalized}}} \mathbb{Z}/|W_S P|\mathbb{Z} \rightarrow 0,$$

where $W_S P := N_S P/P$.

Φ comes from restricting the mark homomorphism of $A(S)$, and Ψ is given by the $[P]_{\mathcal{F}}$ -coordinate functions

$$\Psi_P(f) := \sum_{\bar{s} \in W_S P} f_{(s)P} \pmod{|W_S P|}$$

when P is a fully normalized representative of the conjugacy class $[P]_{\mathcal{F}}$ in \mathcal{F} . Here $\Psi_P = \Psi_{P'}$ if $P \sim_{\mathcal{F}} P'$ are both fully normalized.

This generalizes previous results by Dress and others (see [5], [4, Section 1] or [12]) concerning the mark homomorphism and congruence relations for Burnside rings of finite groups, which also constitutes most of the proof of theorem B. Though it is easier to prove and less surprising that theorem A, we still draw attention to theorem B here because of how useful such a characterisation of Burnside rings in terms of marks can be.

Acknowledgement. First and foremost I thank Matthew Gelvin for some good discussions on \mathcal{F} -stable sets. It was Matthew who originally conjectured theorem A, and he kept insisting that it should be true, though for a while he was the only one believing so. I also thank Kasper K.S. Andersen who calculated (using Magma) a large amount of examples in order to support or refute Matthew’s conjecture. It was those examples that gave me the idea for lemma 4.7, which formed the missing link in the proof of theorem A.

2. FUSION SYSTEMS

The next few pages contain a very short introduction to fusion systems. The aim is to introduce the terminology from the theory of fusion systems that will be used in the paper, and to establish the relevant notation. For a proper introduction to fusion systems see for instance Part I of “Fusion Systems in Algebra and Topology” by Aschbacher, Kessar and Oliver, [1].

Definition 2.1. A fusion system \mathcal{F} over a p -group S , is a category where the objects are the subgroups of S , and for all $P, Q \leq S$ the morphisms must satisfy:

- (i) Every morphism $\varphi \in \text{Mor}_{\mathcal{F}}(P, Q)$ is an injective group homomorphism, and the composition of morphisms in \mathcal{F} is just composition of group homomorphisms.
- (ii) $\text{Hom}_S(P, Q) \subseteq \text{Mor}_{\mathcal{F}}(P, Q)$, where

$$\text{Hom}_S(P, Q) = \{c_s \mid s \in N_S(P, Q)\}$$

is the set of group homomorphisms $P \rightarrow Q$ induced by S -conjugation.

- (iii) For every morphism $\varphi \in \text{Mor}_{\mathcal{F}}(P, Q)$, the group isomorphisms $\varphi: P \rightarrow \varphi P$ and $\varphi^{-1}: \varphi P \rightarrow P$ are elements of $\text{Mor}_{\mathcal{F}}(P, \varphi P)$ and $\text{Mor}_{\mathcal{F}}(\varphi P, P)$ respectively.

We also write $\text{Hom}_{\mathcal{F}}(P, Q)$ or just $\mathcal{F}(P, Q)$ for the morphism set $\text{Mor}_{\mathcal{F}}(P, Q)$; and the group $\mathcal{F}(P, P)$ of automorphisms is denoted by $\text{Aut}_{\mathcal{F}}(P)$.

The canonical example of a fusion system comes from a finite group G with a given p -subgroup S . The fusion system of G over S , denoted $\mathcal{F}_S(G)$, is the fusion system over S where the morphisms from $P \leq S$ to $Q \leq S$ are the homomorphisms induced by G -conjugation:

$$\text{Hom}_{\mathcal{F}_S(G)}(P, Q) := \text{Hom}_G(P, Q) = \{c_g \mid g \in N_G(P, Q)\}.$$

A particular case is the fusion system $\mathcal{F}_S(S)$ consisting only of the homomorphisms induced by S -conjugation.

Let \mathcal{F} be an abstract fusion system over S . We say that two subgroups $P, Q \leq S$ are \mathcal{F} -conjugate, written $P \sim_{\mathcal{F}} Q$, if they are isomorphic in \mathcal{F} , i.e. there exists a group isomorphism $\varphi \in \mathcal{F}(P, Q)$. \mathcal{F} -conjugation is an equivalence relation, and the set of \mathcal{F} -conjugates to P is denoted by $[P]_{\mathcal{F}}$. The set of all \mathcal{F} -conjugacy classes of subgroups in S is denoted by $Cl(\mathcal{F})$. Similarly, we write $P \sim_S Q$ if P and Q are S -conjugate, the S -conjugacy class of P is written $[P]_S$ or just $[P]$, and we write $Cl(S)$ for the set of S -conjugacy classes of subgroups in S . Since all S -conjugation maps are in \mathcal{F} , any \mathcal{F} -conjugacy class $[P]_{\mathcal{F}}$ can be partitioned into disjoint S -conjugacy classes of subgroups $Q \in [P]_{\mathcal{F}}$.

We say that Q is \mathcal{F} - or S -subconjugate to P if Q is respectively \mathcal{F} - or S -conjugate to a subgroup of P , and we denote this by $Q \lesssim_{\mathcal{F}} P$ or $Q \lesssim_S P$ respectively. In the case where $\mathcal{F} = \mathcal{F}_S(G)$, we have $Q \lesssim_{\mathcal{F}} P$ if and only if Q is G -conjugate to a subgroup of P ; and the \mathcal{F} -conjugates of P , are just those G -conjugates of P which are contained in S .

A subgroup $P \leq S$ is said to be *fully \mathcal{F} -normalized* if $|N_S P| \geq |N_S Q|$ for all $Q \in [P]_{\mathcal{F}}$; and similarly P is *fully \mathcal{F} -centralized* if $|C_S P| \geq |C_S Q|$ for all $Q \in [P]_{\mathcal{F}}$.

Definition 2.2. A fusion system \mathcal{F} over S is said to be *saturated* if the following properties are satisfied for all $P \leq S$:

- (i) If P is fully \mathcal{F} -normalized, then P is fully \mathcal{F} -centralized, and $\text{Aut}_S(P)$ is a Sylow p -subgroup of $\text{Aut}_{\mathcal{F}}(P)$.
- (ii) Every homomorphism $\varphi \in \mathcal{F}(P, S)$ where $\varphi(P)$ is fully \mathcal{F} -centralized, extends to a homomorphism $\varphi \in \mathcal{F}(N_{\varphi}, S)$ where

$$N_{\varphi} := \{x \in N_S(P) \mid \exists y \in S: \varphi \circ c_x = c_y \circ \varphi\}.$$

The saturated fusion systems form a class of particularly nice fusion systems, and the saturation axiom are a way to emulate the Sylow theorems for finite groups. In particular, whenever S is a Sylow p -subgroup of G , then the Sylow theorems imply that the induced fusion system $\mathcal{F}_S(G)$ is saturated (see e.g. [1, Theorem 2.3]).

In this paper, we shall rarely use the defining properties of saturated fusion systems directly. We shall instead mainly use the following lifting property that saturated fusion systems satisfy:

Lemma 2.3 ([11]). *Let \mathcal{F} be saturated. Suppose that $P \leq S$ is fully normalized, then for each $Q \in [P]_{\mathcal{F}}$ there exists a homomorphism $\varphi \in \mathcal{F}(N_S Q, N_S P)$ with $\varphi(Q) = P$.*

For the proof, see lemma 4.5 of [11] or lemma 2.6(c) of [1].

3. BURNSIDE RINGS FOR GROUPS

In this section we consider the Burnside ring of a finite group S , and the semiring of finite S -sets. We recall the structure of the Burnside ring $A(S)$ and how to describe the elements and operations of $A(S)$ in terms of fixed points and the homomorphism of marks. In this section S can be any finite group, but later we shall only need the case where S is a p -group.

We consider finite S -sets up to S -isomorphism, and let $A_+(S)$ denote the set of isomorphism classes. Given a finite S -set X , we denote the isomorphism class of X by $[X] \in A_+(S)$. $A_+(S)$ is a commutative semiring with disjoint union as addition and cartesian product as multiplication, and additively $A_+(S)$ is a free commutative monoid, where the basis consists of the (isomorphism classes) of transitive S sets, i.e. $[S/P]$ where P is a subgroup of S . Two transitive S -sets S/P and S/Q are isomorphic if and only if P is conjugate to Q in S .

To describe the multiplication of the semiring $A_+(S)$, it is enough to know the products of basis elements $[S/P]$ and $[S/Q]$. By taking the product $(S/P) \times (S/Q)$ and considering how it breaks into orbits, one reaches the following double coset formula for the multiplication in $A_+(S)$:

$$(3.1) \quad [S/P] \cdot [S/Q] = \sum_{\bar{s} \in P \backslash S / Q} [S/(P \cap {}^s Q)],$$

where $P \backslash S / Q$ is the set of double cosets PsQ with $s \in S$.

The *Burnside ring of S* , denoted $A(S)$, is constructed as the Grothendieck group of $A_+(S)$, consisting of formal differences of finite S -sets. Additively, $A(S)$ is a free abelian group with the same basis as $A_+(S)$. For each element $X \in A(S)$ we define $c_P(X)$, with $P \leq S$, to be the coefficients when we write X as a linear combination of the basis elements $[S/P]$ in $A(S)$, i.e.

$$X = \sum_{[P] \in Cl(S)} c_P(X) \cdot [S/P].$$

Where $Cl(S)$ denotes the set of S -conjugacy classes of subgroup in S .

The resulting maps $c_P: A(S) \rightarrow \mathbb{Z}$ are group homomorphisms, but they are *not* ring homomorphisms. Note also that an element X is in $A_+(S)$, i.e. X is an S -set, if and only if $c_P(X) \geq 0$ for all $P \leq S$.

Instead of counting orbits, an alternative way of characterising an S -set is counting the fixed points for each subgroup $P \leq S$. For every $P \leq S$ and S -set X , we denote the number of fixed points by $\Phi_P(X) := |X^P|$, and this number only depends on P up to S -conjugation. Since we have

$$|(X \sqcup Y)^P| = |X^P| + |Y^P|, \quad \text{and} \quad |(X \times Y)^P| = |X^P| \cdot |Y^P|$$

for all S -sets X and Y , the *fixed point map* $\Phi_P: A_+(S) \rightarrow \mathbb{Z}$ extends to a ring homomorphism $\Phi_P: A(S) \rightarrow \mathbb{Z}$. On the basis elements $[S/P]$, the number of fixed points is given by

$$\Phi_Q([S/P]) = |(S/P)^Q| = \frac{|N_S(Q, P)|}{|P|},$$

where $N_S(Q, P) = \{s \in S \mid {}^s Q \leq P\}$ is the transporter in S from Q to P . In particular, $\Phi_Q([S/P]) \neq 0$ if and only if $Q \lesssim_S P$ (Q is conjugate to a subgroup of P).

We have one fixed point homomorphism Φ_P per conjugacy class of subgroups in S , and we combine them into the *homomorphism of marks* $\Phi = \Phi^S: A(S) \xrightarrow{\prod_{[P]} \Phi_P} \prod_{[P] \in Cl(S)} \mathbb{Z}$.

This ring homomorphism maps $A(S)$ into the product ring $\tilde{\Omega}(S) := \prod_{[P] \in \text{Cl}(S)} \mathbb{Z}$ which is the so-called *ghost ring* for the Burnside ring $A(S)$.

Results by Dress and others show that the mark homomorphism is injective, and that the cokernel of Φ is the *obstruction group* $\text{Obs}(S) := \prod_{[P] \in \text{Cl}(S)} (\mathbb{Z}/|W_S P|\mathbb{Z})$ – where $W_S P := N_S P/P$. These statements are combined in the following proposition, the proof of which can be found in [5], [4, Chapter 1] and [12].

Proposition 3.1. *Let $\Psi = \Psi^S: \tilde{\Omega}(S) \rightarrow \text{Obs}(S)$ be given by the $[P]$ -coordinate functions*

$$\Psi_P(\xi) := \sum_{\bar{s} \in W_S P} \xi_{\langle s \rangle P} \pmod{|W_S P|}.$$

Here $\xi_{\langle s \rangle P}$ denotes the $[\langle s \rangle P]$ -coordinate of an element $\xi \in \tilde{\Omega}(S) = \prod_{[P] \in \text{Cl}(S)} \mathbb{Z}$.

The following sequence of abelian groups is then exact:

$$0 \rightarrow A(S) \xrightarrow{\Phi} \tilde{\Omega}(S) \xrightarrow{\Psi} \text{Obs}(S) \rightarrow 0.$$

Φ is a ring homomorphism, and Ψ is just a group homomorphism.

The strength of this result is that it enables one to perform calculations for the Burnside ring $A(S)$ inside the much nicer product ring $\tilde{\Omega}(S)$, where we identify each element $X \in A(S)$ with its fixed point vector $(\Phi_Q(X))_{[Q] \in \text{Cl}(S)}$.

4. STABLE SETS FOR A FUSION SYSTEM

Let \mathcal{F} be a fusion system over a p -group S . In this section we rephrase the property of \mathcal{F} -stability in terms of the fixed point homomorphisms, and show in example 4.3 how theorem A can fail for a group G if we only consider S -sets that are restrictions of G -sets, instead of considering all G -stable sets. We also consider two possible definitions for the Burnside ring of a fusion system – these agree if \mathcal{F} is saturated. The proof of theorem A begins in section 4.1 in earnest.

Recall that a finite S -set X is said to be \mathcal{F} -stable if it satisfies (1.2):

${}_{P,\varphi}X$ is isomorphic to ${}_{P,\text{incl}}X$ as P -sets, for all $P \leq S$ and homomorphisms $\varphi: P \rightarrow S$ in \mathcal{F} .

In order to define \mathcal{F} -stability not just for S -sets, but for all elements of the Burnside ring, we extend ${}_{P,\varphi}X$ to all $X \in A(S)$. Given a homomorphism $\varphi \in \mathcal{F}(P, S)$ and an S -set X , the P -set ${}_{P,\varphi}X$ was defined as X with the action restricted along φ , that is $p.x := \varphi(p)x$ for $x \in X$ and $p \in P$. This construction then extends linearly to a ring homomorphism $r_\varphi: A(S) \rightarrow A(P)$, and we denote ${}_{P,\varphi}X := r_\varphi(X)$ for all $X \in A(S)$. In this way (1.2) makes sense for all $X \in A(S)$.

Additionally, it is possible to state \mathcal{F} -stability purely in terms of fixed points and the homomorphism of marks for $A(S)$.

Lemma 4.1 ([7]). *The following are equivalent for all elements $X \in A(S)$:*

- (i) ${}_{P,\varphi}X = {}_{P,\text{incl}}X$ in $A(P)$ for all $\varphi \in \mathcal{F}(P, S)$ and $P \leq S$.
- (ii) $\Phi_P(X) = \Phi_{\varphi P}(X)$ for all $\varphi \in \mathcal{F}(P, S)$ and $P \leq S$.
- (iii) $\Phi_P(X) = \Phi_Q(X)$ for all pairs $P, Q \leq S$ with $P \sim_{\mathcal{F}} Q$.

We shall primarily use (ii) and (iii) to characterize \mathcal{F} -stability.

Proof. Let $\Phi^P: A(P) \rightarrow \widetilde{\Omega}(P)$ be the homomorphism of marks for $A(P)$, and note that $\Phi_R^P(P, \text{incl} X) = \Phi_R(X)$ for all $R \leq P \leq S$.

By the definition of the P -action on ${}_{P,\varphi}X$, we have $({}_{P,\varphi}X)^R = X^{\varphi R}$ for any S -set X and all subgroups $R \leq P$. This generalizes to

$$\Phi_R^P({}_{P,\varphi}X) = \Phi_{\varphi R}(X)$$

for $X \in A(S)$.

Assume (i). Then we immediately get

$$\Phi_P(X) = \Phi_P^P(P, \text{incl} X) = \Phi_P^P({}_{P,\varphi}X) = \Phi_{\varphi P}(X)$$

for all $P \leq S$ and $\varphi \in \mathcal{F}(P, S)$; which proves (i) \Rightarrow (ii).

Assume (ii). Let $P \leq S$ and $\varphi \in \mathcal{F}(P, S)$. By assumption, we have $\Phi_{\varphi R}(X) = \Phi_R(X)$ for all $R \leq P$, hence

$$\Phi_R^P({}_{P,\varphi}X) = \Phi_{\varphi R}(X) = \Phi_R(X) = \Phi_R^P(P, \text{incl} X).$$

Since Φ^P is injective, we get ${}_{P,\varphi}X = P, \text{incl} X$; so (ii) \Rightarrow (i).

Finally, we have (ii) \Leftrightarrow (iii) because Q is \mathcal{F} -conjugate to P exactly when Q is the image of a map $\varphi \in \mathcal{F}(P, S)$ in the fusion system. \square

Definition 4.2. We let $A_+(\mathcal{F}) \subseteq A_+(S)$ be the set of all the \mathcal{F} -stable sets, and by property (iii) the sums and products of stable elements are still stable, so $A_+(\mathcal{F})$ is a subsemiring of $A_+(S)$.

Suppose that $\mathcal{F} = \mathcal{F}_S(G)$ is the fusion system for a group with $S \in \text{Syl}_p(G)$. Let $X \in A_+(G)$ be a G -set, and let ${}_S X$ be the same set with the action restricted to the Sylow p -subgroup S . If we let $P \leq S$ and $c_g \in \text{Hom}_{\mathcal{F}_S(G)}(P, S)$ be given; then $x \mapsto gx$ is an isomorphism of P -sets ${}_{P, \text{incl}} X \cong {}_{P, c_g} X$. The restriction ${}_{S, \text{incl}} X$ is therefore G -stable.

Restricting the group action from G to S therefore defines a homomorphism of semirings $A_+(G) \rightarrow A_+(\mathcal{F}_S(G))$, but as the following example shows, this map need not be injective nor surjective.

Example 4.3. The symmetric group S_5 on 5 letters has Sylow 2-subgroups isomorphic to the dihedral group D_8 of order 8. We then consider D_8 as embedding in S_5 as one of the Sylow 2-subgroups. Let H, K be respectively Sylow 3- and 5-subgroups of S_5 .

The transitive S_5 -set $[S_5/H]$ contains 40 elements and all the stabilizers have odd order (they are conjugate to H). When we restrict the action to D_8 , the stabilizers therefore become trivial so the D_8 -action is free, hence $[S_5/H]$ restricts to the D_8 -set $5 \cdot [D_8/1]$, that is 5 disjoint copies of the free orbit $[D_8/1]$. Similarly, the transitive S_5 -set $[S_5/K]$ restricts to $3 \cdot [D_8/1]$.

These two restrictions of S_5 -sets are not linearly independent as D_8 -sets – the S_5 -sets $3 \cdot [S_5/H]$ and $5 \cdot [S_5/K]$ both restrict to $15 \cdot [D_8/1]$. If the restrictions of S_5 -sets were to form a free abelian monoid, then the set $[D_8/1]$ would have to be the restriction of an S_5 -set as well; and since $[D_8/1]$ is irreducible as a D_8 -set, it would have to be the restriction of an irreducible (hence transitive) S_5 -set. However S_5 has no subgroup of index 8, hence there is no transitive S_5 with 8 elements.

This shows that the restrictions of S_5 -sets to D_8 do not form a free abelian monoid, and we also see that $[D_8/1]$ is an example of an $\mathcal{F}_{D_8}(S_5)$ -stable set ($\Phi_1([D_8/1]) = 8$ and $\Phi_Q([D_8/1]) = 0$ for $1 \neq Q \leq D_8$) which cannot be given the structure of an S_5 -set.

To define the Burnside ring of a fusion system \mathcal{F} , we have two possibilities: We can consider the semiring of all the \mathcal{F} -stable S -sets and take the Grothendieck group of this. Alternatively, we can first take the Grothendieck group for all S -sets to get the Burnside ring of S , and then afterwards we consider the subring herein consisting of all the \mathcal{F} -stable elements. The following proposition implies that the two definitions coincide for saturated fusion systems.

Proposition 4.4. *Let \mathcal{F} be a fusion system over a p -group S , and consider the subsemiring $A_+(\mathcal{F})$ of \mathcal{F} -stable S -sets in the semiring $A_+(S)$ of finite S -sets.*

This inclusion induces a ring homomorphism from the Grothendieck group of $A_+(\mathcal{F})$ to the Burnside ring $A(S)$, which is injective.

If \mathcal{F} is saturated, then the image of the homomorphism is the subring of $A(S)$ consisting of the \mathcal{F} -stable elements.

Proof. Let Gr be the Grothendieck group of $A_+(\mathcal{F})$, and let $I: Gr \rightarrow A(S)$ be the induced map coming from the inclusion $i: A_+(\mathcal{F}) \hookrightarrow A_+(S)$.

An element of Gr is a formal difference $X - Y$ where X and Y are \mathcal{F} -stable sets. Assume now that $X - Y$ lies in $\ker I$. This means that $i(X) - i(Y) = 0$ in $A(S)$; and since $A_+(S)$ is a free commutative monoid, we conclude that $i(X) = i(Y)$ as S -sets. But i is just the inclusion map, so we must have $X = Y$ in $A_+(\mathcal{F})$ as well, and $X - Y = 0$ in Gr . Hence $I: Gr \rightarrow A(S)$ is injective.

It is clear that the difference of two \mathcal{F} -stable sets is still \mathcal{F} -stable, so $\text{im } I$ lies in the subring of \mathcal{F} -stable elements. If \mathcal{F} is saturated, then the converse holds, and all \mathcal{F} -stable elements of $A(S)$ can be written as a difference of \mathcal{F} -stable sets; however the proof of this must be postponed to corollary 4.11 below. \square

Definition 4.5. Let \mathcal{F} be saturated. We define *the Burnside ring of \mathcal{F}* , denoted $A(\mathcal{F})$, to be the subring consisting of the \mathcal{F} -stable elements in $A(S)$.

Once we have proven corollary 4.11, we will know that $A(\mathcal{F})$ is also the Grothendieck group of the semiring $A_+(\mathcal{F})$ of \mathcal{F} -stable sets.

4.1. Proving theorems A and B. The proof of theorem A falls into several parts: We begin by constructing some \mathcal{F} -stable sets α_P satisfying certain properties – this is the content of 4.6-4.8. We construct one α_P per \mathcal{F} -conjugacy class of subgroups, and these are the \mathcal{F} -stable sets which we will later show are the irreducible stable sets. A special case of the construction was originally used by Broto, Levi and Oliver in [2, Proposition 5.5] to show that every saturated fusion system has a characteristic biset.

In 4.9-4.11 we then proceed to show that the constructed α_P 's are linearly independent, and that they generate the Burnside ring $A(\mathcal{F})$. When proving that the α_P 's generate $A(\mathcal{F})$, the same proof also establishes theorem B.

Finally, we use the fact that the α_P 's form a basis for the Burnside ring, to argue that they form an additive basis already for the semiring $A_+(\mathcal{F})$, completing the proof of theorem A itself.

As mentioned, we first construct an \mathcal{F} -stable set α_P for each \mathcal{F} -conjugation class of subgroups. The idea when constructing α_P is that we start with the single orbit $[S/P]$ which we then stabilize: We run through the subgroups $Q \leq S$ in decreasing order and add orbits to the constructed S -set such that it becomes \mathcal{F} -stable at the conjugacy class of Q in \mathcal{F} . The stabilization procedure is handled in the following technical lemma 4.6, which is then applied in proposition 4.8 to construct the α_P 's.

Recall that $c_P(X)$ denotes the number of (S/P) -orbits in X , and $\Phi_P(X)$ denotes the number of P -fixed points.

Lemma 4.6. *Let \mathcal{F} be a saturated fusion system over a p -group S , and let \mathcal{H} be a collection of subgroups of S such that \mathcal{H} is closed under taking \mathcal{F} -subconjugates, i.e. if $P \in \mathcal{H}$, then $Q \in \mathcal{H}$ for all $Q \lesssim_{\mathcal{F}} P$.*

Assume that $X \in A_+(S)$ is an S -set satisfying $\Phi_P(X) = \Phi_{P'}(X)$ for all pairs $P \sim_{\mathcal{F}} P'$, with $P, P' \notin \mathcal{H}$. Assume furthermore that $c_P(X) = 0$ for all $P \in \mathcal{H}$.

Then there exists an \mathcal{F} -stable set $X' \in A_+(\mathcal{F}) \subseteq A_+(S)$ satisfying $\Phi_P(X') = \Phi_P(X)$ and $c_P(X') = c_P(X)$ for all $P \notin \mathcal{H}$; and also satisfying $c_P(X') = c_P(X)$ for all $P \leq S$ which are fully normalized in \mathcal{F} . In particular, for a $P \in \mathcal{H}$ which is fully normalized, we have $c_P(X') = 0$.

Proof. We proceed by induction on the size of the collection \mathcal{H} . If $\mathcal{H} = \emptyset$, then X is \mathcal{F} -stable by assumption, so $X' := X$ works.

Assume that $\mathcal{H} \neq \emptyset$, and let $P \in \mathcal{H}$ be maximal under \mathcal{F} -subconjugation as well as fully normalized.

Let $P' \sim_{\mathcal{F}} P$. Then there is a homomorphism $\varphi \in \mathcal{F}(N_S P', N_S P)$ with $\varphi(P') = P$ by lemma 2.3 since \mathcal{F} is saturated. The restriction of S -actions to $\varphi(N_S P')$ gives a ring homomorphism $A(S) \rightarrow A(\varphi(N_S P'))$ that preserves the fixed-point homomorphisms Φ_Q for $Q \leq \varphi(N_S P') \leq N_S P$.

If we consider the S -set X as an element of $A(\varphi(N_S P'))$, we can apply the short exact sequence of proposition 3.1 to get $\Psi^{\varphi(N_S P')}(\Phi^{\varphi(N_S P')}(X)) = 0$. In particular, the P -coordinate function satisfies $\Psi_P^{\varphi(N_S P')}(\Phi^{\varphi(N_S P')}(X)) = 0$, that is

$$\sum_{\bar{s} \in \varphi(N_S P')/P} \Phi_{\langle s \rangle P}(X) \equiv 0 \pmod{|\varphi(N_S P')/P|}.$$

Similarly, we have $\Psi^S(\Phi^S(X)) = 0$, where the P' -coordinate $\Psi_{P'}^S(\Phi^S(X)) = 0$ gives us

$$\sum_{\bar{s} \in N_S P'/P'} \Phi_{\langle s \rangle P'}(X) \equiv 0 \pmod{|N_S P'/P'|}.$$

Since P is maximal in \mathcal{H} , we have by assumption $\Phi_Q(X) = \Phi_{Q'}(X)$ for all $Q \sim_{\mathcal{F}} Q'$ where P is \mathcal{F} -conjugate to a *proper* subgroup of Q . Specifically, we have

$$\Phi_{\langle \varphi(s) \rangle P}(X) = \Phi_{\langle \varphi(s) \rangle P'}(X) = \Phi_{\langle s \rangle P'}(X)$$

for all $s \in N_S P'$ with $s \notin P'$. It then follows that

$$\begin{aligned} \Phi_P(X) - \Phi_{P'}(X) &= \sum_{\bar{s} \in \varphi(N_S P')/P} \Phi_{\langle s \rangle P}(X) - \sum_{\bar{s} \in N_S P'/P'} \Phi_{\langle s \rangle P'}(X) \\ &\equiv 0 - 0 \pmod{|W_S P'|}. \end{aligned}$$

We can therefore define $\lambda_{P'} := (\Phi_P(X) - \Phi_{P'}(X))/|W_S P'| \in \mathbb{Z}$.

Using the $\lambda_{P'}$ as coefficients, we construct a new S -set

$$\tilde{X} := X + \sum_{[P']_S \subseteq [P]_{\mathcal{F}}} \lambda_{P'} \cdot [S/P'] \in A(S).$$

Here $[P]_{\mathcal{F}}$ is the collection of subgroups that are \mathcal{F} -conjugate to P . The sum is then taken over one representative from each S -conjugacy class contained in $[P]_{\mathcal{F}}$.

A priori, the $\lambda_{P'}$ might be negative, and as a result \tilde{X} might not be an S -set. In the original construction of [2], this problem is circumvented by adding copies of

$$\sum_{[P']_S \subseteq [P]_{\mathcal{F}}} \frac{|N_S P|}{|N_S P'|} \cdot [S/P']$$

until all the coefficients are non-negative.

It will however be shown in lemma 4.7 below, that under the assumption that $c_{P'}(X) = 0$ for $P' \sim_{\mathcal{F}} P$, then $\lambda_{P'}$ is always non-negative, and $\lambda_{P'} = 0$ if P' is fully normalized. Hence \tilde{X} is already an S -set without further adjustments.

We clearly have $c_Q(\tilde{X}) = c_Q(X)$ for all $Q \not\sim_{\mathcal{F}} P$, in particular for all $Q \notin \mathcal{H}$. Furthermore, if $P' \sim_{\mathcal{F}} P$ is fully normalized, then $c_{P'}(\tilde{X}) = c_{P'}(X) + \lambda_{P'} = c_{P'}(X)$.

Because $\Phi_Q([S/P']) = 0$ unless $Q \lesssim_S P'$, we see that $\Phi_Q(\tilde{X}) = \Phi_Q(X)$ for every $Q \notin \mathcal{H}$. Secondly, we calculate $\Phi_{P'}(\tilde{X})$ for each $P' \sim_{\mathcal{F}} P$:

$$\begin{aligned} \Phi_{P'}(\tilde{X}) &= \Phi_{P'}(X) + \sum_{[\tilde{P}]_S \subseteq [P]_{\mathcal{F}}} \lambda_{\tilde{P}} \cdot \Phi_{P'}([S/\tilde{P}]) \\ &= \Phi_{P'}(X) + \lambda_{P'} \cdot \Phi_{P'}([S/P']) = \Phi_{P'}(X) + \lambda_{P'} |W_S P'| \\ &= \Phi_P(X); \end{aligned}$$

which is independent of the choice of $P' \sim_{\mathcal{F}} P$.

We define $\mathcal{H}' := \mathcal{H} \setminus [P]_{\mathcal{F}}$ as \mathcal{H} with the \mathcal{F} -conjugates of P removed. Because P is maximal in \mathcal{H} , the subcollection \mathcal{H}' again contains all \mathcal{F} -subconjugates of any $H \in \mathcal{H}'$.

By induction we can apply lemma 4.6 to \tilde{X} and the smaller collection \mathcal{H}' . We get an $X' \in A_+(\mathcal{F})$ with $\Phi_Q(X') = \Phi_Q(\tilde{X})$ and $c_Q(X') = c_Q(\tilde{X})$ for all $Q \notin \mathcal{H}'$; and such that $c_Q(X') = 0$ if $Q \in \mathcal{H}'$ is fully normalized.

It follows that $\Phi_Q(X') = \Phi_Q(\tilde{X}) = \Phi_Q(X)$ and $c_Q(X') = c_Q(\tilde{X}) = c_Q(X)$ for all $Q \notin \mathcal{H}$, and we also have $c_Q(X') = 0$ if $Q \in \mathcal{H}$ is fully normalized. \square

Lemma 4.7. *Let \mathcal{F} be a saturated fusion system over a p -group S , and let $P \leq S$ be a fully normalized subgroup.*

Suppose that X is an S -set with $c_{P'}(X) = 0$ for all $P' \sim_{\mathcal{F}} P$, and satisfying that X is already \mathcal{F} -stable for subgroups larger than P , i.e. $|X^R| = |X^{R'}|$ for all $R \sim_{\mathcal{F}} R'$ where P is \mathcal{F} -conjugate to a proper subgroup of R .

Then $|X^P| \geq |X^{P'}|$ for all $P' \sim_{\mathcal{F}} P$.

Proof. Let $Q \sim_{\mathcal{F}} P$ be given. Because P is fully normalized, there exists by lemma 2.3 a homomorphism $\varphi: N_S Q \hookrightarrow N_S P$ in \mathcal{F} , with $\varphi(Q) = P$.

Let A_1, \dots, A_k be the subgroups of $N_S Q$ that strictly contain Q , i.e. $Q < A_i \leq N_S Q$. We put $B_i := \varphi(A_i)$, and thus also have $P < B_i \leq N_S P$. We let C_1, \dots, C_ℓ be the subgroups of $N_S P$ strictly containing P which are not the image (under φ) of some A_i . Hence $B_1, \dots, B_k, C_1, \dots, C_\ell$ are all the different subgroups of $N_S P$ strictly containing P . We denote the set $\{1, \dots, k\}$ of indices by I , and also $J := \{1, \dots, \ell\}$.

Because $c_Q(X) = c_P(X) = 0$ by assumption, no orbit of X is isomorphic to S/Q , hence no element in X^Q has Q as a stabilizer. Let $x \in X^Q$ be any element, and let $K > Q$ be the stabilizer of x ; so $x \in X^K \subseteq X^Q$. Since K is a p -group, there is some intermediate group L with $Q \triangleleft L \leq K$; hence $x \in X^L$ for some $Q < L \leq N_S Q$. We conclude that

$$X^Q = \bigcup_{i \in I} X^{A_i}.$$

With similar reasoning we also get

$$X^P = \bigcup_{i \in I} X^{B_i} \cup \bigcup_{j \in J} X^{C_j}.$$

The proof is then completed by showing

$$|X^P| = \left| \bigcup_{i \in I} X^{B_i} \cup \bigcup_{j \in J} X^{C_j} \right| \geq \left| \bigcup_{i \in I} X^{B_i} \right| \stackrel{(*)}{=} \left| \bigcup_{i \in I} X^{A_i} \right| = |X^Q|.$$

We only need to prove the equality (*).

Showing (*) has only to do with fixed points for the subgroups A_i and B_i ; and because $B_i = \varphi(A_i) \sim_{\mathcal{F}} A_i$ are subgroups that strictly contain P and Q respectively, we have $|X^{B_i}| = |X^{A_i}|$ by assumption.

To get (*) for the unions $\cup A_i$ and $\cup B_i$ we then have to apply the inclusion-exclusion principle:

$$\left| \bigcup_{i \in I} X^{B_i} \right| = \sum_{\emptyset \neq \Lambda \subseteq I} (-1)^{|\Lambda|+1} \left| \bigcap_{i \in \Lambda} X^{B_i} \right| = \sum_{\emptyset \neq \Lambda \subseteq I} (-1)^{|\Lambda|+1} |X^{\langle B_i \rangle_{i \in \Lambda}}|.$$

Here $\langle B_i \rangle_{i \in \Lambda} \leq N_S P$ is the subgroup generated by the elements of B_i 's with $i \in \Lambda \subseteq I$. Recalling that $B_i = \varphi(A_i)$ by definition, we have $\langle B_i \rangle_{i \in \Lambda} = \langle \varphi(A_i) \rangle_{i \in \Lambda} = \varphi(\langle A_i \rangle_{i \in \Lambda})$, and consequently

$$\sum_{\emptyset \neq \Lambda \subseteq I} (-1)^{|\Lambda|+1} |X^{\langle B_i \rangle_{i \in \Lambda}}| = \sum_{\emptyset \neq \Lambda \subseteq I} (-1)^{|\Lambda|+1} |X^{\varphi(\langle A_i \rangle_{i \in \Lambda})}|.$$

Because $Q < A_i \leq N_S Q$, we also have $Q < \langle A_i \rangle_{i \in \Lambda} \leq N_S Q$, by assumption we therefore get $|X^{\varphi(\langle A_i \rangle_{i \in \Lambda})}| = |X^{\langle A_i \rangle_{i \in \Lambda}}|$ for all $\emptyset \neq \Lambda \subseteq I$. It then follows that

$$\sum_{\emptyset \neq \Lambda \subseteq I} (-1)^{|\Lambda|+1} |X^{\varphi(\langle A_i \rangle_{i \in \Lambda})}| = \sum_{\emptyset \neq \Lambda \subseteq I} (-1)^{|\Lambda|+1} |X^{\langle A_i \rangle_{i \in \Lambda}}| = \dots = \left| \bigcup_{i \in I} X^{A_i} \right|,$$

where we use the inclusion-exclusion principle in reverse. We have thus shown the equality $\left| \bigcup_{i \in I} X^{B_i} \right| = \left| \bigcup_{i \in I} X^{A_i} \right|$ as required. \square

Applying the technical lemma 4.6, we can now construct the irreducible \mathcal{F} -stable sets α_P for $P \leq S$ as described in the following proposition. That the α_P 's are in fact irreducible, or even that they are unique, will not be shown until the proof of theorem A itself.

Proposition 4.8. *Let \mathcal{F} be a saturated fusion system over a p -group S .*

For each \mathcal{F} -conjugacy class $[P]_{\mathcal{F}} \in Cl(\mathcal{F})$ of subgroups, there is an \mathcal{F} -stable set $\alpha_P \in A_+(\mathcal{F})$ such that

- (i) $\Phi_Q(\alpha_P) = 0$ unless Q is \mathcal{F} -subconjugate to P .
- (ii) $c_{P'}(\alpha_P) = 1$ and $\Phi_{P'}(\alpha_P) = |W_S P'|$ when P' is fully normalized and $P' \sim_{\mathcal{F}} P$.
- (iii) $c_Q(\alpha_P) = 0$ when Q is fully normalized and $Q \not\sim_{\mathcal{F}} P$.

Proof. Let $P \leq S$ be fully \mathcal{F} -normalized. We let $X \in A_+(S)$ be the S -set

$$X := \sum_{[P']_S \subseteq [P]_{\mathcal{F}}} \frac{|N_S P|}{|N_S P'|} \cdot [S/P'] \in A_+(S).$$

X then satisfies that $\Phi_Q(X) = 0$ unless $Q \lesssim_S P'$ for some $P' \sim_{\mathcal{F}} P$, in which case we have $Q \lesssim_{\mathcal{F}} P$. For all $P', P'' \in [P]_{\mathcal{F}}$ we have $\Phi_{P''}([S/P']) = 0$ unless $P'' \sim_S P'$; and consequently

$$\Phi_{P'}(X) = \frac{|N_S P|}{|N_S P'|} \cdot \Phi_{P'}([S/P']) = \frac{|N_S P|}{|N_S P'|} \cdot |W_S P'| = |W_S P|$$

which doesn't depend on $P' \sim_{\mathcal{F}} P$.

Let \mathcal{H} be the collection of all Q which are \mathcal{F} -conjugate to a *proper* subgroup of P , then $\Phi_Q(X) = \Phi_{Q'}(X)$ for all pairs $Q \sim_{\mathcal{F}} Q'$ not in \mathcal{H} . Using lemma 4.6 we get some $\alpha_P \in A_+(\mathcal{F})$ with the required properties. \square

Properties (ii) and (iii) make it really simple to decompose a linear combination X of the α_P 's. The coefficient of α_P in X is just the number of $[S/P]$ -orbits in X as an S -set - when P is fully normalized. This is immediate since α_P contains exactly one copy of $[S/P]$, and no other α_Q contains $[S/P]$.

In particular we have:

Corollary 4.9. *The α_P 's in proposition 4.8 are linearly independent.*

In order to prove that the α_P 's generate all \mathcal{F} -stable sets, we will first show that the α_P 's generate all the \mathcal{F} -stable elements in the Burnside ring. As a tool for proving this, we define a ghost ring for the Burnside ring $A(\mathcal{F})$; and as consequence of how the proof proceeds, we end up showing an analogue of proposition 3.1 for saturated fusion systems, describing how the Burnside ring $A(\mathcal{F})$ lies embedded in the ghost ring - this is the content of theorem B.

Definition 4.10. Recall how the ghost ring $\tilde{\Omega}(S)$ for the Burnside ring of a group is defined as the product ring $\prod_{[P]_S \in Cl(S)} \mathbb{Z}$ where the coordinates correspond to the S -conjugacy classes of subgroups. For the ring $A(\mathcal{F})$, we now similarly define *the ghost ring* $\tilde{\Omega}(\mathcal{F})$ as a product ring $\prod_{[P]_{\mathcal{F}} \in Cl(\mathcal{F})} \mathbb{Z}$ with coordinates corresponding to the \mathcal{F} -conjugacy classes of subgroups.

The surjection of indexing sets $Cl(S) \rightarrow Cl(\mathcal{F})$ which sends an S -conjugacy class $[P]_S$ to its \mathcal{F} -conjugacy class $[P]_{\mathcal{F}}$, induces a homomorphism $\tilde{\Omega}(\mathcal{F}) \hookrightarrow \tilde{\Omega}(S)$ that embeds $\tilde{\Omega}(\mathcal{F})$ as the subring of vectors which are constant on each \mathcal{F} -conjugacy class.

Since $A(\mathcal{F})$ is the subring of \mathcal{F} -stable elements in $A(S)$, we can restrict the mark homomorphism $\Phi^S: A(S) \rightarrow \tilde{\Omega}(S)$ to the subring $A(\mathcal{F})$ and get an injective ring homomorphism $\Phi^{\mathcal{F}}: A(\mathcal{F}) \rightarrow \tilde{\Omega}(\mathcal{F})$ - which is the *homomorphism of marks* for $A(\mathcal{F})$.

To model the cokernel of $\Phi^{\mathcal{F}}$ we define $Obs(\mathcal{F})$ as

$$Obs(\mathcal{F}) := \prod_{\substack{[P] \in Cl(\mathcal{F}) \\ P \text{ f.n.}}} (\mathbb{Z}/|W_S P|\mathbb{Z}),$$

where 'f.n.' is short for 'fully normalized', so we take fully normalized representatives of the conjugacy classes in \mathcal{F} .

Theorem B. *Let \mathcal{F} be a saturated fusion system over a p -group S , and let $A(\mathcal{F})$ be the Burnside ring of \mathcal{F} – i.e. the subring consisting of the \mathcal{F} -stable elements in the Burnside ring of S .*

We then have a short-exact sequence

$$0 \rightarrow A(\mathcal{F}) \xrightarrow{\Phi} \tilde{\Omega}(\mathcal{F}) \xrightarrow{\Psi} \text{Obs}(\mathcal{F}) \rightarrow 0.$$

where $\Phi = \Phi^{\mathcal{F}}$ is the homomorphism of marks, and $\Psi = \Psi^{\mathcal{F}}: \tilde{\Omega}(\mathcal{F}) \rightarrow \text{Obs}(\mathcal{F})$ is a group homomorphism given by the $[P]$ -coordinate functions

$$\Psi_P(\xi) := \sum_{\bar{s} \in W_S P} \xi_{\langle s \rangle P} \pmod{|W_S P|}$$

when P is a fully normalized representative of the conjugacy class $[P]$ in \mathcal{F} . Here $\Psi_P = \Psi_{P'}$ if $P \sim_{\mathcal{F}} P'$ are both fully normalized.

Proof. We choose some total order of the conjugacy classes $[P], [Q] \in \text{Cl}(\mathcal{F})$ such that $|P| > |Q|$ implies $[P] < [Q]$, i.e. we take the subgroups in decreasing order. It holds in particular that $Q \lesssim_{\mathcal{F}} P$ implies $[P] \leq [Q]$.

With respect to the ordering above, the group homomorphism Ψ is given by a lower triangular matrix with 1's in the diagonal, hence Ψ is surjective. The mark homomorphism $\Phi = \Phi^{\mathcal{F}}$ is the restriction of the injective ring homomorphism $\Phi^S: A(S) \rightarrow \tilde{\Omega}(S)$, so Φ is injective.

We know from the group case, proposition 3.1, that $\Psi^S \circ \Phi^S = 0$. By construction we have $(\Psi)_P = (\Psi^S)_P$ for the coordinate functions when P is fully normalized; and Φ is the restriction of Φ^S . We conclude that $\Psi \circ \Phi = 0$ as well. It remains to be shown that $\text{im } \Phi$ is actually all of $\ker \Psi$.

Consider the subgroup $H := \text{Span}\{\alpha_P \mid [P] \in \text{Cl}(\mathcal{F})\}$ spanned by the α_P 's in $A(\mathcal{F})$, and consider also the restriction $\Phi|_H$ of the mark homomorphism $\Phi: A(\mathcal{F}) \rightarrow \tilde{\Omega}(\mathcal{F})$.

$\Phi|_H$ is described by a square matrix M in terms of the ordered bases of $H = \text{Span}\{\alpha_P$'s} and $\tilde{\Omega}(\mathcal{F})$. Because $M_{[Q],[P]} := \Phi_Q(\alpha_P)$ is zero unless $P \sim_{\mathcal{F}} Q$ or $|P| > |Q|$, we conclude that M is a lower triangular matrix. The diagonal entries of M are

$$M_{[P],[P]} = \Phi_P(\alpha_P) = |W_S P|,$$

when P is fully normalized.

All the diagonal entries are non-zero, so the cokernel of $\Phi|_H$ is finite of order

$$|\text{coker } \Phi|_H| = \prod_{[P] \in \text{Cl}(\mathcal{F})} M_{[P],[P]} = \prod_{\substack{[P] \in \text{Cl}(\mathcal{F}) \\ P \text{ f.n.}}} |W_S P|.$$

Since $\Phi|_H$ is a restriction of Φ , it follows that $|\text{coker } \Phi| \leq |\text{coker } \Phi|_H|$. At the same time, $\Psi \circ \Phi = 0$ implies that $|\text{coker } \Phi| \geq |\text{Obs}(\mathcal{F})|$.

We do however have

$$|\text{Obs}(\mathcal{F})| = \prod_{\substack{[P] \in \text{Cl}(\mathcal{F}) \\ P \text{ f.n.}}} |W_S P| = |\text{coker } \Phi|_H|.$$

The only possibility is that $\ker \Psi = \text{im } \Phi = \text{im } \Phi|_H$, completing the proof of theorem B. \square

From the last equality $\text{im } \Phi = \text{im } \Phi|_H$, and the fact that Φ is injective, it also follows that $A(\mathcal{F}) = H$ so the α_P 's span all of $A(\mathcal{F})$. Combining this with corollary 4.9 we get:

Corollary 4.11. *The α_P 's form an additive basis for the Burnside ring $A(\mathcal{F})$.*

The corollary tells us that any element $X \in A(\mathcal{F})$ can be written uniquely as an integral linear combination of the α_P 's. In particular, any \mathcal{F} -stable set can be written as a linear combination of α_P 's, and if the coefficients are all non-negative, then we have a linear combination in $A_+(\mathcal{F})$.

Theorem A. *Let \mathcal{F} be a saturated fusion system over a p -group S .*

The sets α_P in proposition 4.8 are all the irreducible \mathcal{F} -stable sets, and every \mathcal{F} -stable set splits uniquely (up to S -isomorphism) as a disjoint union of the α_P 's.

Hence the semiring $A_+(\mathcal{F})$ of \mathcal{F} -stable sets is additively a free commutative monoid with rank equal to the number of conjugacy classes of subgroups in \mathcal{F} .

Proof. Let $\alpha_P \in A_+(\mathcal{F})$ for each conjugacy class $[P] \in Cl(\mathcal{F})$ be given as in proposition 4.8. Let $X \in A_+(\mathcal{F})$ be any \mathcal{F} -stable S -set.

Since the α_P 's form a basis for $A(\mathcal{F})$ by corollary 4.11, we can write X uniquely as

$$X = \sum_{[P] \in Cl(\mathcal{F})} \lambda_P \cdot \alpha_P$$

with $\lambda_P \in \mathbb{Z}$.

Suppose that P is fully normalized, then $c_P(\alpha_Q) = 1$ if $P \sim_{\mathcal{F}} Q$, and $c_P(\alpha_Q) = 0$ otherwise. As a consequence of this, we have

$$c_P(X) = \sum_{[Q] \in Cl(\mathcal{F})} \lambda_Q \cdot c_P(\alpha_Q) = \lambda_P$$

whenever P is fully normalized.

Because X is an S -set, we see that $\lambda_P = c_P(X) \geq 0$. Hence the linear combination $X = \sum_{[P] \in Cl(\mathcal{F})} \lambda_P \cdot \alpha_P$ has nonnegative coefficients, i.e. it is a linear combination in the semiring $A_+(\mathcal{F})$.

As a special case, if we have another element α'_P in $A(\mathcal{F})$ satisfying the properties of proposition 4.8, then the fact that $\lambda_Q = c_Q(\alpha'_P)$ for all fully normalized $Q \leq S$, shows that $\lambda_P = 1$ and $\lambda_Q = 0$ for $Q \not\sim_{\mathcal{F}} P$. Thus the linear combination above simplifies to $\alpha'_P = \alpha_P$. Hence the α_P 's are uniquely determined by the properties of proposition 4.8. \square

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TRANSFER AND CHARACTERISTIC IDEMPOTENTS FOR SATURATED FUSION SYSTEMS

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ABSTRACT. We construct a transfer map to the p -local Burnside ring of a saturated fusion system \mathcal{F} from the p -local Burnside ring of the underlying p -group S . Using such transfer maps, we give a new explicit construction of the characteristic idempotent of \mathcal{F} – the unique idempotent in the p -local double Burnside ring of S satisfying properties of Linckelmann and Webb. We describe this idempotent both in terms of fixed points and as a linear combination of transitive bisets. Additionally, using fixed points we determine the map for Burnside rings given by multiplication with the characteristic idempotent, and show that this is the transfer map previously constructed. Applying these results, we show that for every saturated fusion system the ring generated by all (non-idempotent) characteristic elements in the p -local double Burnside ring is isomorphic to the p -local “single” Burnside ring of the fusion system, and we disprove a conjecture by Park-Ragnarsson-Stancu on the composition product of fusion systems.

1. INTRODUCTION

Saturated fusion systems are abstract models for the p -local structure of finite groups. The canonical example comes from a finite group G with Sylow p -subgroup S . The fusion system $\mathcal{F}_S(G)$ associated to G (and S) is a category whose objects are the subgroups of S and where the morphisms between subgroups are the homomorphisms induced by conjugation by elements of G . As shown by Ragnarsson-Stancu in [12, 13], there is a one-to-one correspondence between the saturated fusion systems on a finite p -group S and their associated characteristic idempotents in $A(S, S)_{(p)}$, the p -localized double Burnside ring of S .

In this paper we introduce a transfer map $\pi: A(S)_{(p)} \rightarrow A(\mathcal{F})_{(p)}$ between Burnside rings for a saturated fusion system \mathcal{F} and its underlying p -group S . By using this transfer map we give a new explicit construction of the characteristic idempotent $\omega_{\mathcal{F}}$ for a saturated fusion system \mathcal{F} . This enables us to calculate the fixed points and coefficients of $\omega_{\mathcal{F}}$ and give a precise description of the products $\omega_{\mathcal{F}} \circ X$ and $X \circ \omega_{\mathcal{F}}$ for any element X of the double Burnside ring. We give an application of these results to a conjecture by Park-Ragnarsson-Stancu on the composition product of saturated fusion systems.

In more detail, we first consider the transfer map for Burnside rings of fusion systems: The Burnside ring $A(S)$ for a finite p -group S is the Grothendieck group formed from the monoid of isomorphism classes of finite S -sets, with disjoint union as addition and cartesian product as multiplication. Let

$$\Phi: A(S) \rightarrow \prod_{\substack{Q \leq S \\ \text{up to } S\text{-conj.}}} \mathbb{Z}$$

be the homomorphism of marks, i.e., the injective ring homomorphism whose Q -coordinate $\Phi_Q(X)$ counts the number of fixed points $|X^Q|$ when X is an S -set. Given a fusion system \mathcal{F} on S , we say that a finite S -set X , or a general element of $A(S)$, is \mathcal{F} -stable if the action

Supported by the Danish National Research Foundation through the Centre for Symmetry and Deformation (DNRF92).

on X is invariant under conjugation in \mathcal{F} – see section 3.1. The \mathcal{F} -stable elements form a subring of $A(S)$ which we call *the Burnside ring of \mathcal{F}* and denote by $A(\mathcal{F})$.

Theorem A. *Let \mathcal{F} be a saturated fusion system on a finite p -group S . We let $A(\mathcal{F})_{(p)}$ denote the p -localized Burnside ring of \mathcal{F} as a subring of the p -localized Burnside ring $A(S)_{(p)}$ for S . Then there is a transfer map $\pi: A(S)_{(p)} \rightarrow A(\mathcal{F})_{(p)}$, which is a homomorphism of $A(\mathcal{F})_{(p)}$ -modules and which restricts to the identity on $A(\mathcal{F})_{(p)}$. In terms of the fixed point homomorphisms the transfer map π satisfies*

$$\Phi_Q(\pi(X)) = \frac{1}{|[Q]_{\mathcal{F}}|} \sum_{Q' \in [Q]_{\mathcal{F}}} \Phi_{Q'}(X),$$

where $[Q]_{\mathcal{F}}$ is the conjugacy class of Q in \mathcal{F} .

If we apply the π to the transitive S -sets S/P for $P \leq S$, we get elements $\beta_P := \pi(S/P)$, which form a $\mathbb{Z}_{(p)}$ -basis for the p -localized Burnside ring $A(\mathcal{F})_{(p)}$ by proposition 4.5, and where $\beta_P = \beta_Q$ if and only if P and Q are conjugate in \mathcal{F} . In proposition 4.7, we show that when \mathcal{F} arises from a finite group G with Sylow p -subgroup S , then the basis elements β_P are closely related to the transitive G -sets G/P for $P \leq S$, and the p -localized Burnside ring $A(\mathcal{F})_{(p)}$ is isomorphic to the part of $A(G)_{(p)}$ where all stabilizers are p -subgroups.

The (double) Burnside module $A(S, T)$ is defined for a pair of p -groups similarly to the Burnside ring of a group, except that we consider isomorphism classes of (S, T) -bisets, which are sets equipped with both a right S -action and a left T -action that commute with each other. The Burnside module $A(S, T)$ is then the Grothendieck group of the monoid formed by isomorphism classes of finite (S, T) -bisets with disjoint union as addition. The (S, T) -bisets correspond to sets with a left $(T \times S)$ -action, and the transitive bisets correspond to transitive sets $(T \times S)/D$ for subgroups $D \leq T \times S$. Note that we do not make the usual requirement that the bisets have a free left action, and the results below hold for non-free bisets as well.

For every triple of p -groups S, T, U we have a composition map $\circ: A(T, U) \times A(S, T) \rightarrow A(S, U)$ given on bisets by $Y \circ X := Y \times_T X = Y \times X / \sim$ where $(yt, x) \sim (y, tx)$ for all $y \in Y, x \in X$, and $t \in T$. For each $D \leq T \times S$ we have a fixed point homomorphism $\Phi_D: A(S, T) \rightarrow \mathbb{Z}$, but it is only a homomorphism of abelian groups. An element $X \in A(S, T)$ is still fully determined by the number of fixed points $\Phi_D(X)$ for $D \leq T \times S$. Subgroups in $T \times S$ of particular interest are the graphs of homomorphisms $\varphi: P \rightarrow T$ for $P \leq S$, where the graph of $\varphi: P \rightarrow T$ is the subgroup $\Delta(P, \varphi) := \{(\varphi(g), g) \mid g \in P\}$. The transitive $(T \times S)$ -set $(T \times S)/\Delta(P, \varphi)$ corresponds to a transitive (S, T) -biset whose isomorphism class we denote by $[P, \varphi]_S^T$.

Given a saturated fusion system \mathcal{F} , a particularly nice class of elements in the p -localized double Burnside ring $A(S, S)_{(p)}$ are the \mathcal{F} -characteristic elements, which satisfy the following properties put down by Linckelmann-Webb: An element $X \in A(S, S)_{(p)}$ is \mathcal{F} -characteristic if it is

\mathcal{F} -generated: X is a linear combination of the (S, S) -bisets $[P, \varphi]_S^S$ where $\varphi: P \rightarrow S$ is a morphism of \mathcal{F} ,

Right \mathcal{F} -stable: For all $P \leq S$ and $\varphi \in \mathcal{F}(P, S)$ we have $X \circ [P, \varphi]_P^S = X \circ [P, id]_P^S$ as elements of $A(P, S)_{(p)}$,

Left \mathcal{F} -stable: For all $P \leq S$ and $\varphi \in \mathcal{F}(P, S)$ we have $[\varphi P, \varphi^{-1}]_S^P \circ X = [P, id]_S^P \circ X$ as elements of $A(P, S)_{(p)}$,

and an additional technical condition to ensure that X is not degenerate.

In this paper we give a new proof that every saturated fusion system \mathcal{F} on S has an associated element $\omega_{\mathcal{F}} \in A(S, S)_{(p)}$ that is both \mathcal{F} -characteristic and idempotent. To construct $\omega_{\mathcal{F}}$ we consider the product fusion system $\mathcal{F} \times \mathcal{F}$ on $S \times S$ and apply the transfer map of theorem A to $S \times S / \Delta(S, id)$. The resulting element $\beta_{\Delta(S, id)}$ then turns out to be both \mathcal{F} -characteristic and idempotent when considered as an element of $A(S, S)_{(p)}$. The new construction of the characteristic idempotent for instance enables us to calculate the fixed points of $\omega_{\mathcal{F}}$:

Theorem B. *Let \mathcal{F} be a saturated fusion system on a finite p -group S . Then there exists a unique \mathcal{F} -characteristic idempotent $\omega_{\mathcal{F}} \in A(S, S)_{(p)}$, and it satisfies:*

For all graphs $\Delta(P, \varphi) \leq S \times S$ with $\varphi \in \mathcal{F}(P, S)$, we have

$$\Phi_{\Delta(P, \varphi)}(\omega_{\mathcal{F}}) = \frac{|S|}{|\mathcal{F}(P, S)|};$$

and $\Phi_D(\omega_{\mathcal{F}}) = 0$ for all other subgroups $D \leq S \times S$. Consequently, if we write $\omega_{\mathcal{F}}$ in the basis of $A(S, S)_{(p)}$, we get the expression

$$\omega_{\mathcal{F}} = \sum_{\substack{[\Delta(P, \varphi)]_{S \times S} \\ \text{with } \varphi \in \mathcal{F}(P, S)}} \frac{|S|}{\Phi_{\Delta(P, \varphi)}([P, \varphi]_S^S)} \left(\sum_{P \leq Q \leq S} \frac{|\{\psi \in \mathcal{F}(Q, S) \mid \psi|_P = \varphi\}|}{|\mathcal{F}(Q, S)|} \cdot \mu(P, Q) \right) [P, \varphi]_S^S,$$

where the outer sum is taken over $(S \times S)$ -conjugacy classes of subgroups, and where μ is the Möbius function for the poset of subgroups in S .

A closer look on the way theorem A is applied to construct $\omega_{\mathcal{F}}$ reveals an even closer relationship between the transfer map and the characteristic idempotent, and we get a precise description of what happens when other elements are multiplied by $\omega_{\mathcal{F}}$:

Theorem C. *Let \mathcal{F}_1 and \mathcal{F}_2 be saturated fusion systems on finite p -groups S_1 and S_2 respectively, and let $\omega_1 \in A(S_1, S_1)_{(p)}$ and $\omega_2 \in A(S_2, S_2)_{(p)}$ be the characteristic idempotents.*

For every element of the Burnside module $X \in A(S_1, S_2)_{(p)}$, the product $\omega_2 \circ X \circ \omega_1$ is right \mathcal{F}_1 -stable and left \mathcal{F}_2 -stable, and satisfies

$$\Phi_D(\omega_2 \circ X \circ \omega_1) = \frac{1}{|[D]_{\mathcal{F}_2 \times \mathcal{F}_1}|} \sum_{D' \in [D]_{\mathcal{F}_2 \times \mathcal{F}_1}} \Phi_{D'}(X),$$

for all subgroups $D \leq S_2 \times S_1$, where $[D]_{\mathcal{F}_2 \times \mathcal{F}_1}$ is the isomorphism class of D in the product fusion system $\mathcal{F}_2 \times \mathcal{F}_1$ on $S_2 \times S_1$.

In particular, corollary 5.8 recovers the transfer map of theorem A for a saturated fusion system \mathcal{F} as multiplication by $\omega_{\mathcal{F}}$ on the Burnside module $A(1, S)_{(p)}$ generated by finite left S -sets. Let $A(\mathcal{F}_1, \mathcal{F}_2)_{(p)}$ denote the right \mathcal{F}_1 -stable and left \mathcal{F}_2 -stable elements of $A(S_1, S_2)_{(p)}$. Then the characteristic idempotents ω_1 and ω_2 act trivially on $A(\mathcal{F}_1, \mathcal{F}_2)_{(p)}$, and theorem C gives a transfer homomorphism of modules over the double Burnside rings $A(\mathcal{F}_1, \mathcal{F}_1)_{(p)}$ and $A(\mathcal{F}_2, \mathcal{F}_2)_{(p)}$ as described in proposition 5.10.

For a saturated fusion system \mathcal{F} on S , the double Burnside ring $A(\mathcal{F}, \mathcal{F})_{(p)}$ is the subring of $A(S, S)_{(p)}$ consisting of all the elements that are both left and right \mathcal{F} -stable. An even smaller subring is the collection of all the elements that are \mathcal{F} -generated as well as \mathcal{F} -stable. We denote this subring $A^{\text{char}}(\mathcal{F})_{(p)}$ since a generic \mathcal{F} -generated, \mathcal{F} -stable element is actually \mathcal{F} -characteristic. Hence we have a sequence of inclusions of subrings

$$A^{\text{char}}(\mathcal{F})_{(p)} \subseteq A(\mathcal{F}, \mathcal{F})_{(p)} \subseteq A(S, S)_{(p)}.$$

The last inclusion is not unital since $\omega_{\mathcal{F}}$ is the multiplicative identity of the first two rings, and $[S, id]_S^S$ is the identity of $A(S, S)_{(p)}$. According to proposition 6.3, $A^{\text{char}}(\mathcal{F})_{(p)}$ has a $\mathbb{Z}_{(p)}$ -basis consisting of elements $\beta_{\Delta(P, id)}$, which only depend on $P \leq S$ up to \mathcal{F} -conjugation, and each element of $A^{\text{char}}(\mathcal{F})_{(p)}$, written

$$X = \sum_{\substack{P \leq S \\ \text{up to } \mathcal{F}\text{-conj.}}} c_{\Delta(P, id)} \beta_{\Delta(P, id)},$$

is \mathcal{F} -characteristic if and only if $c_{\Delta(S, id)}$ is invertible in $\mathbb{Z}_{(p)}$.

For every (S, S) -biset X , we can quotient out the right S -action in order to get X/S as a left S -set. Quotienting out the right S -action preserves disjoint union and extends to a collapse map $q: A(S, S)_{(p)} \rightarrow A(S)_{(p)}$, and by restriction to subrings we get maps

$$\begin{array}{ccccc} A^{\text{char}}(\mathcal{F})_{(p)} & \subseteq & A(\mathcal{F}, \mathcal{F})_{(p)} & \subseteq & A(S, S)_{(p)} \\ & \searrow & \downarrow & & \downarrow \\ & & A(\mathcal{F})_{(p)} & \subseteq & A(S)_{(p)} \end{array}$$

where \mathcal{F} -stable bisets are collapsed to \mathcal{F} -stable sets. In general the collapse map does not respect the multiplication of the double Burnside ring, but combining the techniques of theorems A and C we show that on $A^{\text{char}}(\mathcal{F})_{(p)}$ the collapse map is not only a ring homomorphism but actually an isomorphism of rings!

Theorem D. *Let \mathcal{F} be a saturated fusion system on a finite p -group S .*

Then the collapse map $q: A^{\text{char}}(\mathcal{F})_{(p)} \rightarrow A(\mathcal{F})_{(p)}$, which quotients out the right S -action, is an isomorphism of rings, and it sends the basis element $\beta_{\Delta(P, id)}$ of $A(\mathcal{F}, \mathcal{F})_{(p)}$ to the basis element β_P of $A(\mathcal{F})_{(p)}$.

This generalizes a similar result for groups where the Burnside ring $A(S)$ embeds in the double Burnside ring $A(S, S)$ with the transitive S -set S/P corresponding to the transitive biset $[P, id]_S^S$. As an immediate consequence of theorem D we get an alternative proof that the characteristic idempotent $\omega_{\mathcal{F}}$ is unique: Corollary 6.6 shows that $\beta_{\Delta(S, id)} = \omega_{\mathcal{F}}$ is the only non-zero idempotent of $A^{\text{char}}(\mathcal{F})_{(p)}$ by proving that 0 and S/S are the only idempotents of $A(\mathcal{F})_{(p)}$.

The final section of this paper applies theorem C to disprove a conjecture by Park-Ragnarsson-Stancu, [11], on the composition product of fusion systems. Let \mathcal{F} be a saturated fusion system on a p -group S , and let \mathcal{H}, \mathcal{K} be saturated fusion subsystems on subgroups $R, T \leq S$ respectively. In the terminology of Park-Ragnarsson-Stancu, we then say that \mathcal{F} is the composition product of \mathcal{H} and \mathcal{K} , written $\mathcal{F} = \mathcal{H}\mathcal{K}$, if $S = RT$ and for all subgroups $P \leq T$ it holds that every morphism $\varphi \in \mathcal{F}(P, R)$ can be written as a composition $\varphi = \psi\rho$ where ψ is a morphism of \mathcal{H} and ρ is a morphism of \mathcal{K} .

Park-Ragnarsson-Stancu conjectured that $\mathcal{F} = \mathcal{H}\mathcal{K}$ is equivalent to the following equation of characteristic idempotents:

$$(1.1) \quad [R, id]_S^R \circ \omega_{\mathcal{F}} \circ [T, id]_T^S = \omega_{\mathcal{H}} \circ [R \cap T, id]_T^R \circ \omega_{\mathcal{K}}$$

A special case of the conjecture was proven in [11], in the case where $R = S$ and \mathcal{K} is weakly normal in \mathcal{F} , and the general conjecture was inspired by the group case, where $H, K \leq G$ satisfy $G = HK$ if and only if there is an isomorphism of (K, H) -bisets $G \cong H \times_{H \cap K} K$. By direct calculation via theorem C we can now characterize all cases where (1.1) holds:

Theorem E. *Let \mathcal{F} be a saturated fusion system on a p -group S , and suppose that \mathcal{H}, \mathcal{K} are saturated fusion subsystems of \mathcal{F} on subgroups $R, T \leq S$ respectively.*

Then the characteristic idempotents satisfy

$$(1.2) \quad [R, id]_S^R \circ \omega_{\mathcal{F}} \circ [T, id]_T^S = \omega_{\mathcal{H}} \circ [R \cap T, id]_T^R \circ \omega_{\mathcal{K}}$$

if and only if $\mathcal{F} = \mathcal{H}\mathcal{K}$ and for all $Q \leq R \cap T$ we have

$$(1.3) \quad |\mathcal{F}(Q, S)| = \frac{|\mathcal{H}(Q, R)| \cdot |\mathcal{K}(Q, T)|}{|\text{Hom}_{\mathcal{H} \cap \mathcal{K}}(Q, R \cap T)|}.$$

In particular (1.2) always implies $\mathcal{F} = \mathcal{H}\mathcal{K}$, but the converse is not true in general. In example 7.1, the alternating group A_6 gives rise to a composition product $\mathcal{F} = \mathcal{H}\mathcal{K}$ where (1.3) fails – hence we get a counter-example to the general conjecture of Park-Ragnarsson-Stancu.

At the same time, proposition 7.2 proves a special case of the conjecture where \mathcal{K} is weakly normal in \mathcal{F} , which is a generalization of the case proved by Park-Ragnarsson-Stancu.

Earlier results on Burnside rings for fusion systems. An earlier definition for the Burnside ring of a fusion system, was given by Diaz-Libman in [5]. The advantage of the Diaz-Libman definition of the Burnside ring is that it is constructed in close relation to a nice orbit category for the centric subgroups in a saturated fusion system \mathcal{F} . However, by construction the Burnside ring of Diaz-Libman doesn't see the non-centric subgroup at all, in contrast to the definition of $A(\mathcal{F})$ used in this paper where we have basis elements corresponding to all the subgroups. In proposition 4.8, we compare the two definitions and show that if we quotient out the non-centric part of $A(\mathcal{F})_{(p)}$ we recover the centric Burnside ring of Diaz-Libman, and we relate the basis elements given by Diaz-Libman to the basis elements β_P used in this paper.

Theorem A and the construction of characteristic idempotents in this paper is strongly inspired by an algorithm by Broto-Levi-Oliver. Originally, in [4], Broto-Levi-Oliver gave a procedure for constructing a characteristic biset Ω from a saturated fusion system \mathcal{F} , and using such a biset, they then constructed a classifying spectrum for \mathcal{F} . In [12] Ragnarsson took a characteristic biset as constructed by Broto-Levi-Oliver, and proceeded to refine this biset to get an idempotent. This proof used a Cauchy sequence argument in the p -completion $A(S, S)_p^\wedge$ of the double Burnside ring in order to show that a characteristic idempotent exists. A later part of [12] showed that $\omega_{\mathcal{F}}$ is unique and that in fact $\omega_{\mathcal{F}}$ lies in the p -localization $A(S, S)_{(p)}$ as a subring of the p -completion. The new construction of $\omega_{\mathcal{F}}$ given in this paper takes the original procedure by Broto-Levi-Oliver and refines it in order to construct $\omega_{\mathcal{F}}$ directly as an element of $A(S, S)_{(p)}$ – without needing to work in the p -completion. Furthermore, this refined procedure generalizes in order to give us the transfer map of theorem A.

Finally, we note that the formula for the fixed points of $\omega_{\mathcal{F}}$ given in theorem B coincides with the work done independently by Boltje-Danz in [3]. The calculations by Boltje-Danz are done by working in their ghost ring for the double Burnside ring and applying the steps of Ragnarsson's proof for the uniqueness of $\omega_{\mathcal{F}}$. This way they are able to calculate what the fixed points of $\omega_{\mathcal{F}}$ have to be, assuming that $\omega_{\mathcal{F}}$ exists. In this paper, the fixed points follow as an immediate consequence of the way we construct $\omega_{\mathcal{F}}$.

Outline. Section 2 recalls the definition and basic properties of saturated fusion systems, and establishes the related notation used throughout the rest of the paper. Section 3 gives

a similar treatment to the Burnside ring of a finite group as well as the Burnside ring for a saturated fusion system. Section 4 is the first main section of the paper, where we consider the structure of the p -localization $A(\mathcal{F})_{(p)}$ of the Burnside ring for a saturated fusion system \mathcal{F} on a finite p -group S . In particular, we construct a stabilization map that sends every finite S -set to an \mathcal{F} -stable element in a canonical way, and we prove theorem A. The other main section, section 5, is subdivided in three parts. In 5.1 we recall the double Burnside ring of a group. In 5.2 we apply the stabilization map above for the fusion system $\mathcal{F} \times \mathcal{F}$ in order to construct the characteristic idempotent for \mathcal{F} and prove theorem B. In 5.3 we prove theorem C and study the strong relation between the stabilization homomorphism of theorem A and multiplying with the characteristic idempotent. In section 6 we prove theorem D relating the \mathcal{F} -characteristic elements to the Burnside ring of \mathcal{F} . Finally, section 7 concerns the composition product of fusion systems and theorem E.

Acknowledgements. First I would like to thank Kári Ragnarsson who suggested that I look at [11] and the conjecture therein – which in turn inspired sections 5.3 to 7. I also thank Radu Stancu and Serge Bouc for some good and fruitful discussions during my visit to Amiens, in particular thanks to Serge for pointing out the existence of [10]. Thanks go to Matthew Gelvin for corrections to the paper and for help finding the counterexample of section 7. Last but not least, I thank my advisor Jesper Grodal for his continued suggestions, feedback and support.

2. FUSION SYSTEMS

The next few pages contain a very short introduction to fusion systems, which were originally introduced by Puig under the name “full Frobenius systems.” The aim is to introduce the terminology from the theory of fusion systems that will be used in the paper, and to establish the relevant notation. For a proper introduction to fusion systems see, for instance, Part I of “Fusion Systems in Algebra and Topology” by Aschbacher, Kessar and Oliver, [2].

Definition 2.1. A *fusion system* \mathcal{F} on a p -group S , is a category where the objects are the subgroups of S , and for all $P, Q \leq S$ the morphisms must satisfy:

- (i) Every morphism $\varphi \in \text{Mor}_{\mathcal{F}}(P, Q)$ is an injective group homomorphism, and the composition of morphisms in \mathcal{F} is just composition of group homomorphisms.
- (ii) $\text{Hom}_S(P, Q) \subseteq \text{Mor}_{\mathcal{F}}(P, Q)$, where

$$\text{Hom}_S(P, Q) = \{c_s \mid s \in N_S(P, Q)\}$$

is the set of group homomorphisms $P \rightarrow Q$ induced by S -conjugation.

- (iii) For every morphism $\varphi \in \text{Mor}_{\mathcal{F}}(P, Q)$, the group isomorphisms $\varphi: P \rightarrow \varphi P$ and $\varphi^{-1}: \varphi P \rightarrow P$ are elements of $\text{Mor}_{\mathcal{F}}(P, \varphi P)$ and $\text{Mor}_{\mathcal{F}}(\varphi P, P)$ respectively.

We also write $\text{Hom}_{\mathcal{F}}(P, Q)$ or just $\mathcal{F}(P, Q)$ for the morphism set $\text{Mor}_{\mathcal{F}}(P, Q)$; and the group $\mathcal{F}(P, P)$ of automorphisms is denoted by $\text{Aut}_{\mathcal{F}}(P)$.

The canonical example of a fusion system comes from a finite group G with a given p -subgroup S . The fusion system of G on S , denoted $\mathcal{F}_S(G)$, is the fusion system on S where the morphisms from $P \leq S$ to $Q \leq S$ are the homomorphisms induced by G -conjugation:

$$\text{Hom}_{\mathcal{F}_S(G)}(P, Q) := \text{Hom}_G(P, Q) = \{c_g \mid g \in N_G(P, Q)\}.$$

A particular case is the fusion system $\mathcal{F}_S(S)$ consisting only of the homomorphisms induced by S -conjugation.

Let \mathcal{F} be an abstract fusion system on S . We say that two subgroups $P, Q \leq S$ are \mathcal{F} -conjugate, written $P \sim_{\mathcal{F}} Q$, if they are isomorphic in \mathcal{F} , i.e., there exists a group isomorphism $\varphi \in \mathcal{F}(P, Q)$. \mathcal{F} -conjugation is an equivalence relation, and the set of \mathcal{F} -conjugates to P is denoted by $[P]_{\mathcal{F}}$. The set of all \mathcal{F} -conjugacy classes of subgroups in S is denoted by $Cl(\mathcal{F})$. Similarly, we write $P \sim_S Q$ if P and Q are S -conjugate, the S -conjugacy class of P is written $[P]_S$ or just $[P]$, and we write $Cl(S)$ for the set of S -conjugacy classes of subgroups in S . Since all S -conjugation maps are in \mathcal{F} , any \mathcal{F} -conjugacy class $[P]_{\mathcal{F}}$ can be partitioned into disjoint S -conjugacy classes of subgroups $Q \in [P]_{\mathcal{F}}$.

We say that Q is \mathcal{F} - or S -subconjugate to P if Q is respectively \mathcal{F} - or S -conjugate to a subgroup of P , and we denote this by $Q \lesssim_{\mathcal{F}} P$ or $Q \lesssim_S P$ respectively. In the case where $\mathcal{F} = \mathcal{F}_S(G)$, we have $Q \lesssim_{\mathcal{F}} P$ if and only if Q is G -conjugate to a subgroup of P ; in this case the \mathcal{F} -conjugates of P are just those G -conjugates of P that are contained in S .

A subgroup $P \leq S$ is said to be *fully \mathcal{F} -normalized* if $|N_S P| \geq |N_S Q|$ for all $Q \in [P]_{\mathcal{F}}$; similarly P is *fully \mathcal{F} -centralized* if $|C_S P| \geq |C_S Q|$ for all $Q \in [P]_{\mathcal{F}}$.

Definition 2.2. A fusion system \mathcal{F} on S is said to be *saturated* if the following properties are satisfied for all $P \leq S$:

- (i) If P is fully \mathcal{F} -normalized, then P is fully \mathcal{F} -centralized, and $\text{Aut}_S(P)$ is a Sylow p -subgroup of $\text{Aut}_{\mathcal{F}}(P)$.
- (ii) Every homomorphism $\varphi \in \mathcal{F}(P, S)$ with $\varphi(P)$ fully \mathcal{F} -centralized extends to a homomorphism $\varphi \in \mathcal{F}(N_{\varphi}, S)$, where

$$N_{\varphi} := \{x \in N_S(P) \mid \exists y \in S: \varphi \circ c_x = c_y \circ \varphi\}.$$

The saturation axioms are a way of emulating the Sylow theorems for finite groups; in particular, whenever S is a Sylow p -subgroup of G , then the Sylow theorems imply that the induced fusion system $\mathcal{F}_S(G)$ is saturated (see e.g. [2, Theorem 2.3]).

In this paper, we shall rarely use the defining properties of saturated fusion systems directly. We shall instead mainly use the following lifting property, which saturated fusion systems satisfy:

Lemma 2.3 ([15]). *Let \mathcal{F} be saturated. If $P \leq S$ is fully normalized, then for each $Q \in [P]_{\mathcal{F}}$ there exists a homomorphism $\varphi \in \mathcal{F}(N_S Q, N_S P)$ with $\varphi(Q) = P$.*

For the proof, see lemma 4.5 of [15] or lemma 2.6(c) of [2].

3. BURNSIDE RINGS FOR GROUPS AND FUSION SYSTEMS

In this section we recall the Burnside ring of a finite group S and how to describe its structure in terms of the homomorphism of marks, which embeds the Burnside ring into a suitable ghost ring. We also recall the Burnside ring of a saturated fusion system \mathcal{F} , in the sense of [14], which has a similar mark homomorphism and ghost ring.

Let S be a finite group, not necessarily a p -group. Then the isomorphism classes of finite S -sets form a semiring with disjoint union as addition and cartesian product as multiplication. The Burnside ring of S , denoted $A(S)$, is then defined as the additive Grothendieck group of the semiring, and $A(S)$ inherits the multiplication as well. Given a finite S -set X , we let $[X]$ denote the isomorphism class of X as an element of $A(S)$. The isomorphism classes $[S/P]$ of transitive S -sets form an additive basis for $A(S)$, and two transitive sets S/P and S/Q are isomorphic if and only if the subgroups P and Q are conjugate in S .

For each element $X \in A(S)$ we define $c_P(X)$, with $P \leq S$, to be the coefficients when we write X as a linear combination of the basis elements $[S/P]$ in $A(S)$, i.e.

$$X = \sum_{[P] \in Cl(S)} c_P(X) \cdot [S/P],$$

where $Cl(S)$ denotes the set of S -conjugacy classes of subgroup in S . The resulting maps $c_P: A(S) \rightarrow \mathbb{Z}$ are group homomorphisms, but they are *not* ring homomorphisms.

To describe the multiplication of $A(S)$, it is enough to know the products of basis elements $[S/P]$ and $[S/Q]$. By taking the cartesian product $(S/P) \times (S/Q)$ and considering how it breaks into orbits, one reaches the following double coset formula for the multiplication in $A(S)$:

$$(3.1) \quad [S/P] \cdot [S/Q] = \sum_{\bar{s} \in P \backslash S / Q} [S/(P \cap {}^s Q)],$$

where $P \backslash S / Q$ is the set of double cosets PsQ with $s \in S$.

Instead of counting orbits, an alternative way of characterising a finite S -set is counting the fixed points for each subgroup $P \leq S$. For every $P \leq S$ and S -set X , we denote the number of P -fixed points by $\Phi_P(X) := |X^P|$. This number only depends on P up to S -conjugation. Since we have

$$|(X \sqcup Y)^P| = |X^P| + |Y^P| \quad \text{and} \quad |(X \times Y)^P| = |X^P| \cdot |Y^P|$$

for all S -sets X and Y , the *fixed point map* Φ_P for S -sets extends to a ring homomorphism $\Phi_P: A(S) \rightarrow \mathbb{Z}$. On the basis elements $[S/P]$, the number of fixed points is given by

$$(3.2) \quad \Phi_Q([S/P]) = |(S/P)^Q| = \frac{|N_S(Q, P)|}{|P|},$$

where $N_S(Q, P) = \{s \in S \mid {}^s Q \leq P\}$ is the transporter in S from Q to P . In particular, $\Phi_Q([S/P]) \neq 0$ if and only if $Q \lesssim_S P$ (Q is subconjugate to P).

We have one fixed point homomorphism Φ_P per conjugacy class of subgroups in S , and we combine them into the *homomorphism of marks* $\Phi = \Phi^S: A(S) \xrightarrow{\prod_{[P]} \Phi_P} \prod_{[P] \in Cl(S)} \mathbb{Z}$. This ring homomorphism maps $A(S)$ into the product ring $\tilde{\Omega}(S) := \prod_{[P] \in Cl(S)} \mathbb{Z}$, the so-called *ghost ring* for the Burnside ring $A(S)$.

Results by tom Dieck and others show that the mark homomorphism is injective, and that the cokernel of Φ is the *obstruction group* $Obs(S) := \prod_{[P] \in Cl(S)} (\mathbb{Z}/|W_S P| \mathbb{Z})$, where $W_S P := N_S P / P$. These statements are combined in the following proposition, the proof of which can be found in [7, Chapter 1], [8], and [16].

Proposition 3.1. *Let $\Psi = \Psi^S: \tilde{\Omega}(S) \rightarrow Obs(S)$ be given by the $[P]$ -coordinate functions*

$$\Psi_P(\xi) := \sum_{\bar{s} \in W_S P} \xi_{\langle s \rangle P} \pmod{|W_S P|}.$$

Here $\xi_{\langle s \rangle P}$ denotes the $[\langle s \rangle P]$ -coordinate of an element $\xi \in \tilde{\Omega}(S) = \prod_{[P] \in Cl(S)} \mathbb{Z}$.

The following sequence of abelian groups is then exact:

$$0 \rightarrow A(S) \xrightarrow{\Phi} \tilde{\Omega}(S) \xrightarrow{\Psi} Obs(S) \rightarrow 0.$$

Φ is a ring homomorphism, but Ψ is just a group homomorphism.

The homomorphism of marks enables us to perform calculations for the Burnside ring $A(S)$ inside the much nicer product ring $\tilde{\Omega}(S)$, where we identify each element $X \in A(S)$ with its fixed point vector $(\Phi_Q(X))_{[Q] \in Cl(S)}$.

3.1. The Burnside ring of a saturated fusion system. Let S be a finite p -group, and suppose that \mathcal{F} is a saturated fusion system on S . We say that a finite S -set is \mathcal{F} -stable if the action is unchanged up to isomorphism whenever we act through morphisms of \mathcal{F} . More precisely, if $P \leq S$ is a subgroup and $\varphi: P \rightarrow S$ is a homomorphism in \mathcal{F} , we can turn X into a P -set by using φ to define the action $g.x := \varphi(g)x$ for $g \in P$. We denote the resulting P -set by ${}_{P,\varphi}X$. In particular when $incl: P \rightarrow S$ is the inclusion map, ${}_{P,incl}X$ has the usual restriction of the S -action to P . Restricting the action of S -sets along φ extends to a ring homomorphism $r_\varphi: A(S) \rightarrow A(P)$, and we let ${}_{P,\varphi}X$ denote the image $r_\varphi(X)$ for all elements $X \in A(S)$.

We then say that an element $X \in A(S)$ is \mathcal{F} -stable if it satisfies

$$(3.3) \quad {}_{P,\varphi}X = {}_{P,incl}X \text{ inside } A(P), \text{ for all } P \leq S \text{ and homomorphisms } \varphi: P \rightarrow S \text{ in } \mathcal{F}.$$

Alternatively, one can characterize \mathcal{F} -stability in terms of fixed points and the mark homomorphism, and the following three properties are equivalent for all $X \in A(S)$:

- (i) X is \mathcal{F} -stable.
- (ii) $\Phi_P(X) = \Phi_{\varphi P}(X)$ for all $\varphi \in \mathcal{F}(P, S)$ and $P \leq S$.
- (iii) $\Phi_P(X) = \Phi_Q(X)$ for all pairs $P, Q \leq S$ with $P \sim_{\mathcal{F}} Q$.

A proof of this claim can be found in [9, Proposition 3.2.3] or [14]. We shall primarily use (ii) and (iii) to characterize \mathcal{F} -stability.

It follows from property (iii) that the \mathcal{F} -stable elements form a subring of $A(S)$. We define the *Burnside ring of \mathcal{F}* to be the subring $A(\mathcal{F}) \subseteq A(S)$ consisting of all the \mathcal{F} -stable elements. Equivalently, we can consider the actual S -sets that are \mathcal{F} -stable: The \mathcal{F} -stable sets form a semiring, and we define $A(\mathcal{F})$ to be the Grothendieck group hereof. These two constructions give rise to the same ring $A(\mathcal{F})$ – see [14]. As is the case for the Burnside ring of a group, $A(\mathcal{F})$ has an additive basis, where the basis elements are in one-to-one correspondence with the \mathcal{F} -conjugacy classes of subgroups in S .

For each $X \in A(\mathcal{F})$ the fixed point map $\Phi_P(X)$ only depends on P up to \mathcal{F} -conjugation. The homomorphism of marks for $A(S)$ therefore restricts to the subring $A(\mathcal{F})$ as an injective ring homomorphism

$$\Phi^{\mathcal{F}}: A(S) \xrightarrow{\prod_{[P]_{\mathcal{F}}} \Phi_P} \prod_{[P]_{\mathcal{F}} \in Cl(\mathcal{F})} \mathbb{Z},$$

where $Cl(\mathcal{F})$ denotes the set of \mathcal{F} -conjugacy classes of subgroups in S . We call this map *the homomorphism of marks* for $A(\mathcal{F})$, and the ring $\tilde{\Omega}(\mathcal{F}) := \prod_{[P]_{\mathcal{F}} \in Cl(\mathcal{F})} \mathbb{Z}$ is *the ghost ring* for $A(\mathcal{F})$.

As for the Burnside ring of a group, we also have an explicit description of the cokernel of $\Phi^{\mathcal{F}}$ as the group

$$Obs(\mathcal{F}) := \prod_{\substack{[P]_{\mathcal{F}} \in Cl(\mathcal{F}) \\ P \text{ f.n.}}} (\mathbb{Z}/|W_S P|\mathbb{Z}),$$

where P is taken to be a fully normalized representative for each \mathcal{F} -conjugacy class of subgroups. According to [14], we have a short-exact sequence similar to proposition 3.1:

Proposition 3.2. *Let $\Psi = \Psi^{\mathcal{F}}: \tilde{\Omega}(\mathcal{F}) \rightarrow \text{Obs}(\mathcal{F})$ be given by the $[P]_{\mathcal{F}}$ -coordinate functions*

$$\Psi_P(\xi) := \sum_{\bar{s} \in W_S P} \xi_{\langle s \rangle P} \pmod{|W_S P|},$$

when P is fully \mathcal{F} -normalized, and $\xi_{\langle s \rangle P}$ denotes the $[\langle s \rangle P]_{\mathcal{F}}$ -coordinate of an element $\xi \in \tilde{\Omega}(\mathcal{F}) = \prod_{[P] \in \text{Cl}(\mathcal{F})} \mathbb{Z}$.

The following sequence of abelian groups is then exact:

$$0 \rightarrow A(\mathcal{F}) \xrightarrow{\Phi} \tilde{\Omega}(\mathcal{F}) \xrightarrow{\Psi} \text{Obs}(\mathcal{F}) \rightarrow 0.$$

Φ is a ring homomorphism, but Ψ is just a group homomorphism.

4. THE p -LOCALIZED BURNSIDE RING

Let \mathcal{F} be a saturated fusion system on a p -group S . In this section we show that there is a well-defined stabilization map $A(S)_{(p)} \rightarrow A(\mathcal{F})_{(p)}$ between p -localized Burnside rings. This map is shown to be a homomorphism of $A(\mathcal{F})_{(p)}$ -modules, and it has a simple expression in terms of the mark homomorphism for $A(S)_{(p)}$. Using the stabilization homomorphism, we give a new basis for $A(\mathcal{F})_{(p)}$. It was shown in [14] that the irreducible \mathcal{F} -stable sets form a basis for $A(\mathcal{F})$, but very little is known about their actual structure. The new basis for $A(\mathcal{F})_{(p)}$, though it only exists after p -localization, is easily described in terms of the homomorphism of marks. We use this basis in section 5, for the product fusion system $\mathcal{F} \times \mathcal{F}$ on $S \times S$, to give a new construction of the so-called *characteristic idempotent* for the saturated fusion system \mathcal{F} . In section 4.1 we compare $A(\mathcal{F})_{(p)}$, including its basis, with the centric Burnside ring of \mathcal{F} defined by Diaz and Libman in [5]. When \mathcal{F} is realized by a group G , we also relate $A(\mathcal{F})_{(p)}$ to the p -subgroup part of $A(G)_{(p)}$.

It is useful to have a procedure for constructing an \mathcal{F} -stable set from a general S -set. Such a procedure was used by Broto, Levi and Oliver in [4] to show that every saturated fusion system has at least one “characteristic biset,” a set with left and right S -actions satisfying properties suggested by Linckelmann and Webb. A similar procedure was used in [14], to construct all irreducible \mathcal{F} -stable S -sets. Both constructions follow the same general idea: To begin with, we are given a finite S -set X (or in general an element of the Burnside ring). We then consider each \mathcal{F} -conjugacy class of subgroups in S in decreasing order and add further S -orbits to X until the set becomes \mathcal{F} -stable. To construct the irreducible \mathcal{F} -stable sets, we start with a transitive S -set $[S/P]$; to construct a characteristic biset, we start with S itself considered as an (S, S) -biset.

The construction changes the number of elements and orbits in the set X that we stabilize, and the number of added orbits depends heavily on the set that we start with – if X is already \mathcal{F} -stable we need not add anything at all. Because of this, we expect the stabilized sets to behave quite differently from the sets we start with, for instance, the stabilization procedure does not even preserve addition.

In this section we adjust the construction of [4, 14] such that instead of just adding orbits to stabilize a set, we subtract orbits as well, in a way such that all changes cancel “up to \mathcal{F} -conjugation.” This results in a nicely behaved stabilization procedure that works for all S -sets, with one disadvantage: we must work in the p -localization $A(S)_{(p)}$ instead of $A(S)$.

The following lemmas 4.1 and 4.2 are needed to show that the later calculations work in $A(S)_{(p)}$, i.e., that we never divide by p . Lemma 4.1 is also interesting in itself since it

shows that for any fully normalized subgroup $P \leq S$, the number of \mathcal{F} -conjugates to P is the same as the number of S -conjugates up to a p' -factor.

Lemma 4.1. *Let \mathcal{F} be a saturated fusion system on S , and let $P \leq S$ be fully \mathcal{F} -normalized. Then the number of \mathcal{F} -conjugates of P is equal to $\frac{|S|}{|N_S P|} \cdot k$, where $p \nmid k$.*

Equivalently, $|\mathcal{F}(P, S)| = \frac{|S|}{|C_S P|} \cdot k'$, with $p \nmid k'$.

Proof. Recall that $[P]_{\mathcal{F}}$ denotes the set of subgroups in S that are \mathcal{F} -conjugate to P . We then have $|\mathcal{F}(P, S)| = |\text{Aut}_{\mathcal{F}}(P)| \cdot |[P]_{\mathcal{F}}|$ for all $P \leq S$. When P is fully \mathcal{F} -normalized, we furthermore get

$$|\text{Aut}_{\mathcal{F}}(P)| = |\text{Aut}_S(P)| \cdot k'' = \frac{|N_S P|}{|C_S P|} \cdot k''$$

where $p \nmid k''$ since \mathcal{F} is saturated. It follows that the two statements in the lemma are equivalent for $P \leq S$ fully normalized.

We proceed by induction on the index $|S : P|$. If $P = S$, then $|[S]_{\mathcal{F}}| = 1 = \frac{|S|}{|N_S S|} \cdot 1$. Assume therefore that $P < S$ is fully normalized; since $P \neq S$, we then have $P < N_S P$. The \mathcal{F} -conjugacy class $[P]_{\mathcal{F}}$ is a disjoint union of the S -conjugacy classes $[Q]_S$ where $Q \sim_{\mathcal{F}} P$. The S -conjugacy class $[Q]_S$ has $|S|/|N_S Q|$ elements; and $\frac{|S|}{|N_S Q|}$ is divisible by $\frac{|S|}{|N_S P|}$ since P is fully normalized. In particular, $\frac{|S|}{|N_S P|}$ divides $|[P]_{\mathcal{F}}|$.

Furthermore, we have $|[Q]_S| \cdot \frac{|N_S P|}{|S|} = \frac{|N_S P|}{|N_S Q|} \equiv 0 \pmod{p}$ whenever $Q \sim_{\mathcal{F}} P$ isn't fully normalized. It follows that

$$\begin{aligned} |[P]_{\mathcal{F}}| \cdot \frac{|N_S P|}{|S|} &= \sum_{[Q]_S \subseteq [P]_{\mathcal{F}}} |[Q]_S| \cdot \frac{|N_S P|}{|S|} \\ &\equiv \sum_{\substack{[Q]_S \subseteq [P]_{\mathcal{F}} \\ Q \text{ f.n.}}} |[Q]_S| \cdot \frac{|N_S P|}{|S|} = |[P]_{\mathcal{F}}^{\text{f.n.}}| \cdot \frac{|N_S P|}{|S|} \pmod{p}, \end{aligned}$$

where “f.n.” is short for “fully normalized,” and $[P]_{\mathcal{F}}^{\text{f.n.}}$ is the set of $Q \sim_{\mathcal{F}} P$ that are fully normalized. We conclude that $|[P]_{\mathcal{F}}| = \frac{|S|}{|N_S P|} \cdot k$, with $p \nmid k$, if and only if $|[P]_{\mathcal{F}}^{\text{f.n.}}| = \frac{|S|}{|N_S P|} \cdot k'$, with $p \nmid k'$.

Suppose that $Q \sim_{\mathcal{F}} P$ is fully normalized. Since P is fully normalized, we have a homomorphism $\varphi \in \mathcal{F}(N_S Q, N_S P)$ with $\varphi(Q) = P$ by lemma 2.3; and since Q is fully normalized, φ is an isomorphism. It follows that every $Q \in [P]_{\mathcal{F}}^{\text{f.n.}}$ is a normal subgroup of exactly one element of $[N_S P]_{\mathcal{F}}$, namely $N_S Q \in [N_S P]_{\mathcal{F}}$.

Let $K \sim_{\mathcal{F}} N_S P$. We let $[P]_{\mathcal{F}}^{\triangleleft K}$ denote the set of $Q \sim_{\mathcal{F}} P$ such that $Q \triangleleft K$. Any such $Q \triangleleft K$ is in particular fully normalized since $|K| = |N_S P|$. Any \mathcal{F} -isomorphism $N_S P \xrightarrow{\sim} K$ gives a bijection $[P]_{\mathcal{F}}^{\triangleleft N_S P} \xrightarrow{\sim} [P]_{\mathcal{F}}^{\triangleleft K}$.

The set $[P]_{\mathcal{F}}^{\text{f.n.}}$ is thus seen to be the disjoint union of the sets $[P]_{\mathcal{F}}^{\triangleleft K}$ where $K \sim_{\mathcal{F}} N_S P$, and these sets all have the same number of elements as $[P]_{\mathcal{F}}^{\triangleleft N_S P}$:

$$|[P]_{\mathcal{F}}^{\text{f.n.}}| = \sum_{K \in [N_S P]_{\mathcal{F}}} |[P]_{\mathcal{F}}^{\triangleleft K}| = |[N_S P]_{\mathcal{F}}| \cdot |[P]_{\mathcal{F}}^{\triangleleft N_S P}|.$$

Let $K \sim_{\mathcal{F}} N_S P$ be fully normalized, then there is some $Q \in [P]_{\mathcal{F}}^{\triangleleft K}$. We have $Q \sim_{\mathcal{F}} P$, and Q is fully normalized with $N_S Q = K$ that is itself fully normalized. By letting Q take the place of P , we can therefore assume that $N_S P$ is fully normalized.

Any two elements $Q, R \in [P]_{\mathcal{F}}^{\triangleleft N_S P}$ are mapped $Q \xrightarrow{\sim} R$ by some \mathcal{F} -automorphism of $N_S P$ (since $N_S P$ is the normalizer of both Q and R); hence $\text{Aut}_{\mathcal{F}}(N_S P)$ acts transitively on $[P]_{\mathcal{F}}^{\triangleleft N_S P}$. Let $X \leq \text{Aut}_{\mathcal{F}}(N_S P)$ be the subgroup stabilizing P under this action; then

$$|[P]_{\mathcal{F}}^{\triangleleft N_S P}| = |\text{Aut}_{\mathcal{F}}(N_S P) : X|.$$

The number of elements in $[P]_{\mathcal{F}}^{\text{f.n.}}$ is therefore equal to

$$|[P]_{\mathcal{F}}^{\text{f.n.}}| = |[N_S P]_{\mathcal{F}}| \cdot |\text{Aut}_{\mathcal{F}}(N_S P) : X|.$$

We know that $\frac{|S|}{|N_S P|}$ divides $[P]_{\mathcal{F}}^{\text{f.n.}}$; and by the induction assumption we have $|[N_S P]_{\mathcal{F}}| = \frac{|S|}{|N_S(N_S P)|} \cdot k$, where $p \nmid k$, since $N_S P$ is fully normalized. We can therefore conclude that $\frac{|N_S(N_S P)|}{|N_S P|}$ divides the index $|\text{Aut}_{\mathcal{F}}(N_S P) : X|$.

We now consider the following diagram of subgroups of $\text{Aut}_{\mathcal{F}}(N_S P)$:

$$\begin{array}{ccc} & \text{Aut}_{\mathcal{F}}(N_S P) & \\ & \swarrow \quad \searrow & \\ X & & \text{Aut}_S(N_S P) \\ & \swarrow \quad \searrow & \\ & X \cap \text{Aut}_S(N_S P) & \end{array}$$

The index $|\text{Aut}_{\mathcal{F}}(N_S P) : \text{Aut}_S(N_S P)|$ is coprime to p since $N_S P$ is fully normalized and \mathcal{F} is saturated. We have $C_S(N_S P) \leq C_S P \leq N_S P$, which tells us that $C_S(N_S P) = Z(N_S P)$; and consequently

$$\text{Aut}_S(N_S P) \cong N_S(N_S P)/Z(N_S P).$$

From the definition of X , we get that

$$\begin{aligned} X \cap \text{Aut}_S(N_S P) &= \{\varphi \in \text{Aut}_{\mathcal{F}}(N_S P) \mid \varphi P = P\} \cap \{c_s \in \text{Aut}_{\mathcal{F}}(N_S P) \mid s \in N_S(N_S P)\} \\ &= \{c_s \in \text{Aut}_{\mathcal{F}}(N_S P) \mid s \in N_S P\} = \text{Inn}(N_S P) \cong N_S P/Z(N_S P). \end{aligned}$$

The index $|\text{Aut}_S(N_S P) : X \cap \text{Aut}_S(N_S P)|$ is therefore equal to $\frac{|N_S(N_S P)|}{|N_S P|}$.

The right side of the subgroup diagram now shows that the highest power of p dividing the index $|\text{Aut}_{\mathcal{F}}(N_S P) : X \cap \text{Aut}_S(N_S P)|$ is $\frac{|N_S(N_S P)|}{|N_S P|}$. The highest power of p dividing $|\text{Aut}_{\mathcal{F}}(N_S P) : X|$ is thus at most $\frac{|N_S(N_S P)|}{|N_S P|}$ – and we already know that this power of p divides $|\text{Aut}_{\mathcal{F}}(N_S P) : X|$. We conclude that $|\text{Aut}_{\mathcal{F}}(N_S P) : X| = \frac{|N_S(N_S P)|}{|N_S P|} \cdot k'$ for some k' coprime to p , and we finally have

$$\begin{aligned} |[P]_{\mathcal{F}}^{\text{f.n.}}| &= |[N_S P]_{\mathcal{F}}| \cdot |\text{Aut}_{\mathcal{F}}(N_S P) : X| \\ &= \frac{|S|}{|N_S(N_S P)|} \cdot \frac{|N_S(N_S P)|}{|N_S P|} \cdot k k' = \frac{|S|}{|N_S P|} \cdot k k'; \end{aligned}$$

and $p \nmid k k'$. □

Lemma 4.2. *Let $P, Q \leq S$, then $|[Q]_{\mathcal{F}}|$ divides $|[Q']_S|$ in $\mathbb{Z}_{(p)}$ for all $Q' \sim_{\mathcal{F}} Q$; and furthermore*

$$\frac{1}{|[Q]_{\mathcal{F}}|} \sum_{Q' \in [Q]_{\mathcal{F}}} \Phi_{Q'}([S/P]) = \sum_{[Q']_S \subseteq [Q]_{\mathcal{F}}} \frac{|[Q']_S|}{|[Q]_{\mathcal{F}}|} \Phi_{Q'}([S/P]) = \frac{|\mathcal{F}(Q, P)| \cdot |S|}{|P| \cdot |\mathcal{F}(Q, S)|} \in \mathbb{Z}_{(p)}.$$

Proof. By lemma 4.1 we can express the number of \mathcal{F} -conjugates as $|[Q]_{\mathcal{F}}| = \frac{|S|}{|N_S Q_0|} \cdot k$, with $p \nmid k$, where $Q_0 \sim_{\mathcal{F}} Q$ is fully normalized. At the same time, the number of S -conjugates of Q' is given by $|[Q']_S| = \frac{|S|}{|N_S Q'|}$. Since $|N_S Q'| \leq |N_S Q_0|$, it then follows that $|[Q]_{\mathcal{F}}|$ divides $|[Q']_S|$ in $\mathbb{Z}_{(p)}$.

We try to simplify the sum in the lemma:

$$\begin{aligned} \sum_{[Q']_S \subseteq [Q]_{\mathcal{F}}} \frac{|[Q']_S|}{|[Q]_{\mathcal{F}}|} \Phi_{Q'}([S/P]) &= \frac{1}{|[Q]_{\mathcal{F}}|} \sum_{[Q']_S \subseteq [Q]_{\mathcal{F}}} \frac{|S|}{|N_S(Q')|} \cdot \frac{|N_S(Q', P)|}{|P|} \\ &= \frac{|S|}{|P| \cdot |[Q]_{\mathcal{F}}|} \sum_{[Q']_S \subseteq [Q]_{\mathcal{F}}} \frac{|N_S(Q', P)|}{|N_S(Q')|} \\ &= \frac{|S|}{|P| \cdot |[Q]_{\mathcal{F}}|} \sum_{[Q']_S \subseteq [Q]_{\mathcal{F}}} |\{R \in [Q']_S \mid R \leq P\}| \\ &= \frac{|S|}{|P| \cdot |[Q]_{\mathcal{F}}|} |\{R \in [Q]_{\mathcal{F}} \mid R \leq P\}| \\ &= \frac{|\mathcal{F}(Q, P)| \cdot |S|}{|P| \cdot |\mathcal{F}(Q, S)|}. \end{aligned}$$

The last equality follows from multiplying with $|\text{Aut}_{\mathcal{F}}(Q)|$ in both the numerator and the denominator. \square

Given any element X in the p -localized Burnside ring $A(S)_{(p)}$, we stabilize X according to the following idea: We run through the subgroups $Q \leq S$ in decreasing order and subtract/add orbits to X such that it becomes \mathcal{F} -stable at the conjugacy class of Q in \mathcal{F} , i.e., such that $\Phi_{Q'}(X) = \Phi_Q(X)$ for all $Q' \sim_{\mathcal{F}} Q$. Here we take care to “add as many orbits as we remove” at each step. The actual work of the stabilization procedure is handled in the following technical lemma 4.3, which is then applied in theorem A to construct the stabilization map $A(S)_{(p)} \rightarrow A(\mathcal{F})_{(p)}$.

Recall that $c_P(X)$ denotes the coefficient of $[S/P]$ when X is written in the standard basis of $A(S)_{(p)}$, and $\Phi_P: A(S)_{(p)} \rightarrow \mathbb{Z}_{(p)}$ for $P \leq S$ denote the fixed point homomorphisms.

Lemma 4.3. *Let \mathcal{F} be a saturated fusion system on a p -group S , and let \mathcal{H} be a collection of subgroups of S such that \mathcal{H} is closed under taking \mathcal{F} -subconjugates, i.e., if $P \in \mathcal{H}$, then $Q \in \mathcal{H}$ for all $Q \lesssim_{\mathcal{F}} P$. Assume that $X \in A(S)_{(p)}$ has the property that $\Phi_P(X) = \Phi_{P'}(X)$ for all pairs $P \sim_{\mathcal{F}} P'$, with $P, P' \notin \mathcal{H}$.*

Then there exists a uniquely determined element $\pi X \in A(\mathcal{F})_{(p)} \leq A(S)_{(p)}$ satisfying the following three properties:

- (i) $\Phi_P(\pi X) = \Phi_P(X)$ and $c_P(\pi X) = c_P(X)$ for all $P \notin \mathcal{H}$, $P \leq S$.
- (ii) For all $P \leq S$ we have

$$\sum_{[P']_S \subseteq [P]_{\mathcal{F}}} c_{P'}(\pi X) = \sum_{[P']_S \subseteq [P]_{\mathcal{F}}} c_{P'}(X).$$

(iii) For every $P \leq S$:

$$\Phi_P(\pi X) = \sum_{[P']_S \subseteq [P]_{\mathcal{F}}} \frac{|[P']_S|}{|[P]_{\mathcal{F}}|} \Phi_{P'}(X) = \frac{1}{|[P]_{\mathcal{F}}|} \sum_{P' \in [P]_{\mathcal{F}}} \Phi_{P'}(X).$$

Here $[P]_{\mathcal{F}}$ denotes the set of \mathcal{F} -conjugates of P . In the sums we pick one representative P' for each S -conjugacy class $[P']_S$ contained in $[P]_{\mathcal{F}}$, and by lemma 4.2 the fractions $\frac{|[P']_S|}{|[P]_{\mathcal{F}}|}$ make sense in $\mathbb{Z}_{(p)}$.

Property (i) ensures that we do not destroy the part of X that has already been stabilized. Property (ii) is the requirement that the total number of orbits is constant for each \mathcal{F} -conjugacy class of subgroups. We are only allowed to “replace” an orbit $[S/P]$ by another orbit $[S/P']$ where $P' \sim_{\mathcal{F}} P$. Finally, property (iii) tells us exactly what happens to the mark homomorphism when we stabilize: We simply take the mean of the fixed points for each conjugacy class in \mathcal{F} . Property (iii) also implies that the resulting \mathcal{F} -stable element πX is independent of the choice of collection \mathcal{H} , as long as the chosen collection \mathcal{H} satisfies the assumptions of the lemma.

Proof. We proceed by induction on the size of \mathcal{H} . If $\mathcal{H} = \emptyset$, then X is \mathcal{F} -stable by assumption. Hence the first two properties are vacuously true with $\pi X := X$. Furthermore, since $X \in A(\mathcal{F})_{(p)}$, we have $\Phi_P(X) = \Phi_{P'}(X)$ for all pairs $P \sim_{\mathcal{F}} P'$, and therefore

$$\frac{1}{|[P]_{\mathcal{F}}|} \sum_{P' \in [P]_{\mathcal{F}}} \Phi_{P'}(X) = \frac{\Phi_P(X)}{|[P]_{\mathcal{F}}|} \sum_{P' \in [P]_{\mathcal{F}}} 1 = \Phi_P(X).$$

We therefore assume that $\mathcal{H} \neq \emptyset$, and we let $P \in \mathcal{H}$ be maximal under \mathcal{F} -subconjugation as well as fully normalized.

Let $P' \sim_{\mathcal{F}} P$. Then there is a homomorphism $\varphi \in \mathcal{F}(N_S P', N_S P)$ with $\varphi(P') = P$ by lemma 2.3 since \mathcal{F} is saturated. The restriction of S -actions to the subgroup $\varphi(N_S P')$ gives a ring homomorphism $A(S)_{(p)} \rightarrow A(\varphi(N_S P'))_{(p)}$ that preserves the fixed-point homomorphisms Φ_Q for $Q \leq \varphi(N_S P') \leq N_S P$.

If we consider X as an element of $A(\varphi(N_S P'))$, we can apply the short exact sequence of proposition 3.1 to get $\Psi^{\varphi(N_S P')}(\Phi(X)) = 0$. In particular, the P -coordinate function satisfies $\Psi_P^{\varphi(N_S P')}(\Phi(X)) = 0$, that is,

$$\sum_{\bar{s} \in \varphi(N_S P')/P} \Phi_{\langle s \rangle P}(X) \equiv 0 \pmod{|\varphi(N_S P')/P|}.$$

Similarly, we have $\Psi^S(\Phi^S(X)) = 0$, where the P' -coordinate $\Psi_{P'}^S(\Phi^S(X)) = 0$ gives us

$$\sum_{\bar{s} \in N_S P'/P'} \Phi_{\langle s \rangle P'}(X) \equiv 0 \pmod{|N_S P'/P'|}.$$

Since P is maximal in \mathcal{H} , we have by assumption $\Phi_Q(X) = \Phi_{Q'}(X)$ for all $Q \sim_{\mathcal{F}} Q'$ where P is \mathcal{F} -conjugate to a *proper* subgroup of Q . Specifically, we have

$$\Phi_{\langle \varphi(s) \rangle P}(X) = \Phi_{\langle \varphi(s) \rangle P'}(X) = \Phi_{\langle s \rangle P'}(X)$$

for all $s \in N_S P'$ with $s \notin P'$. It then follows that

$$\begin{aligned} \Phi_P(X) - \Phi_{P'}(X) &= \sum_{\bar{s} \in \varphi(N_S P')/P} \Phi_{\langle s \rangle P}(X) - \sum_{\bar{s} \in N_S P'/P'} \Phi_{\langle s \rangle P'}(X) \\ &\equiv 0 - 0 \pmod{|W_S P'|}. \end{aligned}$$

We can therefore define $\lambda_{P'} := (\Phi_P(X) - \Phi_{P'}(X))/|W_S P'| \in \mathbb{Z}_{(p)}$. We now recall from lemma 4.1 that $|[P]_{\mathcal{F}}| = \frac{|S|}{|N_S P|} \cdot k$ where $p \nmid k$, and since k is invertible in $\mathbb{Z}_{(p)}$, we can define

$$c := \left(\sum_{[P']_S \subseteq [P]_{\mathcal{F}}} \lambda_{P'} \right) / k \in \mathbb{Z}_{(p)},$$

as well as $\mu_{P'} := \lambda_{P'} - \frac{|W_S P|}{|W_S P'|} c \in \mathbb{Z}_{(p)}$. We use the $\mu_{P'}$ as coefficients to construct a new element

$$X' := X + \sum_{[P']_S \subseteq [P]_{\mathcal{F}}} \mu_{P'} \cdot [S/P'] \in A(S)_{(p)}.$$

We then at least have $c_Q(X') = c_Q(X)$ for all $Q \not\sim_{\mathcal{F}} P$. The definition of c ensures that

$$\begin{aligned} \sum_{[P']_S \subseteq [P]_{\mathcal{F}}} \frac{|W_S P|}{|W_S P'|} c &= c \cdot \sum_{[P']_S \subseteq [P]_{\mathcal{F}}} \frac{|N_S P|}{|N_S P'|} = c \cdot \frac{|N_S P|}{|S|} \sum_{[P']_S \subseteq [P]_{\mathcal{F}}} |[P']_S| \\ &= c \cdot \frac{|N_S P|}{|S|} \cdot |[P]_{\mathcal{F}}| = c \cdot k = \sum_{[P']_S \subseteq [P]_{\mathcal{F}}} \lambda_{P'}; \end{aligned}$$

which in turn gives us

$$\begin{aligned} (4.1) \quad & \sum_{[P']_S \subseteq [P]_{\mathcal{F}}} c_{P'}(X') - \sum_{[P']_S \subseteq [P]_{\mathcal{F}}} c_{P'}(X) = \sum_{[P']_S \subseteq [P]_{\mathcal{F}}} \mu_{P'} \\ &= \sum_{[P']_S \subseteq [P]_{\mathcal{F}}} \lambda_{P'} - \sum_{[P']_S \subseteq [P]_{\mathcal{F}}} \frac{|W_S P|}{|W_S P'|} c = 0. \end{aligned}$$

Next we recall that $\Phi_Q([S/P']) = 0$ unless $Q \lesssim_S P'$, which implies that $\Phi_Q(X') = \Phi_Q(X)$ for every $Q \notin \mathcal{H}$. We then calculate $\Phi_{P'}(X')$ for each $P' \sim_{\mathcal{F}} P$:

$$\begin{aligned} (4.2) \quad \Phi_{P'}(X') &= \Phi_{P'}(X) + \sum_{[P'']_S \subseteq [P]_{\mathcal{F}}} \mu_{P''} \cdot \Phi_{P'}([S/P'']) \\ &= \Phi_{P'}(X) + \mu_{P'} \cdot \Phi_{P'}([S/P']) = \Phi_{P'}(X) + \mu_{P'} |W_S P'| \\ &= \Phi_{P'}(X) + \lambda_{P'} |W_S P'| - \frac{|W_S P|}{|W_S P'|} c \cdot |W_S P'| \\ &= \Phi_P(X) - |W_S P| c; \end{aligned}$$

which is independent of the choice of $P' \in [P]_{\mathcal{F}}$.

We define $\mathcal{H}' := \mathcal{H} \setminus [P]_{\mathcal{F}}$ as \mathcal{H} with the \mathcal{F} -conjugates of P removed. Because P is maximal in \mathcal{H} , the subcollection \mathcal{H}' again contains all \mathcal{F} -subconjugates of any $H \in \mathcal{H}'$.

From (4.2) we get that $\Phi_Q(X) = \Phi_{Q'}(X)$ for all $Q \sim_{\mathcal{F}} Q'$ and $Q, Q' \notin \mathcal{H}'$. By induction we can therefore apply lemma 4.3 to X' and the smaller collection \mathcal{H}' . We get an element $\pi X' \in A(\mathcal{F})_{(p)}$ satisfying

- (i) $\Phi_Q(\pi X') = \Phi_Q(X')$ and $c_Q(\pi X') = c_Q(X')$ for all $Q \notin \mathcal{H}'$.
- (ii) For all $Q \leq S$ we have

$$\sum_{[Q']_S \subseteq [Q]_{\mathcal{F}}} c_{Q'}(\pi X') = \sum_{[Q']_S \subseteq [Q]_{\mathcal{F}}} c_{Q'}(X').$$

(iii) For every $Q \leq S$:

$$\Phi_Q(\pi X') = \frac{1}{|[Q]_{\mathcal{F}}|} \sum_{Q' \in [Q]_{\mathcal{F}}} \Phi_{Q'}(X').$$

We claim that $\pi X := \pi X'$ satisfies the properties of the lemma for X and \mathcal{H} as well.

We immediately have that $\Phi_Q(\pi X') = \Phi_Q(X') = \Phi_Q(X)$ and $c_Q(\pi X') = c_Q(X') = c_Q(X)$ for all $Q \notin \mathcal{H}$, so property (i) is satisfied. Since $c_Q(X') = c_Q(X)$ when $Q \not\sim_{\mathcal{F}} P$, we get for all $Q \in \mathcal{H}'$ that

$$\sum_{[Q']_S \subseteq [Q]_{\mathcal{F}}} c_{Q'}(\pi X') = \sum_{[Q']_S \subseteq [Q]_{\mathcal{F}}} c_{Q'}(X') = \sum_{[Q']_S \subseteq [Q]_{\mathcal{F}}} c_{Q'}(X).$$

Furthermore, since $P \notin \mathcal{H}'$, we have $c_{P'}(\pi X') = c_{P'}(X')$ for $P' \sim_{\mathcal{F}} P$. Using (4.1) we then get

$$\sum_{[P']_S \subseteq [P]_{\mathcal{F}}} c_{P'}(\pi X') = \sum_{[P']_S \subseteq [P]_{\mathcal{F}}} c_{P'}(X') = \sum_{[P']_S \subseteq [P]_{\mathcal{F}}} c_{P'}(X).$$

This proves that (ii) is satisfied. Since $c_Q(X') = c_Q(X)$ when $Q \not\sim_{\mathcal{F}} P$, we have $\Phi_Q(X') = \Phi_Q(X)$ for all Q that are not \mathcal{F} -subconjugate to P . Consequently we have

$$\begin{aligned} \Phi_Q(\pi X') &= \frac{1}{|[Q]_{\mathcal{F}}|} \sum_{Q' \in [Q]_{\mathcal{F}}} \Phi_{Q'}(X') \\ &= \frac{1}{|[Q]_{\mathcal{F}}|} \sum_{Q' \in [Q]_{\mathcal{F}}} \Phi_{Q'}(X), \end{aligned}$$

when Q is not \mathcal{F} -subconjugate to P . We need the small lemma 4.2 below to show that every $P' \sim_{\mathcal{F}} P$ satisfies

$$\frac{1}{|[Q]_{\mathcal{F}}|} \sum_{Q' \in [Q]_{\mathcal{F}}} \Phi_{Q'}([S/P']) = \frac{|\mathcal{F}(Q, P)| \cdot |S|}{|P| \cdot |\mathcal{F}(Q, S)|} \in \mathbb{Z}_{(p)}$$

for all $Q \leq S$. In the case where Q is subconjugate to P in \mathcal{F} , we can then use both (4.1) and (4.2) to show that

$$\begin{aligned} \Phi_Q(\pi X') &= \frac{1}{|[Q]_{\mathcal{F}}|} \sum_{Q' \in [Q]_{\mathcal{F}}} \Phi_{Q'}(X') \\ &= \frac{1}{|[Q]_{\mathcal{F}}|} \sum_{Q' \in [Q]_{\mathcal{F}}} \Phi_{Q'}(X) + \sum_{[P']_S \subseteq [P]_{\mathcal{F}}} \mu_{P'} \left(\frac{1}{|[Q]_{\mathcal{F}}|} \sum_{Q' \in [Q]_{\mathcal{F}}} \Phi_{Q'}([S/P']) \right) \\ &= \frac{1}{|[Q]_{\mathcal{F}}|} \sum_{Q' \in [Q]_{\mathcal{F}}} \Phi_{Q'}(X) + \sum_{[P']_S \subseteq [P]_{\mathcal{F}}} \mu_{P'} \cdot \frac{|\mathcal{F}(Q, P)| \cdot |S|}{|P| \cdot |\mathcal{F}(Q, S)|} \\ &= \frac{1}{|[Q]_{\mathcal{F}}|} \sum_{Q' \in [Q]_{\mathcal{F}}} \Phi_{Q'}(X) + 0; \end{aligned}$$

which proves that $\pi X'$ satisfies (iii). \square

Theorem A. *Let \mathcal{F} be a saturated fusion system on a finite p -group S . We let $A(\mathcal{F})_{(p)}$ denote the p -localized Burnside ring of \mathcal{F} as a subring of the p -localized Burnside ring $A(S)_{(p)}$ for S . Then there is a transfer map $\pi: A(S)_{(p)} \rightarrow A(\mathcal{F})_{(p)}$, which is a homomorphism of $A(\mathcal{F})_{(p)}$ -modules and which restricts to the identity on $A(\mathcal{F})_{(p)}$. In terms of the fixed point homomorphisms the transfer map π satisfies*

$$\Phi_Q(\pi(X)) = \frac{1}{|[Q]_{\mathcal{F}}|} \sum_{Q' \in [Q]_{\mathcal{F}}} \Phi_{Q'}(X),$$

where $[Q]_{\mathcal{F}}$ is the conjugacy class of Q in \mathcal{F} .

Proof. To construct $\pi(X)$ we apply lemma 4.3 to X and the collection \mathcal{H} of all subgroups in S . This results in a stable element $\pi(X) \in A(\mathcal{F})_{(p)}$ satisfying

$$\Phi_Q(\pi(X)) = \frac{1}{|[Q]_{\mathcal{F}}|} \sum_{Q' \in [Q]_{\mathcal{F}}} \Phi_{Q'}(X)$$

as wanted. If we apply π to an element X that is already \mathcal{F} -stable, then

$$\Phi_Q(\pi(X)) = \frac{1}{|[Q]_{\mathcal{F}}|} \sum_{Q' \in [Q]_{\mathcal{F}}} \Phi_{Q'}(X) = \frac{1}{|[Q]_{\mathcal{F}}|} \sum_{Q' \in [Q]_{\mathcal{F}}} \Phi_Q(X) = \Phi_Q(X),$$

so $\pi(X) = X$. Hence π is the identity map when restricted to $A(\mathcal{F})_{(p)}$.

If $X \in A(\mathcal{F})_{(p)}$ and $Y \in A(S)_{(p)}$, then since the fixed point homomorphisms preserve products, we have

$$\begin{aligned} \Phi_Q(\pi(XY)) &= \frac{1}{|[Q]_{\mathcal{F}}|} \sum_{Q' \in [Q]_{\mathcal{F}}} \Phi_{Q'}(XY) = \frac{1}{|[Q]_{\mathcal{F}}|} \sum_{Q' \in [Q]_{\mathcal{F}}} \Phi_{Q'}(X)\Phi_{Q'}(Y) \\ &= \Phi_Q(X) \cdot \frac{1}{|[Q]_{\mathcal{F}}|} \sum_{Q' \in [Q]_{\mathcal{F}}} \Phi_{Q'}(Y) = \Phi_Q(X) \cdot \Phi_Q(\pi(Y)). \end{aligned}$$

This shows that $\pi(XY) = X \cdot \pi(Y)$, and by a similar argument, π preserves addition. Hence π is a homomorphism of $A(\mathcal{F})_{(p)}$ -modules. \square

Remark 4.4. As stated in lemma 4.3 the stabilization homomorphism $\pi: A(\mathcal{F})_{(p)} \rightarrow A(S)_{(p)}$ also satisfies

$$\sum_{[P']_S \subseteq [P]_{\mathcal{F}}} c_{P'}(\pi(X)) = \sum_{[P']_S \subseteq [P]_{\mathcal{F}}} c_{P'}(X).$$

Hence π replaces orbits of X within each \mathcal{F} -conjugation class, but doesn't otherwise add or remove orbits from X . This fact will be important for describing the action of the characteristic idempotent on bisets in theorem C of section 5.

We know that the transitive S sets $[S/P]$ form a basis for $A(S)_{(p)}$. We now apply the projection $\pi: A(S)_{(p)} \rightarrow A(\mathcal{F})_{(p)}$ to this basis, and we get a new basis for the p -localized Burnside ring $A(\mathcal{F})_{(p)}$.

Proposition 4.5. *Let $\beta_P \in A(\mathcal{F})_{(p)}$ be defined by $\beta_P := \pi([S/P])$. In terms of the homomorphism of marks, β_P is then given by*

$$\Phi_Q(\beta_P) = \frac{|\mathcal{F}(Q, P)| \cdot |S|}{|P| \cdot |\mathcal{F}(Q, S)|} \in \mathbb{Z}_{(p)},$$

hence β_P only depends on P up to \mathcal{F} -conjugation.

The elements β_P defined this way form a $\mathbb{Z}_{(p)}$ -basis for $A(\mathcal{F})_{(p)}$.

Proof. By theorem A and lemma 4.2 the element $\beta_P := \pi([S/P])$ satisfies

$$\Phi_Q(\beta_P) = \frac{1}{|[Q]_{\mathcal{F}}|} \sum_{Q' \in [Q]_{\mathcal{F}}} \Phi_{Q'}([S/P]) = \frac{|\mathcal{F}(Q, P)| \cdot |S|}{|P| \cdot |\mathcal{F}(Q, S)|} \in \mathbb{Z}_{(p)},$$

where the last expression, and thus β_P , only depends on P up to conjugation in \mathcal{F} .

Because the transitive S -sets $[S/P]$ for $P \leq S$ generate $A(S)_{(p)}$, and since π is surjective, the elements β_P must generate all of $A(\mathcal{F})_{(p)}$.

We now order the \mathcal{F} -conjugacy classes $[P]_{\mathcal{F}}$ according to decreasing order of P , and the mark homomorphism $\Phi: \text{Span}\{\beta_P\} \rightarrow \tilde{\Omega}(\mathcal{F})_{(p)}$ is then represented by a matrix M with entries

$$M_{Q,P} = \frac{|\mathcal{F}(Q, P)| \cdot |S|}{|P| \cdot |\mathcal{F}(Q, S)|}.$$

If Q is not \mathcal{F} -subconjugate to P , then $M_{Q,P} = 0$; so M is a lower triangular matrix with diagonal entries

$$M_{P,P} = \frac{|\mathcal{F}(P, P)| \cdot |S|}{|P| \cdot |\mathcal{F}(P, S)|} \neq 0.$$

Since all diagonal entries are non-zero, we conclude that the β_P are linearly independent over $\mathbb{Z}_{(p)}$. \square

The mark homomorphism $\Phi: A(S)_{(p)} \rightarrow \tilde{\Omega}(S)_{(p)}$ embeds the Burnside ring of S into its ghost ring, and since we know the value of $\Phi_Q(\beta_P)$ from proposition 4.5, we know the image of β_P inside $\tilde{\Omega}(S)_{(p)}$. We might then wonder whether we can pull back our knowledge from $\tilde{\Omega}(S)_{(p)}$ to $A(S)_{(p)}$ and write β_P explicitly as a linear combination of transitive S -sets.

In [10], David Gluck gives a method on how to do exactly this. Because $A(S)_{(p)}$ embeds in the ghost ring $\tilde{\Omega}(S)_{(p)}$ as a subring of finite index, if we take the tensor product with \mathbb{Q} , we get an isomorphism $\Phi: A(S) \otimes \mathbb{Q} \xrightarrow{\cong} \tilde{\Omega}(S) \otimes \mathbb{Q}$. What [10] then contains is an expression for the inverse isomorphism. Let $e_Q := (0, \dots, 0, 1, 0, \dots, 0)$ be the standard basis element of $\tilde{\Omega}(S) \otimes \mathbb{Q}$ corresponding to the subgroup $Q \leq S$.

The inverse $\Phi^{-1}: \tilde{\Omega}(S) \otimes \mathbb{Q} \rightarrow A(S) \otimes \mathbb{Q}$ is then given by

$$(4.3) \quad \Phi^{-1}(e_Q) = \frac{1}{|N_S Q|} \sum_{R \leq Q} \mu(R, Q) \cdot |R| \cdot [S/R],$$

where μ is the Möbius-function for the poset of subgroups in S .

Since we know the image $\Phi(\beta_P)$, we can apply the isomorphism above to get an expression for β_P inside $A(S) \otimes \mathbb{Q}$; and because $A(S)_{(p)}$ is embedded in $A(S) \otimes \mathbb{Q}$, the expression holds in $A(S)_{(p)}$ as well.

Proposition 4.6. *For each $P \leq S$, the element $\beta_P \in A(\mathcal{F})_{(p)}$ is given by the following expression when written as a $\mathbb{Z}_{(p)}$ -linear combination of transitive S -sets:*

$$\begin{aligned} \beta_P &= \sum_{[R]_S} \frac{1}{\Phi_R([S/R])} \left(\sum_{R \leq Q \leq S} \Phi_Q(\beta_P) \cdot \mu(R, Q) \right) [S/R] \\ &= \sum_{[R]_S} \frac{|R| \cdot |S|}{|N_S R| \cdot |P|} \left(\sum_{R \leq Q \leq S} \frac{|\mathcal{F}(Q, P)|}{|\mathcal{F}(Q, S)|} \cdot \mu(R, Q) \right) [S/R]. \end{aligned}$$

Proof. By proposition 4.5 we have

$$\Phi(\beta_P) = \sum_{[Q]_S} \Phi_Q(\beta_P) \cdot e_Q = \sum_{Q \leq S} \frac{|N_S Q|}{|S|} \cdot \Phi_Q(\beta_P) \cdot e_Q.$$

We then apply the formula (4.3) for the inverse of Φ and get

$$\begin{aligned} \beta_P &= \Phi^{-1} \left(\sum_{Q \leq S} \frac{|N_S Q|}{|S|} \cdot \Phi_Q(\beta_P) \cdot e_Q \right) \\ &= \sum_{Q \leq S} \frac{|N_S Q|}{|S|} \cdot \Phi_Q(\beta_P) \cdot \frac{1}{|N_S Q|} \left(\sum_{R \leq Q \leq S} \mu(R, Q) \cdot |R| \cdot [S/R] \right) \\ &= \sum_{R \leq S} \frac{|R|}{|S|} \left(\sum_{R \leq Q \leq S} \Phi_Q(\beta_P) \cdot \mu(R, Q) \right) [S/R] \\ &= \sum_{[R]_S} \frac{|R|}{|N_S R|} \left(\sum_{R \leq Q \leq S} \Phi_Q(\beta_P) \cdot \mu(R, Q) \right) [S/R]. \quad \square \end{aligned}$$

4.1. Equivalent Burnside rings. In this section we compare the ring $A(\mathcal{F})_{(p)}$, with the β_P -basis, to other Burnside rings related to the saturated fusion system \mathcal{F} . First we consider the case where \mathcal{F} is realized by a group G : We see that $A(\mathcal{F})_{(p)}$ is isomorphic to the ring $A(G; p)_{(p)}$ generated by G -sets $[G/P]$ where $P \leq G$ is a p -group, and the basis element β_P almost corresponds to the transitive G -set $[G/P]$. After that, we consider the Burnside ring $A^{cent}(\mathcal{F})$ introduced by Antonio Diaz and Assaf Libman in [5], which is defined using only the centric subgroups of \mathcal{F} : We show that after p -localization $A^{cent}(\mathcal{F})_{(p)}$ is isomorphic to the “centric part” of $A(\mathcal{F})_{(p)}$, again with the basis elements corresponding to each other in a suitable way. Both of these isomorphisms are originally due to Diaz-Libman in [6] as example 3.9 and theorem A, respectively. New in this section is the fact that the bases of the rings correspond as well.

Proposition 4.7. *Suppose that S is a Sylow p -subgroup of G , and let $\mathcal{F} := \mathcal{F}_S(G)$. Define $A(G; p)$ to be the subring of $A(G)$ where all isotropy subgroups are p -groups.*

Then the transitive G -set $[G/S]$ is invertible in $A(G; p)_{(p)}$, and we get an isomorphism of rings $A(\mathcal{F})_{(p)} \cong A(G; p)_{(p)}$ by

$$\beta_P \mapsto \frac{[G/P]}{[G/S]}.$$

This isomorphism is in a way the best we could hope for, since the basis element β_P only depends on the fusion data in $\mathcal{F}_S(G)$, while the transitive G -set $[G/P]$ depends on the actual group G . If we replace G with a product $G' = G \times H$ where H is a p' -group, then the fusion system $\mathcal{F}_S(G')$ and β_P are the same for G' as for G , but the transitive set $[G'/P]$ has increased in size by a factor $|H|$. However, as we see in the proof below, the quotient $\frac{[G/P]}{[G/S]}$ depends only on the fusion system and not on G .

Proof. We first show that $[G/S]$ is invertible in $A(G)_{(p)}$. For every $Q \leq S$ that is fully \mathcal{F} -normalized, we have

$$\Phi_Q([G/S]) = \frac{|N_G(Q, S)|}{|S|} = \frac{|N_G Q| \cdot |\{Q' \leq S \mid Q' \sim_{\mathcal{F}} Q\}|}{|S|}.$$

By lemma 4.1, we have $|\{Q' \leq S \mid Q' \sim_{\mathcal{F}} Q\}| = \frac{|S|}{|N_S Q|} \cdot k$ with $p \nmid k$. We thus get

$$\Phi_Q([G/S]) = \frac{|N_G Q|}{|N_S Q|} \cdot k,$$

which is invertible in $\mathbb{Z}_{(p)}$ since Q is fully \mathcal{F} -normalized.

If $H \leq G$ is not a p -group, then $\Phi_H(X) = 0$ for all $X \in A(G; p)_{(p)}$. We also know that every p -subgroup of G is conjugate to a subgroup of S by Sylow's theorems, and therefore the mark homomorphism for $A(G)_{(p)}$ restricts to an inclusion

$$\Phi: A(G; p)_{(p)} \rightarrow \prod_{[Q]_{\mathcal{F}}} \mathbb{Z}_{(p)} = \tilde{\Omega}(\mathcal{F})_{(p)},$$

and $A(G; p)_{(p)}$ has finite index in $\tilde{\Omega}(\mathcal{F})_{(p)}$ for rank reasons.

Because $\Phi_Q([G/S])$ is invertible in $\mathbb{Z}_{(p)}$, $[G/S]$ is invertible in the ghost ring $\tilde{\Omega}(\mathcal{F})_{(p)}$. It follows that multiplication with $[G/S]$ is a bijection $\tilde{\Omega}(\mathcal{F})_{(p)} \rightarrow \tilde{\Omega}(\mathcal{F})_{(p)}$, which sends $A(G; p)_{(p)}$ into itself. Since $A(G; p)_{(p)}$ has finite index in $\tilde{\Omega}(\mathcal{F})_{(p)}$, multiplication with $[G/S]$ must then also be a bijection of $A(G; p)_{(p)}$ to itself, hence $[G/S]$ is invertible in $A(G; p)_{(p)}$.

It thus makes sense to consider the elements $\frac{[G/P]}{[G/S]}$ for $P \leq S$, and we calculate

$$\Phi_Q \left(\frac{[G/P]}{[G/S]} \right) = \frac{|N_G(Q, P)| \cdot |S|}{|P| \cdot |N_G(Q, S)|} = \frac{|\mathcal{F}(Q, P)| \cdot |S|}{|P| \cdot |\mathcal{F}(Q, S)|} = \Phi_Q(\beta_P).$$

It follows that $\frac{[G/P]}{[G/S]} = \beta_P$ as elements of $\tilde{\Omega}(\mathcal{F})_{(p)}$, giving the isomorphism $A(G; p)_{(p)} \cong A(\mathcal{F})_{(p)}$. \square

The Burnside ring defined by Diaz-Libman in [5] for a saturated fusion system \mathcal{F} , is constructed in terms of an orbit category over the \mathcal{F} -centric subgroups of S . A subgroup $P \leq S$ is \mathcal{F} -centric if all \mathcal{F} -conjugates $P' \sim_{\mathcal{F}} P$ are self-centralizing, i.e., $C_S(P') \leq P'$. We denote the Diaz-Libman Burnside ring by $A^{cent}(\mathcal{F})$, and it comes equipped with an additive basis ξ_P indexed by the \mathcal{F} -conjugacy classes of \mathcal{F} -centric subgroups. As shown in [5] there is also an injective homomorphism of marks

$$\Phi^{cent}: A^{cent}(\mathcal{F}) \rightarrow \prod_{\substack{[P]_{\mathcal{F}} \\ P \text{ is } \mathcal{F}\text{-centric}}} \mathbb{Z}$$

with finite cokernel, and on basis elements Φ^{cent} is given by

$$\Phi_Q^{cent}(\xi_P) = \frac{|Z(Q)| \cdot |\mathcal{F}(Q, P)|}{|P|}.$$

Proposition 4.8. *Let \mathcal{F} be a saturated fusion system on a p -group S , and write $N \leq A(\mathcal{F})_{(p)}$ for the $\mathbb{Z}_{(p)}$ -submodule generated by β_P for non- \mathcal{F} -centric P . Then N is an ideal in the Burnside ring $A(\mathcal{F})_{(p)}$, and there is a ring isomorphism $A(\mathcal{F})_{(p)}/N \cong A^{cent}(\mathcal{F})_{(p)}$ with the Burnside ring of Diaz-Libman. The basis element ξ_S is invertible in $A^{cent}(\mathcal{F})_{(p)}$, and the isomorphism is given by*

$$\overline{\beta_P} \mapsto \frac{\xi_P}{\xi_S}$$

for \mathcal{F} -centric $P \leq S$.

Proof. If P is \mathcal{F} -centric, then any subgroup containing P is \mathcal{F} -centric as well, hence the collection of *non*- \mathcal{F} -centric subgroup is closed under \mathcal{F} -conjugation and taking subgroups. By the double coset formula (3.1) for $A(S)_{(p)}$, the $\mathbb{Z}_{(p)}$ -submodule generated by the elements $[S/P]$ with P non- \mathcal{F} -centric is an ideal in $A(S)_{(p)}$. Let us denote this ideal $M \leq A(S)_{(p)}$.

The stabilization map $\pi: A(S)_{(p)} \rightarrow A(\mathcal{F})_{(p)}$ is a homomorphism of $A(\mathcal{F})_{(p)}$ -modules, so the image $N := \pi(M)$ is an ideal of $A(\mathcal{F})_{(p)}$, and at the same time N is the $\mathbb{Z}_{(p)}$ -submodule generated by the elements $\pi([S/P]) = \beta_P$ where P is non- \mathcal{F} -centric. By proposition 4.5, we have $\Phi_Q(\beta_P) = 0$ whenever Q is \mathcal{F} -centric and P is not. Hence the homomorphism

$$A(\mathcal{F})_{(p)} \xrightarrow{\Phi} \prod_{[P]_{\mathcal{F}}} \mathbb{Z}_{(p)} \rightarrow \prod_{\substack{[P]_{\mathcal{F}} \\ P \text{ is } \mathcal{F}\text{-centric}}} \mathbb{Z}_{(p)}$$

send N to 0, and therefore induces a ring homomorphism

$$\Phi: A(\mathcal{F})_{(p)}/N \rightarrow \prod_{\substack{[P]_{\mathcal{F}} \\ P \text{ is } \mathcal{F}\text{-centric}}} \mathbb{Z}_{(p)}.$$

Let $\overline{\beta_P}$ denote the equivalence class of β_P in $A(\mathcal{S})_{(p)}/N$ when P is \mathcal{F} -centric. The quotient ring $A(\mathcal{F})_{(p)}/N$ then has a basis consisting of $\overline{\beta_P}$ for each \mathcal{F} -centric P up to \mathcal{F} -conjugation.

The rest of this proof follows the same lines as the proof of proposition 4.7: For the basis element ξ_S of $A^{cent}(\mathcal{F})_{(p)}$ the image under the mark homomorphism has the form $\Phi_Q^{cent}(\xi_S) = \frac{|Z(Q)|}{|S|} \cdot |\mathcal{F}(Q, S)|$, which by lemma 4.1 is invertible in $\mathbb{Z}_{(p)}$. Hence ξ_S is invertible in the ghost ring

$$\prod_{\substack{[P]_{\mathcal{F}} \\ P \text{ is } \mathcal{F}\text{-centric}}} \mathbb{Z}_{(p)},$$

and since Φ^{cent} has finite cokernel, it follows that ξ_S is invertible in $A^{cent}(\mathcal{F})_{(p)}$ as well. It therefore makes sense to form the fractions $\frac{\xi_P}{\xi_S}$. Applying the fixed point homomorphisms to these fractions, we then get

$$\Phi_Q^{cent} \left(\frac{\xi_P}{\xi_S} \right) = \frac{|Z(Q)| \cdot |\mathcal{F}(Q, P)| \cdot |S|}{|P| \cdot |\mathcal{F}(Q, S)| \cdot |Z(Q)|} = \frac{|\mathcal{F}(Q, P)| \cdot |S|}{|P| \cdot |\mathcal{F}(Q, S)|} = \Phi_Q(\beta_P)$$

for all \mathcal{F} -centric subgroup $Q, P \leq S$. This shows that the ring homomorphism

$$\Phi: A(\mathcal{F})_{(p)}/N \rightarrow \prod_{\substack{[P]_{\mathcal{F}} \\ P \text{ is } \mathcal{F}\text{-centric}}} \mathbb{Z}_{(p)}$$

sends $\overline{\beta_P}$ to $\Phi^{cent}(\frac{\xi_P}{\xi_S})$, which proves that Φ is injective on $A(\mathcal{F})_{(p)}$ and that $\beta_P \mapsto \frac{\xi_P}{\xi_S}$ gives a ring isomorphism $A(\mathcal{F})_{(p)}/N \cong A^{cent}(\mathcal{F})_{(p)}$. \square

5. THE CHARACTERISTIC IDEMPOTENT

In this section we make use of the stabilization homomorphism of theorem A to give new results on the characteristic idempotent for a saturated fusion system. These idempotents were shown by Ragnarsson and Stancu to classify the saturated fusion systems on a given p -group. In section 5.1 we recall the structures of the double Burnside rings and modules, and the category that they form. In section 5.2 we give a new construction of the characteristic idempotent $\omega_{\mathcal{F}}$ for a saturated fusion system \mathcal{F} on S by stabilizing the diagonal subgroup $\Delta(S) \leq S \times S$ with respect to the fusion system $\mathcal{F} \times \mathcal{F}$. As a consequence we discover the value of the mark homomorphism on the idempotent, and

this is the content of theorem B. In section 5.3 we discuss multiplication $X \mapsto \omega_{\mathcal{F}} \circ X$ with the characteristic idempotent – both for elements X of the double Burnside ring of S , but also more generally when X is just some finite set with an S -action. Theorem C describes the action of $\omega_{\mathcal{F}}$ in terms of the homomorphism of marks.

5.1. The category of Burnside modules. For finite groups G and H , a (G, H) -biset is a set with both a left H -action and a right G -action, and such that the two actions commute. A (G, H) -biset X gives rise to a $(H \times G)$ -set by defining $(h, g).x := hxg^{-1}$, and vice versa. The transitive (G, H) -bisets have the form $[(H \times G)/D]$ for subgroups $D \leq H \times G$. The isomorphism classes of finite (G, H) -bisets form a monoid, and the Grothendieck group $A(G, H)$ is called the *Burnside module* of G and H . Additively $A(G, H)$ is isomorphic to $A(H \times G)$ and we have a basis consisting of the transitive bisets $[(H \times G)/D]$ where $D \leq H \times G$ is determined up to $(H \times G)$ -conjugation.

The multiplication for the Burnside modules is different from the non-biset Burnside rings. We have multiplication/composition maps $\circ: A(H, K) \times A(G, H) \rightarrow A(G, K)$, defined for every (G, H) -biset X and (H, K) -biset Y as

$$Y \circ X := Y \times_H X = Y \times X / \sim$$

where $(yh, x) \sim (y, hx)$ for all $y \in Y$, $x \in X$ and $h \in H$. With this composition, the Burnside modules form the Hom-sets of a category with finite groups as objects. The ring of endomorphisms $A(G, G)$ of G is the *double Burnside ring* of G . The identity element of $A(G, G)$ is the group G considered as a (G, G) -biset. On transitive bisets, the composition is given by a double coset formula

$$(5.1) \quad [(K \times H)/D] \circ [(H \times G)/C] = \sum_{\bar{x} \in \pi_2 D \backslash H / \pi_1 C} [(K \times G)/(D * {}^{(x,1)}C)]$$

where the subgroup $B * A$ is defined as $\{(k, g) \in K \times G \mid \exists h \in H: (k, h) \in B, (h, g) \in A\}$ for subgroups $B \leq K \times H$ and $A \leq H \times G$.

Given a homomorphism $\varphi: U \rightarrow H$ with $U \leq G$, the graph $\Delta(U, \varphi) = \{(\varphi u, u) \mid u \in U\}$ is a subgroup of $H \times G$. We introduce the notation $[U, \varphi]_G^H$ as a shorthand for the biset $[(H \times G)/\Delta(U, \varphi)]$, and if the groups G, H are clear from context, we just write $[U, \varphi]$. The bisets $[U, \varphi]$ generate the (G, H) -bisets that have a free left H -action. For these basis elements, (5.1) takes the form

$$(5.2) \quad [T, \psi]_H^K \circ [U, \varphi]_G^H = \sum_{\bar{x} \in T \backslash H / \varphi U} [\varphi^{-1}(T^x) \cap U, \psi c_x \varphi]_G^K.$$

From the isomorphism $A(G, H) \cong A(H \times G)$ of additive groups, the Burnside modules inherit fixed point homomorphisms $\Phi_C: A(G, H) \rightarrow \mathbb{Z}$ for each $(H \times G)$ -conjugacy class of subgroups $C \leq H \times G$. Note however that the fixed point homomorphisms for $A(G, G)$ are *not* ring homomorphisms – they are only homomorphisms of abelian groups.

Given any (G, H) -biset X , we can swap the actions to get an (H, G) -biset X^{op} with $g.x^{\text{op}}.h := h^{-1}.x.g^{-1}$, which extends to a group isomorphism $(-)^{\text{op}}: A(G, H) \rightarrow A(H, G)$. We clearly have $[(H \times G)/D]^{\text{op}} = [(G \times H)/D^{\text{op}}]$ and $\Phi_C(X^{\text{op}}) = \Phi_{C^{\text{op}}}(X)$, where $C^{\text{op}}, D^{\text{op}}$ are the subgroups C, D with the coordinates swapped. Any element of the double Burnside ring $X \in A(G, G)$ that satisfies $X^{\text{op}} = X$ is called *symmetric*.

5.2. A new construction of the characteristic idempotent. Let \mathcal{F} be a fusion system on a p -group S . We then say that an element of the p -localized double Burnside ring $A(S, S)_{(p)}$ is \mathcal{F} -characteristic if it satisfies the Linckelmann-Webb properties: The element

is \mathcal{F} -generated (see 5.1), it is \mathcal{F} -stable (see 5.2), and finally there is a p' -condition for the number of elements (see 5.4).

K. Ragnarsson showed in [12] that for every saturated fusion system \mathcal{F} on a p -group S , there is a unique idempotent $\omega_{\mathcal{F}} \in A(S, S)_{(p)}$ that is \mathcal{F} -characteristic, and [13] shows how \mathcal{F} can be reconstructed from $\omega_{\mathcal{F}}$ (or any \mathcal{F} -characteristic element). To construct the characteristic idempotent of a saturated fusion system, Ragnarsson used a Cauchy-sequence argument in the p -completion $A(S, S)_p^{\wedge}$ to construct $\omega_{\mathcal{F}}$ as an element of $A(S, S)_p^{\wedge}$. Later arguments then showed that $\omega_{\mathcal{F}}$ lives already in the p -localization $A(S, S)_{(p)}$, and that it is unique.

In this section we give a new construction of $\omega_{\mathcal{F}}$ inside $A(S, S)_{(p)}$ directly; in fact $\omega_{\mathcal{F}}$ turns out to be the basis element $\beta_{\Delta(S)}$ of proposition 4.5 with respect to the fusion system $\mathcal{F} \times \mathcal{F}$ on $S \times S$. As a consequence we learn the value of the fixed point homomorphisms on $\omega_{\mathcal{F}}$ as stated in theorem B, and we also gain a (complicated) decomposition of $\omega_{\mathcal{F}}$ into (S, S) -orbits.

Definition 5.1. Let \mathcal{F} be a fusion system on a p -group S . An element $X \in A(S, S)$ is then said to be \mathcal{F} -generated if X is expressed solely in terms of basis elements $[P, \varphi]$ where $\varphi: P \rightarrow S$ is a morphism of \mathcal{F} . The \mathcal{F} -generated elements form a subring $A_{\mathcal{F}}(S, S)$ of the double Burnside ring, and since $[P, \varphi]^{\text{op}} = [\varphi P, \varphi^{-1}]$ for all $\varphi \in \mathcal{F}(P, S)$, the ring $A_{\mathcal{F}}(S, S)$ of \mathcal{F} -generated elements is stable with respect to the reflection $(-)^{\text{op}}$.

Any subgroup of a graph $\Delta(P, \varphi)$ with $\varphi \in \mathcal{F}(P, S)$ has the form $\Delta(R, \varphi|_R)$ for some subgroup $R \leq P$. By (3.2) we thus have $\Phi_D([P, \varphi]) = 0$ unless D is the graph of a morphism in \mathcal{F} . An element $X \in A(S, S)_{(p)}$ is therefore \mathcal{F} -generated if and only if $\Phi_D([P, \varphi]) = 0$ for all subgroup $D \leq S \times S$ that are not graphs from \mathcal{F} .

Definition 5.2. For the Burnside ring of a group $A(S)$ we defined by (3.3) what it means for an S -set to be \mathcal{F} -stable. With bisets we now have both a left and a right actions, hence we get two notions of stability:

Let $\mathcal{F}_1, \mathcal{F}_2$ be fusion systems on p -groups S_1, S_2 respectively. Any $X \in A(S_1, S_2)_{(p)}$ is said to be *right \mathcal{F}_1 -stable* if it satisfies

$$(5.3) \quad X \circ [P, \varphi]_P^{S_1} = X \circ [P, \text{id}]_P^{S_1} \text{ inside } A(P, S_2)_{(p)}, \text{ for all } P \leq S_1 \text{ and } \varphi: P \rightarrow S_1 \text{ in } \mathcal{F}_1.$$

Similarly $X \in A(S_1, S_2)_{(p)}$ is *left \mathcal{F}_2 -stable* if it satisfies

$$(5.4) \quad [\varphi P, \varphi^{-1}]_{S_2}^P \circ X = [P, \text{id}]_{S_2}^P \circ X \text{ inside } A(S_1, P)_{(p)}, \text{ for all } P \leq S_2 \text{ and } \varphi: P \rightarrow S_2 \text{ in } \mathcal{F}_2.$$

Because $([P, \varphi])^{\text{op}} = [\varphi P, \varphi^{-1}]$ when φ is injective, we clearly have that X is right \mathcal{F} -stable if and only if X^{op} is left \mathcal{F} -stable. For the double Burnside ring $A(S, S)_{(p)}$, any element that is both left and right \mathcal{F} -stable is said to be *fully \mathcal{F} -stable* or just *\mathcal{F} -stable*.

As with \mathcal{F} -stability in $A(S)_{(p)}$, we can characterize left and right stability in terms of the homomorphism of marks for the double Burnside ring.

Lemma 5.3. Let $\mathcal{F}_1, \mathcal{F}_2$ be fusion systems on p -groups S_1, S_2 respectively. The following are then equivalent for all $X \in A(S_1, S_2)_{(p)}$:

- (i) X is both right \mathcal{F}_1 -stable and left \mathcal{F}_2 -stable.
- (ii) X considered as an element of $A(S_2 \times S_1)_{(p)}$ is $(\mathcal{F}_2 \times \mathcal{F}_1)$ -stable.
- (iii) $\Phi_D(X) = \Phi_{D'}(X)$ for all subgroups $D, D' \leq S_2 \times S_1$ that are $(\mathcal{F}_2 \times \mathcal{F}_1)$ -conjugate.

The analogue statements for right and left stability follow if we let \mathcal{F}_1 or \mathcal{F}_2 be trivial fusion systems.

For the purposes of this paper it would be sufficient to state lemma 5.3 and later results only for bisets where both actions are free, in which case the proof of lemma 5.3 would be easier. However, all the later proofs are nearly identical in the bifree and non-free cases, so for completeness sake we include the general statements – though the following proof becomes harder.

Proof. The equivalence of (ii) and (iii) is just the characterization of stability in Burnside rings (see page 9).

Suppose that $X \in A(S_1, S_2)_{(p)}$ is both right \mathcal{F}_1 -stable and left \mathcal{F}_2 -stable. Let the map $\varphi \in \text{Hom}_{\mathcal{F}_2 \times \mathcal{F}_1}(D, S_2 \times S_1)$ be any homomorphism in the product fusion system, and let the ring homomorphism $\varphi^*: A(S_2 \times S_1)_{(p)} \rightarrow A(D)_{(p)}$ be the restriction along φ . For subgroups $D \leq C \leq S_2 \times S_1$ we also let incl_D^C denote the inclusion of D in C . We then wish to show that $\varphi^*(X) = (\text{incl}_D^{S_2 \times S_1})^*(X)$. Define D_i to be the projection of D to the group S_i , then by definition of the product fusion system φ has the form $(\varphi_2 \times \varphi_1)|_D$ for suitable morphisms $\varphi_i \in \mathcal{F}_i(D_i, S_i)$. The restriction homomorphism φ^* thus decomposes as

$$\varphi^*: A(S_2 \times S_1)_{(p)} \xrightarrow{(\varphi_2 \times \varphi_1)^*} A(D_2 \times D_1)_{(p)} \xrightarrow{(\text{incl}_D^{D_2 \times D_1})^*} A(D)_{(p)}.$$

On (S_1, S_2) -bisets the composition

$$[\varphi_2 D_2, \varphi_2^{-1}]_{S_2}^{D_2} \circ X \circ [D_1, \varphi_1]_{D_1}^{S_1}$$

is exactly the same as the restriction $(\varphi_2 \times \varphi_1)^*$ of $(S_2 \times S_1)$ -sets, and by the assumed stability of X we therefore get

$$\begin{aligned} (\varphi_2 \times \varphi_1)^*(X) &= [\varphi_2 D_2, \varphi_2^{-1}]_{S_2}^{D_2} \circ X \circ [D_1, \varphi_1]_{D_1}^{S_1} \\ &= [D_2, \text{id}]_{S_2}^{D_2} \circ X \circ [D_1, \text{id}]_{D_1}^{S_1} = (\text{incl}_{D_2 \times D_1}^{S_2 \times S_1})^*(X). \end{aligned}$$

Restricting further to D , we then have $\varphi^*(X) = (\text{incl}_D^{S_2 \times S_1})^*(X)$ as claimed.

Suppose conversely that X is $\mathcal{F}_2 \times \mathcal{F}_1$ -stable. Then in particular we assume that $(\text{id} \times \varphi)^*(X) = (\text{incl}_{S_2 \times P}^{S_2 \times S_1})^*(X)$ for all maps $\varphi \in \mathcal{F}_1(P, S_1)$, hence we have

$$X \circ [P, \varphi]_P^{S_1} = (\text{id} \times \varphi)^*(X) = (\text{incl}_{S_2 \times P}^{S_2 \times S_1})^*(X) = X \circ [P, \text{id}]_P^{S_1}$$

so X is right \mathcal{F}_1 -stable. Similarly we get that X is left \mathcal{F}_2 -stable as well. \square

Let $A^\triangleleft(S, S)_{(p)}$ be the subring of the double Burnside ring generated by left-free bisets, i.e., the subring with basis elements $[P, \varphi]$ where $\varphi: P \rightarrow S$ is any group homomorphism. We then define an augmentation map $\varepsilon(X) := \frac{|X|}{|S|}$ for any biset X . Since $\varepsilon(X \circ Y) = \frac{|X \times_S Y|}{|S|} = \frac{|X||Y|}{|S|^2} = \varepsilon(X)\varepsilon(Y)$, we get a ring homomorphism $\varepsilon: A^\triangleleft(S, S)_{(p)} \rightarrow \mathbb{Z}_{(p)}$.

Definition 5.4. Let \mathcal{F} be a fusion system on a p -group S . An element $X \in A(S, S)_{(p)}$ is said to be *right/left/fully \mathcal{F} -characteristic* if:

- (i) X is \mathcal{F} -generated.
- (ii) X is right/left/fully \mathcal{F} -stable respectively.
- (iii) $\varepsilon(X)$ is invertible in $\mathbb{Z}_{(p)}$.

A fully \mathcal{F} -characteristic element is also just called *\mathcal{F} -characteristic*.

Remark 5.5. We will now give a new proof that every saturated fusion system has a fully \mathcal{F} -characteristic idempotent.

To see that the characteristic idempotent for \mathcal{F} is unique, one can use the uniqueness part of Ragnarsson's proof in [12]. Alternatively, corollary 6.6 establishes the uniqueness of $\omega_{\mathcal{F}}$. Until that corollary is proved, we let $\omega_{\mathcal{F}}$ denote only the particular \mathcal{F} -characteristic idempotent constructed below.

Theorem B. *Let \mathcal{F} be a saturated fusion system on a finite p -group S . Then there exists a (unique) fully \mathcal{F} -characteristic idempotent $\omega_{\mathcal{F}} \in A(S, S)_{(p)}$, and it satisfies:*

For all graphs $\Delta(P, \varphi) \leq S \times S$ with $\varphi \in \mathcal{F}(P, S)$, we have

$$\Phi_{\Delta(P, \varphi)}(\omega_{\mathcal{F}}) = \frac{|S|}{|\mathcal{F}(P, S)|};$$

and $\Phi_D(\omega_{\mathcal{F}}) = 0$ for all other subgroups $D \leq S \times S$. Consequently, if we write $\omega_{\mathcal{F}}$ in the basis of $A(S, S)_{(p)}$, we get the expression

$$\omega_{\mathcal{F}} = \sum_{\substack{[\Delta(P, \varphi)]_{S \times S} \\ \text{with } \varphi \in \mathcal{F}(P, S)}} \frac{|S|}{\Phi_{\Delta(P, \varphi)}([P, \varphi]_S^S)} \left(\sum_{P \leq Q \leq S} \frac{|\{\psi \in \mathcal{F}(Q, S) \mid \psi|_P = \varphi\}|}{|\mathcal{F}(Q, S)|} \cdot \mu(P, Q) \right) [P, \varphi]_S^S,$$

where the outer sum is taken over $(S \times S)$ -conjugacy classes of subgroups, and where μ is the Möbius function for the poset of subgroups in S .

The general strategy of the construction is as follows: We consider the saturated fusion system $\mathcal{F} \times \mathcal{F}_S$ on $S \times S$, where $\mathcal{F}_S := \mathcal{F}_S(S)$ is the trivial fusion system on S . For this product fusion system we then apply the stabilization map of theorem A to $[S, id]$ and get $\beta_{\Delta(S)} \in A(\mathcal{F} \times \mathcal{F}_S)_{(p)}$. By construction $\beta_{\Delta(S)}$ is only left \mathcal{F} -stable, but fixed point calculations will show that $\beta_{\Delta(S)}$ is right stable as well. Finally, using lemma 4.3, we will show that $\beta_{\Delta(S)}$ is idempotent.

Alternatively, we could in theory stabilize $[S, id]$ with respect to $\mathcal{F} \times \mathcal{F}$, in order to immediately get a fully \mathcal{F} -stable element. The fixed point formulas imply that this would give us exactly the same element $\beta_{\Delta(S)}$ as before. However, by stabilizing with respect to a larger fusion system, lemma 4.3 yields less information about the orbits of the stabilized element, hence it would be harder to show that $\beta_{\Delta(S)}$ is idempotent. This is why we use the first, asymmetric approach to the construction.

Proof. Let $\mathcal{F}_S := \mathcal{F}_S(S)$ denote the trivial fusion system on S , then $\mathcal{F} \times \mathcal{F}_S$ is a product of saturated fusion systems and is therefore a saturated fusion system on $S \times S$.

Next we remark that the $(\mathcal{F} \times \mathcal{F}_S)$ -conjugates of a graph $\Delta(P, \varphi)$ with $\varphi \in \mathcal{F}(P, S)$ are all the other graphs $\Delta(P', \psi)$ with $P' \sim_S P$ and $\psi \in \mathcal{F}(P', S)$. Furthermore, the subgroups of the diagonal $\Delta(S) := \Delta(S, id)$ in $S \times S$ are the graphs $\Delta(P, id)$ for $P \leq S$; and consequently the subgroups of $S \times S$ that are $(\mathcal{F} \times \mathcal{F}_S)$ -subconjugate to $\Delta(S)$ are exactly the graphs $\Delta(P, \varphi)$ with $\varphi \in \mathcal{F}(P, S)$.

Recall that the basis element $\beta_{\Delta(S)} \in A(\mathcal{F} \times \mathcal{F}_S)_{(p)}$ of proposition 4.5 is constructed by applying lemma 4.3 to the (S, S) -biset $[S \times S / \Delta(S)] = [S, id]$. For subgroups $D \leq S \times S$ we have $\Phi_D([S, id]) = 0$ unless D is $(S \times S)$ -subconjugate to $\Delta(S)$.

When applying lemma 4.3 to $[S, id]$, we can therefore use the collection of subgroups \mathcal{H} consisting of the graphs $\Delta(P, \varphi)$ with $\varphi \in \mathcal{F}(P, S)$, since $\Phi_D([S, id]) = 0$ for all other subgroups $D \leq S \times S$. As remarked right after lemma 4.3 the stable element that the lemma constructs does not depend on the collection \mathcal{H} used. Hence we still get $\beta_{\Delta(S)}$ even

though we use a smaller collection \mathcal{H} than in section 4 (where \mathcal{H} contained all subgroups). By lemma 4.3, $\beta_{\Delta(S)}$ then satisfies

- (i) $\Phi_D(\beta_{\Delta(S)}) = \Phi_D([S, id]) = 0$ and $c_D(\beta_{\Delta(S)}) = c_D([S, id]) = 0$ for all $D \leq S \times S$ not on the form $\Delta(P, \varphi)$ with $\varphi \in \mathcal{F}(P, S)$.
- (ii) For all $\Delta(P, \varphi)$ with $\varphi \in \mathcal{F}(P, S)$, we have

$$\sum_{[\Delta(P, \psi)]_{S \times S} \subseteq [\Delta(P, \varphi)]_{\mathcal{F} \times \mathcal{F}_S}} c_{\Delta(P, \psi)}(\beta_{\Delta(S)}) = \sum_{[\Delta(P, \psi)]_{S \times S} \subseteq [\Delta(P, \varphi)]_{\mathcal{F} \times \mathcal{F}_S}} c_{\Delta(P, \psi)}([S, id]).$$

By proposition 4.5, the element $\beta_{\Delta(S)}$ also satisfies

$$(5.5) \quad \Phi_{\Delta(P, \varphi)}(\beta_{\Delta(S)}) = \frac{|\mathrm{Hom}_{\mathcal{F} \times \mathcal{F}_S}(\Delta(P, \varphi), \Delta(S, id))| \cdot |S \times S|}{|\Delta(S, id)| \cdot |\mathrm{Hom}_{\mathcal{F} \times \mathcal{F}_S}(\Delta(P, \varphi), S \times S)|}$$

for all $P \leq S$ and $\varphi \in \mathcal{F}(P, S)$.

Property (i) shows that $\beta_{\Delta(S)} \in A(S, S)_{(p)}$ is a linear combination of basis elements $[P, \varphi]$ with $\varphi \in \mathcal{F}(P, S)$. Hence $\beta_{\Delta(S)}$ is \mathcal{F} -generated. As a consequence of (ii), the value of the augmentation map on $\beta_{\Delta(S)}$ is

$$\begin{aligned} \varepsilon(\beta_{\Delta(S)}) &= \sum_{\substack{[\Delta(P, \varphi)]_{S \times S} \\ \text{with } \varphi \in \mathcal{F}(P, S)}} c_{\Delta(P, \varphi)}(\beta_{\Delta(S)}) \cdot \varepsilon([P, \varphi]) \\ &= \sum_{\substack{[\Delta(P, \varphi)]_{S \times S} \\ \text{with } \varphi \in \mathcal{F}(P, S)}} c_{\Delta(P, \varphi)}(\beta_{\Delta(S)}) \cdot \frac{|S|}{|P|} \\ &= \sum_{[P]_S} \left(\sum_{[\Delta(P, \varphi)]_{S \times S} \subseteq [\Delta(P, id)]_{\mathcal{F} \times \mathcal{F}_S}} c_{\Delta(P, \varphi)}(\beta_{\Delta(S)}) \right) \cdot \frac{|S|}{|P|} \\ &= \sum_{[P]_S} \left(\sum_{[\Delta(P, \varphi)]_{S \times S} \subseteq [\Delta(P, id)]_{\mathcal{F} \times \mathcal{F}_S}} c_{\Delta(P, \varphi)}([S, id]) \right) \cdot \frac{|S|}{|P|} \\ &= c_{\Delta(S, id)}([S, id]) \cdot \frac{|S|}{|S|} = 1. \end{aligned}$$

By construction, $\beta_{\Delta(S)}$ is stable as an $(S \times S)$ -set with respect to the fusion system $\mathcal{F} \times \mathcal{F}_S$. Therefore, by lemma 5.3, $\beta_{\Delta(S)}$ is left \mathcal{F} -stable as an element of $A(S, S)_{(p)}$. We have thus proved that $\beta_{\Delta(S)} \in A(S, S)_{(p)}$ is a left characteristic element for \mathcal{F} .

We now consider the value of $\Phi_{\Delta(P, \varphi)}(\beta_{\Delta(S)})$ with $\varphi \in \mathcal{F}(P, S)$ in more detail. First we remark that $\Phi_{\Delta(P, \varphi)}(\beta_{\Delta(S)}) = \Phi_{\Delta(P, id)}(\beta_{\Delta(S)})$ since $\beta_{\Delta(S)}$ is left \mathcal{F} -stable. Then (5.5) gives us

$$\Phi_{\Delta(P, id)}(\beta_{\Delta(S)}) = \frac{|\mathrm{Hom}_{\mathcal{F} \times \mathcal{F}_S}(\Delta(P, id), \Delta(S, id))| \cdot |S \times S|}{|\Delta(S, id)| \cdot |\mathrm{Hom}_{\mathcal{F} \times \mathcal{F}_S}(\Delta(P, id), S \times S)|}.$$

The morphisms of $\mathrm{Hom}_{\mathcal{F} \times \mathcal{F}_S}(\Delta(P, id), S \times S)$ are the pairs (φ, c_s) where $\varphi \in \mathcal{F}(P, S)$ and $c_s \in \mathcal{F}_S(P, S)$, hence

$$|\mathrm{Hom}_{\mathcal{F} \times \mathcal{F}_S}(\Delta(P, id), S \times S)| = |\mathcal{F}_S(P, S)| \cdot |\mathcal{F}(P, S)|.$$

The image of $\Delta(P, id)$ under a morphism $(\varphi, c_s) \in \text{Hom}_{\mathcal{F} \times \mathcal{F}_S}(\Delta(P, id), S \times S)$ is

$$(\varphi, c_s)(\Delta(P, id)) = \{(\varphi(g), c_s(g)) \mid g \in P\} = \Delta({}^sP, \varphi \circ (c_s)^{-1}).$$

This image lies in $\Delta(S, id)$ if and only if $\varphi \circ (c_s)^{-1} = id$, i.e., if $\varphi = c_s$. The number of morphisms in $\text{Hom}_{\mathcal{F} \times \mathcal{F}_S}(\Delta(P, id), \Delta(S, id))$ is therefore simply $|\mathcal{F}_S(P, S)|$.

Returning to the expression for $\Phi_{[P, id]}(\beta_{\Delta(S)})$ we then have

$$\Phi_{\Delta(P, id)}(\beta_{\Delta(S)}) = \frac{|\mathcal{F}_S(P, S)| \cdot |S \times S|}{|\Delta(S, id)| \cdot (|\mathcal{F}_S(P, S)| \cdot |\mathcal{F}(P, S)|)} = \frac{|S|}{|\mathcal{F}(P, S)|},$$

which only depends on the \mathcal{F} -conjugacy class of P . We conclude that for all (P, φ) with $\varphi \in \mathcal{F}(P, S)$, and (Q, ψ) with $\psi \in \mathcal{F}(Q, S)$, and such that $P \sim_{\mathcal{F}} Q$, we have

$$\Phi_{\Delta(P, \varphi)}(\beta_{\Delta(S)}) = \Phi_{\Delta(Q, \psi)}(\beta_{\Delta(S)}).$$

Recalling that $\Phi_D(\beta_{\Delta(S)}) = 0$ when $D \leq S \times S$ is not a graph $\Delta(Q, \psi)$ with $\psi \in \mathcal{F}(Q, S)$, lemma 5.3 says that $\beta_{\Delta(S)}$ is fully \mathcal{F} -stable and not just left \mathcal{F} -stable.

We have proven that $\beta_{\Delta(S)}$ is fully \mathcal{F} -characteristic, so we now need to show that $\beta_{\Delta(S)}$ is actually idempotent. Since $\beta_{\Delta(S)}$ is right \mathcal{F} -stable, we have $\beta_{\Delta(S)} \circ [P, \varphi] = \beta_{\Delta(S)} \circ [P, id]$ for all $\varphi \in \mathcal{F}(P, S)$. We can therefore calculate

$$\begin{aligned} \beta_{\Delta(S)} \circ \beta_{\Delta(S)} &= \beta_{\Delta(S)} \circ \left(\sum_{\substack{[\Delta(P, \varphi)]_{S \times S} \\ \text{with } \varphi \in \mathcal{F}(P, S)}} c_{\Delta(P, \varphi)}(\beta_{\Delta(S)}) \cdot [P, \varphi] \right) \\ &= \sum_{\substack{[\Delta(P, \varphi)]_{S \times S} \\ \text{with } \varphi \in \mathcal{F}(P, S)}} c_{\Delta(P, \varphi)}(\beta_{\Delta(S)}) \cdot (\beta_{\Delta(S)} \circ [P, \varphi]) \\ &= \sum_{[P]_S} \left(\sum_{[\Delta(P, \varphi)]_{S \times S} \subseteq [\Delta(P, id)]_{\mathcal{F} \times \mathcal{F}_S}} c_{\Delta(P, \varphi)}(\beta_{\Delta(S)}) \right) \cdot (\beta_{\Delta(S)} \circ [P, id]) \\ &\stackrel{(ii)}{=} \sum_{[P]_S} \left(\sum_{[\Delta(P, \varphi)]_{S \times S} \subseteq [\Delta(P, id)]_{\mathcal{F} \times \mathcal{F}_S}} c_{\Delta(P, \varphi)}([S, id]) \right) \cdot (\beta_{\Delta(S)} \circ [P, id]) \\ &= c_{\Delta(S, id)}([S, id]) \cdot (\beta_{\Delta(S)} \circ [S, id]) = \beta_{\Delta(S)}; \end{aligned}$$

so $\beta_{\Delta(S)}$ is a characteristic idempotent for \mathcal{F} .

Finally, proposition 4.6 gives the coefficients of $\beta_{\Delta(S)}$ in terms of the basis in $A(S, S)_{(p)}$:

$$\begin{aligned} c_{\Delta(P, \varphi)}(\beta_{\Delta(S)}) &= \frac{1}{\Phi_{\Delta(P, \varphi)}([P, \varphi])} \left(\sum_{D \geq \Delta(P, \varphi)} \Phi_D(\omega_{\mathcal{F}}) \cdot \mu(\Delta(P, \varphi), D) \right) \\ &= \frac{1}{\Phi_{\Delta(P, \varphi)}([P, \varphi])} \left(\sum_{\Delta(Q, \psi) \geq \Delta(P, \varphi)} \frac{|S|}{|\mathcal{F}(Q, S)|} \cdot \mu(\Delta(P, \varphi), \Delta(Q, \psi)) \right) \\ &= \frac{|S|}{\Phi_{\Delta(P, \varphi)}([P, \varphi])} \left(\sum_{Q \geq P} \frac{|\{\psi \in \mathcal{F}(Q, S) \mid \psi|_P = \varphi\}|}{|\mathcal{F}(Q, S)|} \cdot \mu(P, Q) \right), \end{aligned}$$

which expresses the characteristic idempotent $\beta_{\Delta(S)}$ as the linear combination in the theorem. \square

Corollary 5.6. *Let \mathcal{F} be a saturated fusion system on a p -group S , and let $\omega_{\mathcal{F}}$ be the characteristic idempotent constructed in theorem B.*

If $X \in A(S, T)_{(p)}$ is right \mathcal{F} -stable, then $X \circ \omega_{\mathcal{F}} = X$. Similarly if $X \in A(T, S)_{(p)}$ is left \mathcal{F} -stable, then $\omega_{\mathcal{F}} \circ X = X$.

Proof. To calculate the product $X \circ \omega_{\mathcal{F}}$ when X is right \mathcal{F} -stable, we apply the same technique used in theorem B to show that $\beta_{\Delta(S)}$ is idempotent:

$$\begin{aligned} X \circ \omega_{\mathcal{F}} &= \sum_{\substack{[\Delta(P, \varphi)]_{S \times S} \\ \text{with } \varphi \in \mathcal{F}(P, S)}} c_{\Delta(P, \varphi)}(\omega_{\mathcal{F}}) \cdot (X \circ [P, \varphi]) \\ &= \sum_{[P]_S} \left(\sum_{[\Delta(P, \varphi)]_{S \times S} \subseteq [\Delta(P, id)]_{\mathcal{F} \times \mathcal{F}_S}} c_{\Delta(P, \varphi)}(\omega_{\mathcal{F}}) \right) \cdot (X \circ [P, id]) \\ &= \sum_{[P]_S} \left(\sum_{[\Delta(P, \varphi)]_{S \times S} \subseteq [\Delta(P, id)]_{\mathcal{F} \times \mathcal{F}_S}} c_{\Delta(P, \varphi)}([S, id]) \right) \cdot (X \circ [P, id]) \\ &= c_{\Delta(S, id)}([S, id]) \cdot (X \circ [S, id]) = X. \end{aligned}$$

From theorem B we have

$$\Phi_{\Delta(P, \varphi)}(\omega_{\mathcal{F}}) = \frac{|S|}{|\mathcal{F}(P, S)|} = \Phi_{\Delta(\varphi P, \varphi^{-1})}(\omega_{\mathcal{F}}) = \Phi_{\Delta(P, \varphi)}(\omega_{\mathcal{F}}^{\text{op}}),$$

which implies that $\omega_{\mathcal{F}}^{\text{op}} = \omega_{\mathcal{F}}$. Hence, if $X \in A(T, S)_{(p)}$ is left \mathcal{F} -stable, then equivalently $X^{\text{op}} \in A(S, T)_{(p)}$ is right \mathcal{F} -stable and

$$\omega_{\mathcal{F}} \circ X = (X^{\text{op}} \circ \omega_{\mathcal{F}}^{\text{op}})^{\text{op}} = (X^{\text{op}})^{\text{op}} = X$$

follows by the right \mathcal{F} -stable case above. \square

5.3. The action of the characteristic idempotent. In this section we explore how a characteristic idempotent $\omega_{\mathcal{F}}$ acts by multiplication on elements of the double Burnside ring and other Burnside modules. Theorem C gives a precise description of the action of $\omega_{\mathcal{F}}$ in terms of the fixed point maps, and in this way we recover the stabilization homomorphism of theorem A: The Burnside ring $A(S)_{(p)}$ is isomorphic to the Burnside module $A(1, S)_{(p)}$, and through this isomorphism the stabilization homomorphism of theorem A is given by multiplication with $\omega_{\mathcal{F}}$ from the left.

We warm up with a result about basis elements for Burnside modules $A(S_1, S_2)_{(p)}$, where S_1 and S_2 are p -groups. We already know that a transitive (S_1, S_2) -set $(S_2 \times S_1)/D$ only depends on D up to $(S_2 \times S_1)$ -conjugation, and now we show that when we multiply $(S_2 \times S_1)/D$ by characteristic idempotents the result only depends on the subgroup D up to conjugation in the corresponding saturated fusion systems.

Lemma 5.7. *Let \mathcal{F}_1 and \mathcal{F}_2 be saturated fusion systems on the p -groups S_1 and S_2 respectively, and let $\omega_1 \in A(S_1, S_1)_{(p)}$ and $\omega_2 \in A(S_2, S_2)_{(p)}$ be their respective characteristic idempotents.*

Then for all subgroups $D, C \leq S_2 \times S_1$, if D and C are conjugate in $\mathcal{F}_2 \times \mathcal{F}_1$, we have

$$\omega_2 \circ [(S_2 \times S_1)/D] \circ \omega_1 = \omega_2 \circ [(S_2 \times S_1)/C] \circ \omega_1$$

in $A(S_1, S_2)_{(p)}$.

Proof. Suppose that the subgroups $D, C \leq S_2 \times S_1$ are conjugate in $\mathcal{F}_2 \times \mathcal{F}_1$, and let $\varphi \in \text{Hom}_{\mathcal{F}_2 \times \mathcal{F}_1}(D, C)$ be an isomorphism.

By definition of $\mathcal{F}_2 \times \mathcal{F}_1$, the homomorphism φ extends to $(\varphi_2 \times \varphi_1): D_2 \times D_1 \rightarrow C_2 \times C_1$ where D_i is the projection of D onto S_i , similarly for C_i , and where $\varphi_i \in \mathcal{F}_i(D_i, C_i)$. By assumption, φ is invertible in $\mathcal{F}_2 \times \mathcal{F}_1$, hence the inverse φ^{-1} also extends to a homomorphism $C_2 \times C_1 \rightarrow D_2 \times D_1$, which shows that φ_1 and φ_2 are invertible in \mathcal{F}_1 and \mathcal{F}_2 respectively. With this we have

$$\begin{aligned} [(S_2 \times S_1)/D] &= [D_2, \text{id}]_{D_2}^{S_2} \circ [(D_2 \times D_1)/D] \circ [D_1, \text{id}]_{S_1}^{D_1} \\ &= [D_2, \text{id}]_{D_2}^{S_2} \circ [C_2, \varphi_2^{-1}]_{C_2}^{D_2} \circ [(C_2 \times C_1)/C] \circ [D_1, \varphi_1]_{D_1}^{C_1} \circ [D_1, \text{id}]_{S_1}^{D_1} \\ &= [C_2, \varphi_2^{-1}]_{C_2}^{S_2} \circ [(C_2 \times C_1)/C] \circ [D_1, \varphi_1]_{S_1}^{C_1}. \end{aligned}$$

Since ω_2 is right \mathcal{F}_2 -stable, and ω_1 is left \mathcal{F}_1 -stable, it follows that

$$\begin{aligned} \omega_2 \circ [(S_2 \times S_1)/D] \circ \omega_1 &= \omega_2 \circ [C_2, \varphi_2^{-1}]_{C_2}^{S_2} \circ [(C_2 \times C_1)/C] \circ [D_1, \varphi_1]_{S_1}^{C_1} \circ \omega_1 \\ &= \omega_2 \circ [C_2, \text{id}]_{C_2}^{S_2} \circ [(C_2 \times C_1)/C] \circ [C_1, \text{id}]_{S_1}^{C_1} \circ \omega_1 \\ &= \omega_2 \circ [(S_2 \times S_1)/C] \circ \omega_1. \end{aligned} \quad \square$$

Theorem C. *Let \mathcal{F}_1 and \mathcal{F}_2 be saturated fusion systems on finite p -groups S_1 and S_2 respectively, and let $\omega_1 \in A(S_1, S_1)_{(p)}$ and $\omega_2 \in A(S_2, S_2)_{(p)}$ be their characteristic idempotents as constructed earlier.*

For every element of the Burnside module $X \in A(S_1, S_2)_{(p)}$, the product $\omega_2 \circ X \circ \omega_1$ is right \mathcal{F}_1 -stable and left \mathcal{F}_2 -stable, and satisfies

$$\Phi_D(\omega_2 \circ X \circ \omega_1) = \frac{1}{|[D]_{\mathcal{F}_2 \times \mathcal{F}_1}|} \sum_{D' \in [D]_{\mathcal{F}_2 \times \mathcal{F}_1}} \Phi_{D'}(X),$$

for all subgroups $D \leq S_2 \times S_1$, where $[D]_{\mathcal{F}_2 \times \mathcal{F}_1}$ is the isomorphism class of D in the product fusion system $\mathcal{F}_2 \times \mathcal{F}_1$ on $S_2 \times S_1$.

Note that by the fixed point formula, theorem C states that the map $X \mapsto \omega_{\mathcal{F}_2} \circ X \circ \omega_{\mathcal{F}_1}$ coincides with the stabilization map $A(S_2 \times S_1)_{(p)} \rightarrow A(\mathcal{F}_2 \times \mathcal{F}_1)_{(p)}$ of theorem A.

Proof. Any product $\omega_2 \circ X \circ \omega_1$ is right \mathcal{F}_1 -stable by definition since ω_1 is right \mathcal{F}_1 -stable, and similarly we see that $\omega_2 \circ X \circ \omega_1$ is left \mathcal{F}_2 -stable.

Consider the element $X \in A(S_1, S_2)_{(p)}$ as an element of $A(S_2 \times S_1)_{(p)}$. The fusion system $\mathcal{F}_2 \times \mathcal{F}_1$ on $S_2 \times S_1$ is saturated by [4, Lemma 1.5], and we apply theorem A, to get an $(\mathcal{F}_2 \times \mathcal{F}_1)$ -stable element πX satisfying

$$\Phi_D(\pi X) := \frac{1}{|[D]_{\mathcal{F}_2 \times \mathcal{F}_1}|} \sum_{D' \in [D]_{\mathcal{F}_2 \times \mathcal{F}_1}} \Phi_{D'}(X)$$

for all $D \leq S_2 \times S_1$. By lemma 5.3, πX is left \mathcal{F}_2 -stable and right \mathcal{F}_1 -stable when considered as an element $X \in A(S_1, S_2)_{(p)}$.

Furthermore, by remark 4.4, πX also satisfies

$$\sum_{[D']_{S_2 \times S_1} \subseteq [D]_{\mathcal{F}_2 \times \mathcal{F}_1}} c_{D'}(\pi X) = \sum_{[D']_{S_2 \times S_1} \subseteq [D]_{\mathcal{F}_2 \times \mathcal{F}_1}} c_{D'}(X),$$

or equivalently

$$\sum_{[D']_{S_2 \times S_1} \subseteq [D]_{\mathcal{F}_2 \times \mathcal{F}_1}} c_{D'}(\pi X - X) = 0.$$

Using lemma 5.7 we then have

$$\begin{aligned}
& \omega_2 \circ (\pi X - X) \circ \omega_1 \\
&= \sum_{[D]_{\mathcal{F}_2 \times \mathcal{F}_1}} \left(\sum_{[D']_{S_2 \times S_1} \subseteq [D]_{\mathcal{F}_2 \times \mathcal{F}_1}} c_{D'}(\pi X - X) \cdot (\omega_2 \circ (S_2 \times S_1/D') \circ \omega_1) \right) \\
&= \sum_{[D]_{\mathcal{F}_2 \times \mathcal{F}_1}} \left(\sum_{[D']_{S_2 \times S_1} \subseteq [D]_{\mathcal{F}_2 \times \mathcal{F}_1}} c_{[D']}(\pi X - X) \right) \cdot (\omega_2 \circ (S_2 \times S_1/D) \circ \omega_1) \\
&= \sum_{[D]_{\mathcal{F}_2 \times \mathcal{F}_1}} 0 \cdot (\omega_2 \circ (S_2 \times S_1/D) \circ \omega_1) = 0.
\end{aligned}$$

From which we conclude

$$\omega_2 \circ X \circ \omega_1 = \omega_2 \circ \pi X \circ \omega_1 = \pi X,$$

where the last equality holds by corollary 5.6 since πX is left \mathcal{F}_2 -stable and right \mathcal{F}_1 -stable. \square

Corollary 5.8. *Let \mathcal{F} be a saturated fusion system on a p -group S . A set with a left action of S is the same as a $(1, S)$ -biset, so the Burnside module $A(1, S)_{(p)}$ is isomorphic to the Burnside ring $A(S)_{(p)}$. Through this isomorphism left multiplication with $\omega_{\mathcal{F}}$ in $A(1, S)_{(p)}$ coincides with the stabilization homomorphism $\pi: A(S)_{(p)} \rightarrow A(\mathcal{F})_{(p)}$ of theorem A.*

Proof. The subgroups of $S \times 1$ are all on the form $Q \times 1$ for some $Q \leq S$, and the characteristic idempotent for the unique fusion system on the trivial group is just $[1, id]_1^1 = [pt]_1^1$. By theorem C we then have

$$\Phi_{Q \times 1}(\omega_{\mathcal{F}} \circ X) = \frac{1}{|[Q]_{\mathcal{F}}|} \sum_{Q' \in [Q]_{\mathcal{F}}} \Phi_{Q' \times 1}(X) = \Phi_Q(\pi(X))$$

for all $X \in A(1, S)_{(p)}$, so $\omega_{\mathcal{F}} \circ X = \pi(X)$ as claimed. \square

Definition 5.9. For saturated fusion systems $\mathcal{F}_1, \mathcal{F}_2$ on p -groups S_1, S_2 , we define the Burnside module $A(\mathcal{F}_1, \mathcal{F}_2)_{(p)}$ as the $\mathbb{Z}_{(p)}$ -submodule of $A(S_1, S_2)_{(p)}$ consisting of the elements that are right \mathcal{F}_1 -stable and left \mathcal{F}_2 -stable.

The elements $\omega_{\mathcal{F}_2} \circ [(S_2 \times S_1)/D] \circ \omega_{\mathcal{F}_1}$ generate $A(\mathcal{F}_1, \mathcal{F}_2)_{(p)}$ over $\mathbb{Z}_{(p)}$. By the fixed point calculation of theorem C the element $\omega_{\mathcal{F}_2} \circ [(S_2 \times S_1)/D] \circ \omega_{\mathcal{F}_1}$ actually corresponds to the element $\beta_D \in A(\mathcal{F}_2 \times \mathcal{F}_1)_{(p)}$, so it follows that the elements $\{\omega_{\mathcal{F}_2} \circ [(S_2 \times S_1)/D] \circ \omega_{\mathcal{F}_1} \mid D \leq S_2 \times S_1\}$ form a $\mathbb{Z}_{(p)}$ -basis for the Burnside module $A(\mathcal{F}_1, \mathcal{F}_2)_{(p)}$. Two subgroups C and D give the same basis element if and only if C and D are conjugate in $\mathcal{F}_2 \times \mathcal{F}_1$. The existence of such basis elements nicely generalizes the basis we have for the Burnside modules of groups.

As for groups, the Burnside modules $A(\mathcal{F}_1, \mathcal{F}_2)_{(p)}$ form the Hom-sets of a category where the objects are all saturated fusion systems on p -groups. We define the (p -localized) double Burnside ring of a saturated fusion system \mathcal{F} to be the ring $A(\mathcal{F}, \mathcal{F})_{(p)}$. The double Burnside ring $A(\mathcal{F}, \mathcal{F})_{(p)}$ has a 1-element: It is simply the characteristic idempotent $\omega_{\mathcal{F}} = \omega_{\mathcal{F}} \circ [S, id] \circ \omega_{\mathcal{F}}$, which is also one of the $\mathbb{Z}_{(p)}$ -basis elements.

Multiplication with characteristic idempotents ω_1 and ω_2 defines a map $A(S_1, S_2)_{(p)} \rightarrow A(\mathcal{F}_1, \mathcal{F}_2)_{(p)}$. In the spirit of theorem A, we proceed to show that this map is a homomorphism of modules.

Proposition 5.10. *Let \mathcal{F}_1 and \mathcal{F}_2 be saturated fusion systems on p -groups S_1 and S_2 respectively, and let $\omega_1 \in A(S_1, S_1)_{(p)}$ and $\omega_2 \in A(S_2, S_2)_{(p)}$ be their characteristic idempotents as constructed earlier.*

Then the map $\pi: A(S_1, S_2)_{(p)} \rightarrow A(\mathcal{F}_1, \mathcal{F}_2)_{(p)}$ given by $\pi(X) := \omega_2 \circ X \circ \omega_1$ is a homomorphism of left $A(\mathcal{F}_2, \mathcal{F}_2)_{(p)}$ -modules and right $A(\mathcal{F}_1, \mathcal{F}_1)_{(p)}$ -modules.

Proof. We only show that π is a homomorphism of right $A(\mathcal{F}_1, \mathcal{F}_1)_{(p)}$ -modules, since the other case is similar.

Let $X \in A(S_1, S_2)_{(p)}$ be given, and let $Z \in A(\mathcal{F}_1, \mathcal{F}_1)_{(p)}$ be a fully \mathcal{F}_1 -stable element of $A(S_1, S_1)_{(p)}$. Then \mathcal{F}_1 -stability ensures that $\omega_1 \circ Z = Z \circ \omega_1 = Z$ by corollary 5.6; hence we get

$$\pi(X \circ Z) = \omega_2 \circ X \circ Z \circ \omega_1 = \omega_2 \circ X \circ \omega_1 \circ Z = \pi(X) \circ Z. \quad \square$$

6. THE BURNSIDE RING EMBEDS IN THE DOUBLE BURNSIDE RING

In this section we show that the “one-sided” Burnside ring $A(\mathcal{F})_{(p)}$ of sections 3.1 and 4 always embeds in the double Burnside ring $A(\mathcal{F}, \mathcal{F})_{(p)}$ for \mathcal{F} defined above. In fact, theorem D states that $A(\mathcal{F})_{(p)}$ is isomorphic to the subring generated by all \mathcal{F} -characteristic elements. Through this isomorphism we can describe the structure of the \mathcal{F} -characteristic elements, and in particular we prove that there is only one \mathcal{F} -characteristic idempotent.

The isomorphism between the “one-sided” Burnside ring and a subring of the double Burnside ring is inspired by a similar result for finite groups, where the Burnside ring $A(G)$ embeds in the double Burnside ring $A(G, G)$. Let us therefore first analyze the situation for Burnside rings of p -groups and see what might be generalized to fusion systems:

Example 6.1. Let S be a finite p -group. Recall the double coset formula (3.1) for the multiplication of basis elements in $A(S)$:

$$[S/P] \cdot [S/Q] = \sum_{\bar{s} \in P \backslash S/Q} [S/(P \cap {}^s Q)].$$

If we then consider the bisets $[P, id]_S^S$ and $[Q, id]_S^S$ for subgroup $P, Q \leq S$ in the double Burnside ring for S , then the double coset formula (5.2) for $A(S, S)$ shows us that

$$[P, id]_S^S \circ [Q, id]_S^S = \sum_{\bar{s} \in P \backslash S/Q} [P^s \cap Q, c_s]_S^S = \sum_{\bar{s} \in P \backslash S/Q} [P \cap {}^s Q, id]_S^S.$$

If we compare the two formulas, we discover that the basis elements $[S/P]$ in $A(S)$ and the basis elements $[P, id]_S^S$ in $A(S, S)$ satisfy exactly the same multiplication formula. Hence we get an injective ring homomorphism $\iota: A(S) \rightarrow A(S, S)_{(p)}$ mapping $[S/P] \mapsto [P, id]$, which embeds $A(S)$ as the subring of $A(S, S)$ generated by $[P, id]$ for $P \leq S$.

The basis elements $[P, id]$ are precisely the basis elements $[(S \times S)/D]$ for $D = \Delta(P, c_s)$ the graph of an S -conjugation map – recall that the subgroup $\Delta(P, c_s)$ is only determined up to $(S \times S)$ -conjugation, so $[P, c_s] = [P, id]$. The subring generated by $[P, id]$ for $P \leq S$, is therefore the ring $A_{\mathcal{F}_S}(S, S)$ of all \mathcal{F}_S -generated elements (see definition 5.1), where \mathcal{F}_S is the trivial fusion system on S . This suggests that we should consider the \mathcal{F} -generated elements for general fusion systems.

Finally the inverse of ι is the map $q: A_{\mathcal{F}_S}(S, S) \rightarrow A(S)$ given on bisets by $X \mapsto X/S$. Here we eliminate the right S -action by quotienting out, equivalently this can be expressed by the multiplication $X \mapsto X \circ [(S \times 1)/(S \times 1)]$ from $A_{\mathcal{F}_S}(S, S)$ to $A(1, S)$. To see that this map is the inverse of ι we simply examine the basis elements and note that

$$q([P, id]_S^S) = q([S \times_P S]) = [(S \times_P S)/S] = [S/P].$$

It is not clear that q preserves the multiplication, but this must be true since $q = \iota^{-1}$. A similar situation occurs in theorem D: We state the theorem for the nice map q where we quotient out the right S -action, but to actually see that q is a ring homomorphism we construct the inverse ι as ring homomorphism from the start.

Definition 6.2. For a saturated fusion system \mathcal{F} on S , the double Burnside ring $A(\mathcal{F}, \mathcal{F})_{(p)}$ was defined to be the subring of $A(S, S)_{(p)}$ consisting of the elements that are both left and right \mathcal{F} -stable. Example 6.1 suggests that we should look at those elements of $A(\mathcal{F}, \mathcal{F})_{(p)}$ that are in addition \mathcal{F} -generated. We therefore define

$$A^{\text{char}}(\mathcal{F})_{(p)} := A(\mathcal{F}, \mathcal{F})_{(p)} \cap A_{\mathcal{F}}(S, S)_{(p)}$$

as the subring formed by all elements that are \mathcal{F} -stable as well as \mathcal{F} -generated. Hence we have a sequence of inclusions of subrings

$$A^{\text{char}}(\mathcal{F})_{(p)} \subseteq A(\mathcal{F}, \mathcal{F})_{(p)} \subseteq A(S, S)_{(p)}.$$

The last inclusion is not unital since $\omega_{\mathcal{F}}$ is the multiplicative identity of the first two rings, and $[S, id]_S^S$ is the identity of $A(S, S)_{(p)}$.

We use the notation $A^{\text{char}}(\mathcal{F})_{(p)}$ for this particular subring because the following proposition shows that $A^{\text{char}}(\mathcal{F})_{(p)}$ is generated, over $\mathbb{Z}_{(p)}$, by all the \mathcal{F} -characteristic elements in $A(S, S)_{(p)}$. Note that not all elements of $A^{\text{char}}(\mathcal{F})_{(p)}$ are \mathcal{F} -characteristic, but the non-characteristic elements of $A^{\text{char}}(\mathcal{F})_{(p)}$ are few, and they form a proper $\mathbb{Z}_{(p)}$ -submodule.

Proposition 6.3. *Let \mathcal{F} be a saturated fusion systems on a p -group S , and let $A^{\text{char}}(\mathcal{F})_{(p)}$ be defined as above. Then $A^{\text{char}}(\mathcal{F})_{(p)}$ is also the subring of $A(S, S)_{(p)}$ generated by the \mathcal{F} -characteristic elements, and it has a $\mathbb{Z}_{(p)}$ -basis consisting of the elements $\beta_{\Delta(P, id)} = \omega_{\mathcal{F}} \circ [P, id] \circ \omega_{\mathcal{F}}$, which are in one-to-one correspondence with the \mathcal{F} -conjugacy classes of subgroup $P \leq S$.*

The characteristic elements of \mathcal{F} are those elements $X \in A^{\text{char}}(\mathcal{F})_{(p)}$ where the coefficient of X at the basis element $\beta_{\Delta(S, id)} = \omega_{\mathcal{F}}$ is invertible in $\mathbb{Z}_{(p)}$.

Proof. We first claim that $A^{\text{char}}(\mathcal{F})_{(p)} = \omega_{\mathcal{F}} \circ A_{\mathcal{F}}(S, S)_{(p)} \circ \omega_{\mathcal{F}}$, where $A_{\mathcal{F}}(S, S)_{(p)}$ is the subring of \mathcal{F} -generated elements in $A(S, S)_{(p)}$. Each element in $\omega_{\mathcal{F}} \circ A_{\mathcal{F}}(S, S)_{(p)} \circ \omega_{\mathcal{F}}$ is \mathcal{F} -stable and a product of \mathcal{F} -generated elements (hence \mathcal{F} -generated as well), so it is contained in $A^{\text{char}}(\mathcal{F})_{(p)}$.

Conversely, suppose $X \in A^{\text{char}}(\mathcal{F})_{(p)}$. Because X is \mathcal{F} -stable, we have $X = \omega_{\mathcal{F}} \circ X \circ \omega_{\mathcal{F}}$ by corollary 5.6, so X lies in the product $\omega_{\mathcal{F}} \circ A_{\mathcal{F}}(S, S)_{(p)} \circ \omega_{\mathcal{F}}$. We conclude that we have $A^{\text{char}}(\mathcal{F})_{(p)} = \omega_{\mathcal{F}} \circ A_{\mathcal{F}}(S, S)_{(p)} \circ \omega_{\mathcal{F}}$ as claimed.

We know that $A_{\mathcal{F}}(S, S)_{(p)}$ is generated by the sets $[P, \varphi]$ with $\varphi \in \mathcal{F}(P, S)$ by definition. Hence $A^{\text{char}}(\mathcal{F})$ is generated by the elements $\omega_{\mathcal{F}} \circ [P, \varphi] \circ \omega_{\mathcal{F}}$ with $\varphi \in \mathcal{F}(P, S)$, and by lemma 5.7 we have $\omega_{\mathcal{F}} \circ [P, \varphi] \circ \omega_{\mathcal{F}} = \omega_{\mathcal{F}} \circ [P, id] \circ \omega_{\mathcal{F}} = \beta_{\Delta(P, id)}$ as an element of $A(\mathcal{F} \times \mathcal{F})_{(p)}$. So the elements $\beta_{\Delta(P, id)}$ generate $A^{\text{char}}(\mathcal{F})_{(p)}$ and are linearly independent over $\mathbb{Z}_{(p)}$ since they are already part of a basis for the double Burnside ring $A(\mathcal{F}, \mathcal{F})_{(p)}$.

Two basis elements $\beta_{\Delta(P, id)}$ and $\beta_{\Delta(Q, id)}$ are equal exactly when $\Delta(P, id)$ and $\Delta(Q, id)$ are $(\mathcal{F} \times \mathcal{F})$ -conjugate, which happens if and only if P and Q are \mathcal{F} -conjugate.

The elements $X \in A^{\text{char}}(\mathcal{F})_{(p)}$ are already \mathcal{F} -stable and \mathcal{F} -generated, so the only extra condition that \mathcal{F} -characteristic elements must satisfy is that $\varepsilon(X)$ is invertible in $\mathbb{Z}_{(p)}$, i.e., $\varepsilon(X) \not\equiv 0 \pmod{p}$ in $\mathbb{Z}_{(p)}$. Any basis element of $A^{\text{char}}(\mathcal{F})_{(p)}$ other than $\beta_{\Delta(S, id)}$ is of the form $\omega_{\mathcal{F}} \circ [P, incl] \circ \omega_{\mathcal{F}}$ with $P < S$. Because $\varepsilon(\omega_{\mathcal{F}}) = 1$, by the proof of theorem B, we therefore have

$$\varepsilon(\omega_{\mathcal{F}} \circ [P, incl] \circ \omega_{\mathcal{F}}) = 1 \cdot \varepsilon([P, incl]) \cdot 1 = \frac{|S|}{|P|} \equiv 0 \pmod{p}$$

for all $P < S$. So whether $\varepsilon(X) \not\equiv 0 \pmod{p}$ depends only on the coefficient of X at the basis element $\beta_{\Delta(S, id)} = \omega_{\mathcal{F}}$. \square

Lemma 6.4. *Let $\iota^S: A(S)_{(p)} \rightarrow A(S, S)_{(p)}$ be the injective ring homomorphism of example 6.1 given by $[S/P] \mapsto [P, id]$. For every $X \in A(S)_{(p)}$ and subgroup $D \leq S \times S$, we have $\Phi_D(\iota^S(X)) = 0$ unless D is $(S \times S)$ -conjugate to $\Delta(Q, id)$ for some $Q \leq S$. In that case*

$$\Phi_{\Delta(Q, id)}(\iota^S(X)) = \Phi_Q(X) \cdot |C_S(Q)|.$$

Furthermore, $\iota^S(X)$ is symmetric for all $X \in A(S)_{(p)}$, i.e., $\iota^S(X)^{\text{op}} = \iota^S(X)$.

Proof. By linearity in $X \in A(S)_{(p)}$, it is enough to prove the lemma for basis elements $[S/P] \in A(S)_{(p)}$, where $P \leq S$. The symmetry is obvious since $\iota^S([S/P]) = [P, id]$, which is symmetric.

Since $\iota^S([S/P]) = [P, id]$, we apply the formula (3.2) for the fixed-point homomorphisms on basis elements: For $D \leq S \times S$ we have $\Phi_D([P, id]) = 0$ unless D is $(S \times S)$ -subconjugate to $\Delta(P, id)$. The subgroups of $\Delta(P, id)$ are $\Delta(R, id)$ for $R \leq P$, hence D has to be of the form $\Delta(Q, c_s)$ for $Q \leq S$ and $s \in S$, which is $(S \times S)$ -conjugate to $\Delta(Q, id)$. For the graph $\Delta(Q, id)$ we then have

$$\begin{aligned} \Phi_{\Delta(Q, id)}(\iota^S([S/P])) &= \frac{|N_{S \times S}(\Delta(Q, id), \Delta(P, id))|}{|\Delta(P, id)|} \\ &= \frac{|\{(s, t) \mid s, t \in N_S(Q, P) \text{ and } c_s = c_t \in \text{Hom}_S(Q, P)\}|}{|P|} \\ &= \frac{|N_S(Q, P)|}{|P|} \cdot |C_S(Q)| = \Phi_Q([S/P]) \cdot |C_S(Q)|. \end{aligned} \quad \square$$

Lemma 6.5. *Let \mathcal{F} be a saturated fusion system on a p -group S . For all basis elements $\beta_P \in A(\mathcal{F})_{(p)}$ it holds that*

$$\omega_{\mathcal{F}} \circ \iota^S(\beta_P) \circ \omega_{\mathcal{F}} = \omega_{\mathcal{F}} \circ \iota^S(\beta_P) = \iota^S(\beta_P) \circ \omega_{\mathcal{F}} = \beta_{\Delta(P, id)}.$$

By linearity, we get for all $X \in A(\mathcal{F})_{(p)}$ that $\omega_{\mathcal{F}} \circ \iota^S(X) \circ \omega_{\mathcal{F}} = \omega_{\mathcal{F}} \circ \iota^S(X) = \iota^S(X) \circ \omega_{\mathcal{F}}$.

Proof. Because the basis element $\beta_{\Delta(P, id)} \in A^{\text{char}}(\mathcal{F})_{(p)}$ is \mathcal{F} -generated, we have that $\Phi_D(\beta_{\Delta(P, id)}) = 0$ unless D has the form $\Delta(Q, \psi)$ with $\psi \in \mathcal{F}(Q, S)$, and because $\beta_{\Delta(P, id)}$ is \mathcal{F} -stable, we have $\Phi_{\Delta(Q, \psi)}(\beta_{\Delta(P, id)}) = \Phi_{\Delta(Q, id)}(\beta_{\Delta(P, id)})$ when $\psi \in \mathcal{F}(Q, S)$. Considered as an element of $A(\mathcal{F} \times \mathcal{F})_{(p)}$ we know these fixed point values from proposition 4.5:

$$\begin{aligned} \Phi_{\Delta(Q, id)}(\beta_{\Delta(P, id)}) &= \frac{|\text{Hom}_{\mathcal{F} \times \mathcal{F}}(\Delta(Q, id), \Delta(P, id))| \cdot |S \times S|}{|\Delta(P, id)| \cdot |\text{Hom}_{\mathcal{F} \times \mathcal{F}}(\Delta(Q, id), S \times S)|} \\ &= \frac{|\mathcal{F}(Q, P)| \cdot |S|^2}{|P| \cdot |\mathcal{F}(Q, S)|^2} = \Phi_Q(\beta_P) \cdot \frac{|S|}{|\mathcal{F}(Q, S)|}. \end{aligned}$$

For the product $\omega_{\mathcal{F}} \circ \iota^S(\beta_P)$ we apply theorem C to give us

$$\Phi_{\Delta(Q,\psi)}(\omega_{\mathcal{F}} \circ \iota^S(\beta_P)) = \frac{1}{|[\Delta(Q,\psi)]_{\mathcal{F} \times \mathcal{F}_S}|} \sum_{\Delta(Q',\psi') \in [\Delta(Q,\psi)]_{\mathcal{F} \times \mathcal{F}_S}} \Phi_{\Delta(Q',\psi')}(\iota^S(\beta_P)),$$

where \mathcal{F}_S is the trivial fusion system on S . By lemma 6.4, $\Phi_{\Delta(Q',\psi')}(\iota^S(\beta_P)) = 0$ unless $\Delta(Q',\psi')$ is $(S \times S)$ -conjugate to $\Delta(Q',id)$. Since $Q' \sim_S Q$ for all subgroups $\Delta(Q',\psi') \in [\Delta(Q,\psi)]_{\mathcal{F} \times \mathcal{F}_S}$, we conclude that all summands are zero unless $\Delta(Q,id) \in [\Delta(Q,\psi)]_{\mathcal{F} \times \mathcal{F}_S}$. Hence $\Delta(Q,\psi)$ should be conjugate to $\Delta(Q,id)$ inside $\mathcal{F} \times \mathcal{F}_S$, i.e., ψ must lie in \mathcal{F} .

In this case, by left \mathcal{F} -stability of $\omega_{\mathcal{F}} \circ \iota^S(\beta_P)$, we have

$$\Phi_{\Delta(Q,\psi)}(\omega_{\mathcal{F}} \circ \iota^S(\beta_P)) = \Phi_{\Delta(Q,id)}(\omega_{\mathcal{F}} \circ \iota^S(\beta_P)).$$

We still get $\Phi_{\Delta(Q',\psi')}(\iota^S(\beta_P)) = 0$ unless $\Delta(Q',\psi')$ is actually $(S \times S)$ -conjugate to $\Delta(Q',id)$ and $\Delta(Q,id)$. In the calculation of $\Phi_{\Delta(Q,id)}(\omega_{\mathcal{F}} \circ \iota^S(\beta_P))$ we can therefore omit all the summands that are zero, and we get

$$\begin{aligned} & \Phi_{\Delta(Q,\psi)}(\omega_{\mathcal{F}} \circ \iota^S(\beta_P)) \\ &= \Phi_{\Delta(Q,id)}(\omega_{\mathcal{F}} \circ \iota^S(\beta_P)) \\ &= \frac{1}{|[\Delta(Q,id)]_{\mathcal{F} \times \mathcal{F}_S}|} \sum_{\Delta(Q',\psi') \in [\Delta(Q,id)]_{S \times S}} \Phi_{\Delta(Q',\psi')}(\iota^S(\beta_P)) \\ &= \frac{|[\Delta(Q,id)]_{S \times S}|}{|[\Delta(Q,id)]_{\mathcal{F} \times \mathcal{F}_S}|} \cdot \Phi_{\Delta(Q,id)}(\iota^S(\beta_P)) \\ &= \frac{|\text{Hom}_{S \times S}(\Delta(Q,id), S \times S)| \cdot |\text{Aut}_{\mathcal{F} \times \mathcal{F}_S}(\Delta(Q,id))|}{|\text{Aut}_{S \times S}(\Delta(Q,id))| \cdot |\text{Hom}_{\mathcal{F} \times \mathcal{F}_S}(\Delta(Q,id), S \times S)|} \cdot \Phi_Q(\beta_P) \cdot |C_S(Q)| \\ &= \frac{\frac{|S|^2}{|C_S(Q)|^2} \cdot \frac{|N_S(Q)|}{|C_S(Q)|}}{\frac{|N_S(Q)|}{|C_S(Q)|} \cdot (|\mathcal{F}(Q,S)| \cdot \frac{|S|}{|C_S(Q)|})} \cdot \Phi_Q(\beta_P) \cdot |C_S(Q)| \\ &= \Phi_Q(\beta_P) \cdot \frac{|S|}{|\mathcal{F}(Q,S)|} = \Phi_{\Delta(Q,\psi)}(\beta_{\Delta(P,id)}). \end{aligned}$$

This shows that $\omega_{\mathcal{F}} \circ \iota^S(\beta_P) = \beta_{\Delta(P,id)}$; and by symmetry we have

$$\beta_{\Delta(P,id)} = (\beta_{\Delta(P,id)})^{\text{op}} = (\omega_{\mathcal{F}} \circ \iota^S(\beta_P))^{\text{op}} = \iota^S(\beta_P)^{\text{op}} \circ \omega_{\mathcal{F}}^{\text{op}} = \iota^S(\beta_P) \circ \omega_{\mathcal{F}}.$$

Finally, $\omega_{\mathcal{F}} \circ (\iota^S(\beta_P) \circ \omega_{\mathcal{F}}) = \omega_{\mathcal{F}} \circ (\omega_{\mathcal{F}} \circ \iota^S(\beta_P)) = \omega_{\mathcal{F}} \circ \iota^S(\beta_P)$. \square

Theorem D. *Let \mathcal{F} be a saturated fusion system on a finite p -group S .*

Then the collapse map $q: A^{\text{char}}(\mathcal{F})_{(p)} \rightarrow A(\mathcal{F})_{(p)}$, which quotients out the right S -action, is an isomorphism of rings, and it sends the basis element $\beta_{\Delta(P,id)}$ of $A(\mathcal{F}, \mathcal{F})_{(p)}$ to the basis element β_P of $A(\mathcal{F})_{(p)}$.

Proof. For a biset X the quotient X/S is the same as the product $X \times_S pt$, so the collapse map $q: A(S, S)_{(p)} \rightarrow A(S)_{(p)}$ is alternatively given as right-multiplication with the one-point $(1, S)$ -biset $[pt]_1^S$. The one-point biset has $\Phi_D([pt]_1^S) = 1$ for all $D \leq S \times 1$, and by theorem C we then also have $\Phi_D(\omega_{\mathcal{F}} \circ [pt]_1^S) = 1$ for all $D \leq S \times 1$, so $\omega_{\mathcal{F}} \circ [pt]_1^S = [pt]_1^S$.

If we apply the collapse map q to the basis elements $\beta_{\Delta(P, id)} = \omega_{\mathcal{F}} \circ [P, id] \circ \omega_{\mathcal{F}}$ of $A^{\text{char}}(\mathcal{F})_{(p)}$ we therefore get

$$\begin{aligned} q(\beta_{\Delta(P, id)}) &= \omega_{\mathcal{F}} \circ [P, id]_S^S \circ \omega_{\mathcal{F}} \circ [pt]_1^S \\ &= \omega_{\mathcal{F}} \circ [P, id]_S^S \circ [pt]_1^S = \omega_{\mathcal{F}} \circ [S/P]_1^S. \end{aligned}$$

By corollary 5.8 multiplication with $\omega_{\mathcal{F}}$ in $A(1, S)_{(p)}$ is the same as the stabilization map of theorem A, so $q(\beta_{\Delta(P, id)}) = \omega_{\mathcal{F}} \circ [S/P]_1^S = \beta_P$ as elements of $A(S)_{(p)}$.

Now we define a $\mathbb{Z}_{(p)}$ -homomorphism $\iota^{\mathcal{F}}: A(\mathcal{F})_{(p)} \rightarrow A^{\text{char}}(\mathcal{F})_{(p)}$ by

$$\iota^{\mathcal{F}}(X) = \omega_{\mathcal{F}} \circ \iota^S(X) \circ \omega_{\mathcal{F}},$$

and by lemma 6.5 we then have $\iota^{\mathcal{F}}(\beta_P) = \beta_{\Delta(P, id)}$. Because q sends $\beta_{\Delta(P, id)} \in A^{\text{char}}(\mathcal{F})_{(p)}$ to $\beta_P \in A(\mathcal{F})_{(p)}$, and $\iota^{\mathcal{F}}$ sends it back again, the two maps q and $\iota^{\mathcal{F}}$ are inverse isomorphisms of $\mathbb{Z}_{(p)}$ -modules $A(\mathcal{F})_{(p)}$ and $A^{\text{char}}(\mathcal{F})_{(p)}$.

Finally, we recall that ι^S is a ring homomorphism, and apply lemma 6.5 to show that all elements $X, Y \in A(\mathcal{F})_{(p)}$ satisfy

$$(\omega_{\mathcal{F}} \circ \iota^S(X) \circ \omega_{\mathcal{F}}) \circ (\omega_{\mathcal{F}} \circ \iota^S(Y) \circ \omega_{\mathcal{F}}) = \omega_{\mathcal{F}} \circ \iota^S(X) \circ \iota^S(Y) \circ \omega_{\mathcal{F}} = \omega_{\mathcal{F}} \circ \iota^S(XY) \circ \omega_{\mathcal{F}}.$$

Hence $\iota^{\mathcal{F}}$ preserves multiplication, and consequently the inverse $q: A^{\text{char}}(\mathcal{F})_{(p)} \rightarrow A(\mathcal{F})_{(p)}$ does as well. \square

We now apply the ring isomorphism $A^{\text{char}}(\mathcal{F})_{(p)} \cong A(\mathcal{F})_{(p)}$, to determine all idempotents of $A^{\text{char}}(\mathcal{F})_{(p)}$. In particular, this finally completes the proof that a saturated fusion system \mathcal{F} has exactly one characteristic idempotent.

Corollary 6.6. *Let \mathcal{F} be a saturated fusion system on a finite p -group S . The only idempotents of $A(\mathcal{F})_{(p)}$ are 0 and the 1-element $[S/S]$. Hence it follows that $A^{\text{char}}(\mathcal{F})_{(p)}$ has exactly one non-zero idempotent, hence the characteristic idempotent $\omega_{\mathcal{F}}$ is unique.*

Proof. By proposition 3.2 the Burnside ring $A(\mathcal{F})_{(p)}$ fits in a short-exact sequence of $\mathbb{Z}_{(p)}$ -modules

$$0 \rightarrow A(\mathcal{F})_{(p)} \xrightarrow{\Phi} \tilde{\Omega}(\mathcal{F})_{(p)} \xrightarrow{\Psi} \text{Obs}(\mathcal{F})_{(p)} \rightarrow 0.$$

Here Φ is the mark homomorphism, $\text{Obs}(\mathcal{F})_{(p)}$ is the group

$$\text{Obs}(\mathcal{F})_{(p)} = \prod_{\substack{[P]_{\mathcal{F}} \in \text{Cl}(\mathcal{F}) \\ P \text{ f.n.}}} (\mathbb{Z}/|W_S P|\mathbb{Z}),$$

and Ψ is given by the $[P]_{\mathcal{F}}$ -coordinate functions

$$\Psi_P(\xi) := \sum_{\bar{s} \in W_S P} \xi_{\langle s \rangle P} \pmod{|W_S P|},$$

when P is fully \mathcal{F} -normalized, and $\xi_{\langle s \rangle P}$ denotes the $[\langle s \rangle P]_{\mathcal{F}}$ -coordinate of an element $\xi \in \tilde{\Omega}(\mathcal{F})_{(p)} = \prod_{[P] \in \text{Cl}(\mathcal{F})} \mathbb{Z}_{(p)}$.

Let ω be an idempotent in $A(\mathcal{F})_{(p)}$, then since Φ is a ring homomorphism, the fixed point vector $\Phi(\omega)$ must be idempotent in the product ring $\tilde{\Omega}(\mathcal{F})_{(p)}$. Since $\Phi(\omega)$ is an element of a product ring, it is idempotent if and only if each coordinate $\Phi_Q(\omega)$ is idempotent in $\mathbb{Z}_{(p)}$. The only idempotents of $\mathbb{Z}_{(p)}$ are 0 and 1, so $\omega \in A(\mathcal{F})_{(p)}$ is idempotent if and only if we have $\Phi_Q(\omega) \in \{0, 1\}$ for all $Q \leq S$.

Let the top coordinate $\Phi_S(\omega)$ be fixed as either 0 or 1, then we will prove by induction on the index of $Q \leq S$ that there is at most one possibility for the coordinate $\Phi_Q(\omega)$. Suppose that $Q < S$, and that $\Phi_R(\omega)$ is determined for all R with $|R| > |Q|$. Then because $\Psi\Phi = 0$, the fixed points must satisfy

$$\sum_{\bar{s} \in W_S Q} \Phi_{\langle s \rangle Q}(\omega) \equiv 0 \pmod{|W_S Q|},$$

or if we isolate $\Phi_Q(\omega)$:

$$\Phi_Q(\omega) \equiv - \sum_{\substack{\bar{s} \in W_S Q \\ \bar{s} \neq 1}} \Phi_{\langle s \rangle Q}(\omega) \pmod{|W_S Q|}.$$

We have $|\langle s \rangle Q| > |Q|$ for all $s \in N_S Q$ with $s \notin Q$, so all the numbers $\Phi_{\langle s \rangle Q}(\omega)$ are already determined by induction. In addition $Q < S$ implies $Q < N_S Q$, so $|W_S Q| \geq 2$, and thus $\Phi_Q(\omega) = 0$ and $\Phi_Q(\omega) = 1$ cannot both satisfy the congruence relation.

We conclude that once $\Phi_S(\omega)$ is fixed, there is at most one possibility for ω . The empty set $0 = [\emptyset]$ is idempotent and satisfies $\Psi_S(0) = 0$, and the one point set $[S/S]$ is idempotent and satisfies $\Phi_S([S/S]) = 1$, so both possibilities exist. \square

7. ON THE COMPOSITION PRODUCT OF SATURATED FUSION SYSTEMS

In this final section we apply the earlier theorems B and C about characteristic idempotents to a conjecture of Park-Ragnarsson-Stancu in [11] concerning composition products of fusion systems and how to characterize them in terms of characteristic idempotents.

Let \mathcal{F} be a fusion system on a p -group S , and let \mathcal{H}, \mathcal{K} be fusion subsystems on subgroups $R, T \leq S$ respectively. In the terminology of Park-Ragnarsson-Stancu, we then say that \mathcal{F} is the composition product of \mathcal{H} and \mathcal{K} , written $\mathcal{F} = \mathcal{H}\mathcal{K}$, if $S = RT$ and for all subgroups $P \leq T$ it holds that every morphism $\varphi \in \mathcal{F}(P, R)$ can be written as a composition $\varphi = \psi\rho$ such that ψ is a morphism of \mathcal{H} and ρ is a morphism of \mathcal{K} .

For a finite group G with subgroups $H, K \leq G$, we can ask whether $G = HK$, i.e., if every $g \in G$ can be written as $g = hk$ with $h \in H$ and $k \in K$. It turns out that the answer to this question is detected by the structure of G as an (K, H) -biset. With a little thought one can show that $G = HK$ if and only if the (K, H) -biset G is isomorphic to the transitive biset $H \times_{H \cap K} K$. This result for groups inspired Park-Ragnarsson-Stancu to conjecture that $\mathcal{F} = \mathcal{H}\mathcal{K}$ is equivalent to a similar relation between the characteristic idempotents:

$$(7.1) \quad [R, id]_S^R \circ \omega_{\mathcal{F}} \circ [T, id]_T^S = \omega_{\mathcal{H}} \circ [R \cap T, id]_T^R \circ \omega_{\mathcal{K}}$$

Thanks to theorem C we can now directly calculate under which circumstances (7.1) holds, which results in the following theorem.

Theorem E. *Let \mathcal{F} be a saturated fusion system on a p -group S , and suppose that \mathcal{H}, \mathcal{K} are saturated fusion subsystems of \mathcal{F} on subgroups $R, T \leq S$ respectively.*

Then the characteristic idempotents satisfy

$$(7.2) \quad [R, id]_S^R \circ \omega_{\mathcal{F}} \circ [T, id]_T^S = \omega_{\mathcal{H}} \circ [R \cap T, id]_T^R \circ \omega_{\mathcal{K}}$$

if and only if $\mathcal{F} = \mathcal{H}\mathcal{K}$ and for all $Q \leq R \cap T$ we have

$$(7.3) \quad |\mathcal{F}(Q, S)| = \frac{|\mathcal{H}(Q, R)| \cdot |\mathcal{K}(Q, T)|}{|\text{Hom}_{\mathcal{H}\mathcal{K}}(Q, R \cap T)|}.$$

Proof. The element $l.h.s. := [R, id]_S^R \circ \omega_{\mathcal{F}} \circ [T, id]_T^S$ is the characteristic idempotent for \mathcal{F} restricted to $A(T, R)_{(p)}$. For subgroups $D \leq R \times T$ we therefore have $\Phi_D([R, id]_S^R \circ \omega_{\mathcal{F}} \circ [T, id]_T^S) = 0$ unless D has the form $\Delta(P, \varphi)$ with $P \leq T$ and $\varphi \in \mathcal{F}(P, R)$, and for such P and φ we get

$$\Phi_{\Delta(P, \varphi)}([R, id]_S^R \circ \omega_{\mathcal{F}} \circ [T, id]_T^S) = \frac{|S|}{|\mathcal{F}(P, S)|}.$$

For the right hand side we know from theorem C that $r.h.s. := \omega_{\mathcal{H}} \circ [R \cap T, id]_T^R \circ \omega_{\mathcal{K}}$ is equal to the basis element $\beta_{\Delta(R \cap T, id)}$ in $A(\mathcal{H} \times \mathcal{K})_{(p)}$. Hence we have $\Phi_D(\omega_{\mathcal{H}} \circ [R \cap T, id]_T^R \circ \omega_{\mathcal{K}}) = 0$ unless D is $(\mathcal{H} \times \mathcal{K})$ -conjugate to $\Delta(P, id)$ for some $P \leq R \cap T$, i.e., D has the form $\Delta(\rho^{-1}Q, \psi\rho)$ with $Q \leq T \cap R$, $\rho^{-1} \in \mathcal{K}(Q, T)$, and $\psi \in \mathcal{H}(Q, R)$; and if D has this form, then we get

$$\begin{aligned} \Phi_{\Delta(\rho^{-1}Q, \psi\rho)}(r.h.s.) &= \Phi_{\Delta(\rho^{-1}Q, \psi\rho)}(\omega_{\mathcal{H}} \circ [R \cap T, id]_T^R \circ \omega_{\mathcal{K}}) \\ &= \Phi_{\Delta(\rho^{-1}Q, \psi\rho)}(\beta_{\Delta(R \cap T, id)}) = \Phi_{\Delta(Q, id)}(\beta_{\Delta(R \cap T, id)}) \\ &= \frac{|\text{Hom}_{\mathcal{H} \times \mathcal{K}}(\Delta(Q, id), \Delta(R \cap T, id))| \cdot |R \times T|}{|\Delta(R \cap T, id)| \cdot |\text{Hom}_{\mathcal{H} \times \mathcal{K}}(\Delta(Q, id), R \times T)|} \\ &= \frac{|\text{Hom}_{\mathcal{H} \cap \mathcal{K}}(Q, R \cap T)| \cdot |R| \cdot |T|}{|R \cap T| \cdot |\mathcal{H}(Q, R)| \cdot |\mathcal{K}(Q, T)|} \\ &= |RT| \cdot \frac{|\text{Hom}_{\mathcal{H} \cap \mathcal{K}}(Q, R \cap T)|}{|\mathcal{H}(Q, R)| \cdot |\mathcal{K}(Q, T)|}. \end{aligned}$$

Suppose that (7.2) is true. Comparing $\Phi_{\Delta(1, id)}(l.h.s.) = |S|$ and $\Phi_{\Delta(1, id)}(r.h.s.) = |RT|$, we see that we must have $|S| = |RT|$, and consequently $S = RT$. Furthermore we know that $\Phi_{\Delta(P, \varphi)}(l.h.s.) \neq 0$ if $P \leq T$ and $\varphi \in \mathcal{F}(P, R)$, it is therefore a requirement for (7.2) that $\Phi_{\Delta(P, \varphi)}(r.h.s.) \neq 0$ as well, which is the case exactly when $\Delta(P, \varphi)$ has the form $\Delta(\rho^{-1}Q, \psi\rho)$ with $\rho \in \mathcal{K}$ and $\psi \in \mathcal{H}$, hence $\varphi = \psi\rho \in \mathcal{HK}$, so we must have $\mathcal{F} = \mathcal{HK}$. Because $S = RT$, the equality $\Phi_{\Delta(Q, id)}(l.h.s.) = \Phi_{\Delta(Q, id)}(r.h.s.)$ gives us (7.3).

If we conversely suppose that $\mathcal{F} = \mathcal{HK}$, then $\Phi_{\Delta(P, \varphi)}(l.h.s.)$ and $\Phi_{\Delta(P, \varphi)}(r.h.s.)$ are non-zero for the same indices, and because $S = RT$, the only obstacle for equality of fixed points $\Phi_{\Delta(\rho^{-1}Q, \psi\rho)}(l.h.s.) = \Phi_{\Delta(\rho^{-1}Q, \psi\rho)}(r.h.s.)$ is whether it holds that

$$\frac{1}{|\mathcal{F}(Q, S)|} = \frac{|\text{Hom}_{\mathcal{H} \cap \mathcal{K}}(Q, R \cap T)|}{|\mathcal{H}(Q, R)| \cdot |\mathcal{K}(Q, T)|}$$

for all $Q \leq R \cap T$, which is (7.3). \square

Example 7.1. The following example shows that the conjecture of Park-Ragnarsson-Stancu fails in general. We consider the alternating group A_6 , and identify one of its Sylow 2-subgroups with the dihedral group D_8 . The associated fusion system $\mathcal{F} := \mathcal{F}_{D_8}(A_6)$ is the saturated fusion system on D_8 wherein all five subgroups of order 2 are conjugate. Let $R, T \leq D_8$ be the two Klein four-groups inside D_8 , and let $\mathcal{H} = \mathcal{F}_R(R \times \mathbb{Z}/3)$, $\mathcal{K} = \mathcal{F}_T(T \times \mathbb{Z}/3)$ be fusion subsystems of \mathcal{F} on R and T respectively, with $\mathbb{Z}/3$ acting nontrivially on $R \cong T \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. Then \mathcal{H} and \mathcal{K} both contain the order 3 automorphisms of the Klein four-group, and both are saturated.

We claim that $\mathcal{F} = \mathcal{HK}$. First of all $D_8 = RT$ is clear. Next, there is no isomorphism between R and T in \mathcal{F} , so the only subgroups of T that map to R in \mathcal{F} , are the subgroups of order 2 and the trivial group. Suppose $A \leq T$ has order 2. Then every morphism $\varphi \in \mathcal{F}(A, R)$ factors through $Z(D_8) = R \cap T$, and can therefore be factored as $\varphi = \rho\psi$ with $\psi \in \mathcal{K}(A, Z(D_8))$ and $\rho \in \mathcal{H}(Z(D_8), R)$. Hence we have $\mathcal{F} = \mathcal{HK}$.

However (7.3) fails for the intersection $Q := Z(D_8) = R \cap T$ since we get

$$|\mathcal{F}(Z(D_8), S)| = 5 \neq \frac{3 \cdot 3}{1} = \frac{|\mathcal{H}(Z(D_8), R)| \cdot |\mathcal{K}(Z(D_8), T)|}{|(\mathcal{H} \cap \mathcal{K})(Z(D_8), Z(D_8))|}.$$

Consequently, we have $\mathcal{F} = \mathcal{H}\mathcal{K}$ but not (7.2), so the conjecture of Park-Ragnarsson-Stancu is false in general.

The following generalization of [11, Theorem 1.3] is an example of how to apply theorem E to prove a special case of the conjecture.

Proposition 7.2. *Let \mathcal{F} be a saturated fusion system on a p -group S , and let \mathcal{H}, \mathcal{K} be saturated fusion subsystems of \mathcal{F} on subgroups $R, T \leq S$ respectively. Suppose that \mathcal{K} is weakly normal in \mathcal{F} , i.e., \mathcal{K} is saturated and \mathcal{F} -invariant in the sense of [1].*

Then $\mathcal{F} = \mathcal{H}\mathcal{K}$ if and only if the characteristic idempotents satisfy

$$[R, id]_S^R \circ \omega_{\mathcal{F}} \circ [T, id]_T^S = \omega_{\mathcal{H}} \circ [R \cap T, id]_T^R \circ \omega_{\mathcal{K}}.$$

Proof. By theorem E it is sufficient to prove that $\mathcal{F} = \mathcal{H}\mathcal{K}$ implies (7.3), so suppose $\mathcal{F} = \mathcal{H}\mathcal{K}$. The subsystem \mathcal{K} being \mathcal{F} -invariant means that T is strongly closed in \mathcal{F} , and whenever we have $Q, R \leq P \leq T$ and $\varphi \in \mathcal{F}(P, T)$, conjugation by φ induces a bijection $\mathcal{K}(Q, R) \xrightarrow{\varphi(-)\varphi^{-1}} \mathcal{K}(\varphi Q, \varphi R)$.

According to [1, Lemma 3.6], the intersection $\mathcal{H} \cap \mathcal{K}$ is an \mathcal{H} -invariant fusion system on $R \cap T$. Suppose we have subgroups $Q \leq R \cap T$ and $Q' \sim_{\mathcal{H}} Q$, and choose an isomorphism $\varphi \in \mathcal{H}(Q, Q')$. Because T is strongly closed in \mathcal{F} , $R \cap T$ is strongly closed in \mathcal{H} , hence $Q' \leq R \cap T$. By the Frattini property of \mathcal{H} -invariant subsystems, [1, Section 3], φ can be factored as $\varphi = \eta\kappa$ with $\kappa \in (\mathcal{H} \cap \mathcal{K})(Q, R \cap T)$ and $\eta \in \text{Aut}_{\mathcal{H}}(R \cap T)$. If we let $Q'' := \kappa(Q)$, we then have $|\mathcal{K}(Q, T)| = |\mathcal{K}(Q'', T)|$ and $|(\mathcal{H} \cap \mathcal{K})(Q, T)| = |(\mathcal{H} \cap \mathcal{K})(Q'', T)|$. Furthermore, the \mathcal{H} -isomorphism $\eta: Q'' \rightarrow Q'$ is defined on all of $R \cap T$, so the \mathcal{F} -stability of \mathcal{K} and \mathcal{H} -stability of $\mathcal{H} \cap \mathcal{K}$ implies that η induces bijections $\mathcal{K}(Q'', T) \cong \mathcal{K}(Q', T)$ and $(\mathcal{H} \cap \mathcal{K})(Q'', T) \cong (\mathcal{H} \cap \mathcal{K})(Q', T)$.

We will now prove (7.3), and because T is strongly closed in \mathcal{F} , we must show

$$|\mathcal{F}(Q, T)| = \frac{|\mathcal{H}(Q, R \cap T)| \cdot |\mathcal{K}(Q, T)|}{|(\mathcal{H} \cap \mathcal{K})(Q, R \cap T)|}$$

for all $Q \leq R \cap T$. Let therefore $Q \leq R \cap T$ be given. For every homomorphism $\varphi \in \mathcal{F}(Q, T)$, we can factor $\varphi^{-1}: \varphi Q \rightarrow Q$ as $\varphi^{-1} = \eta^{-1}\kappa^{-1}$ with $\eta^{-1} \in \mathcal{H}$ and $\kappa^{-1} \in \mathcal{K}$, or equivalently $\varphi = \kappa\eta$. We will enumerate $\mathcal{F}(Q, T)$ by counting the number of pairs of isomorphisms (κ, η) with $\eta: Q \rightarrow Q'$ in \mathcal{H} and $\kappa: Q' \rightarrow Q''$ in \mathcal{K} . The number of choices for η is $|\mathcal{H}(Q, R \cap T)|$, and for each $\eta: Q \rightarrow Q'$ the number of choices for κ is $|\mathcal{K}(Q', T)|$. Because Q' is isomorphic to Q in \mathcal{H} , the arguments above imply that $|\mathcal{K}(Q', T)| = |\mathcal{K}(Q, T)|$, which is independent of the chosen $\eta \in \mathcal{H}(Q, R \cap T)$. The total number of composable pairs (κ, η) is therefore

$$|\mathcal{H}(Q, T)| \cdot |\mathcal{K}(Q, T)|.$$

Given a pair (κ, η) of composable isomorphisms $Q \xrightarrow{\eta} Q' \xrightarrow{\kappa} Q''$, we then count the number of other pairs $Q \xrightarrow{\eta'} Q''' \xrightarrow{\kappa'} Q''$ that represent the same isomorphism in \mathcal{F} . If (κ, η) and (κ', η') give the same isomorphism $Q \rightarrow Q''$ in \mathcal{F} , then we have $\kappa\eta = \kappa'\eta'$ or equivalently $(\kappa')^{-1}\kappa = \eta'\eta^{-1} \in (\mathcal{H} \cap \mathcal{K})(Q', R \cap T)$. Conversely, given any $\rho \in (\mathcal{H} \cap \mathcal{K})(Q', R \cap T)$, the pair $(\kappa\rho^{-1}, \rho\eta)$ defines the same \mathcal{F} -homomorphism as (κ, η) . The number of pairs representing the same map as (κ, η) is therefore $|(\mathcal{H} \cap \mathcal{K})(Q', R \cap T)| = |(\mathcal{H} \cap \mathcal{K})(Q, R \cap T)|$, which is

independent of the chosen pair (κ, η) . Hence there are $|(\mathcal{H} \cap \mathcal{K})(Q, R \cap T)|$ pairs representing each homomorphism $\varphi \in \mathcal{F}(Q, T)$, so we get

$$|\mathcal{F}(Q, T)| = \frac{|\mathcal{H}(Q, R \cap T)| \cdot |\mathcal{K}(Q, T)|}{|(\mathcal{H} \cap \mathcal{K})(Q, R \cap T)|}$$

as we wanted. \square

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