Internship report

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1 Introduction

This document is a sort of report of my internship with Jesper Grodal at Københavns Universitet in the second semester of the school year 2018-2019.
It covers part of what I studied, not entering into many details or giving many proofs, only giving the ones I find instructive or illuminating, or some interesting or motivating examples that are dealt with in some more detail.
I studied various different but related topics; the main part of this report will be dedicated to one aspect of what I learnt : group cohomology, and various tools associated to it.
This document uses the language of categories, homological algebra, and basic algebraic topology, which will not be redefined. One can find the necessary prerequisites in the given references, if needed.

I wish to thank my advisor, Jesper Grodal, for receiving me for the semester, and helping as well as guiding me throughout, he has been of great help; and the weekly talks with him as well as the many references he gave me have allowed me to discover some great mathematics that go well beyond what is presented in this report.

2 Group (Co)Homology

2.1 Motivation

Suppose you have a manifold M and a discrete group G acting properly on M; you know the de Rham cohomology of M and want to see what more you need to get the de Rham cohomology of M/G.
You then notice that on p-forms, you have a pullback map $p^* : \Omega^p(M/G) \to \Omega^p(M)$, and that it's an injective map, whose image is precisely $\Omega^p(M)^G$, the G-invariant forms on M. You conclude, naively, that there is a natural isomorphism $H^p_{\mathrm{dR}}(M)^G \cong H^p_{\mathrm{dR}}(M/G)$: "of course cohomology commutes with taking G-invariants". This allows you to compute $H^p_{\mathrm{dR}}(M/G)$. For instance, take $M = \mathbb{R}, G = \mathbb{Z}$ acting by translation as usual. From the easy computation of $H^1_{\mathrm{dR}}(\mathbb{R}) = 0$ you conclude that $H^1_{\mathrm{dR}}(S^1) = 0$. This is of course wrong.
It doesn't work because cohomology *doesn't* commute with taking invariants. The lack of commutativity is taken into account by *group cohomology*, which is what we will introduce in what follows.

2.2 Functorial formulation

Cohomology doesn't commute with taking invariants because taking invariants is not an *exact functor*. We introduce in this section the relevant notions and machinery that help us measure how far a functor is from being exact.

Definition 1. Let $\mathcal{A}, \mathcal{B}$ be abelian categories and F an additive functor between them. F is said to be left exact if whenever $0 \to A \to B \to C \to 0$ is exact in $\mathcal{A}$, $0 \to F(A) \to F(B) \to F(C)$ is exact in $\mathcal{B}$. Dually, it is right exact if whenever $0 \to A \to B \to C \to 0$ is exact in $\mathcal{A}$, $F(A) \to F(B) \to F(C) \to 0$ is exact in $\mathcal{B}$. It is exact if it is both left and right exact.

Examples 1.
- Fix an abelian group A. Then $A \otimes_{\mathbb{Z}} - : \mathbf{Ab} \to \mathbf{Ab}$ is right exact, but is not left exact in general (when it is, A is called *flat*)

- Still with our abelian group A, $\hom(A,-) : \mathbf{Ab} \to \mathbf{Ab}$ is left exact, but not right exact in general (when it is, A is called *projective*)
- The relevant example for group cohomology : let G be a group and let $G-\mathbf{Mod}$ denote the category of abelian groups together with an action of G by automorphisms, with G-equivariant maps as arrows, then we have the invariants functor functor

$$(-)^G : G-\mathbf{Mod} \to \mathbf{Ab}$$

  which is left exact but not right exact in general.

- The other relevant example, for group homology, is the co-invariant functor

$$(-)_G : G-\mathbf{Mod} \to \mathbf{Ab}$$

  defined by $M_G = M/\langle g\cdot m - m, g \in G, m \in M\rangle$. This is right exact, but not left exact in general.

More generally, right adjoints are left exact, and dually left adjoints are right exact.
Quite clearly, exact functors commute with (co)homology, but left or right exact functors don't in general, which is exactly what went wrong in the above section. To correct the lack of exactness, we introduce left or right derived functors. We will not give the whole theory here, only an outline; for a more detailed account see for instance [1], [2] or [4].

Proposition 1. *Let $\mathcal{A}$ be an abelian category with enough injectives and $F : \mathcal{A} \to \mathcal{B}$ a left exact functor; then there is a sequence $\{R^nF\}_{n\in\mathbb{N}}$ of additive functors $\mathcal{A} \to \mathcal{B}$, called the right derived functors of F, such that :*

- $R^0F \simeq F$
- *For $n > 1$, R^nF vanishes on injectives*
- *For any short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, there is a sequence of connecting morphisms $\delta^n : R^nF(C) \to R^{n+1}F(A)$ such that the sequence $0 \to R^0F(A) \to R^0F(B) \to R^0F(C) \to R^1F(A) \to \cdots \to R^nF(B) \to R^nF(C) \to R^{n+1}F(A) \to \dots$ is a long exact sequence. Moreover these δ^n are natural in the exact sequence (that is, any morphism of short exact sequences induces a morphism of long exact sequences).*

Moreover these properties characterize the sequence $\{R^nF\}_{n\in\mathbb{N}}$ up to a natural isomorphism commuting with the δ^n.

Since it is an interesting technique, we do give the main idea for the proof of uniqueness of these :

Sketch of proof. It's called dimension-shifting: we proceed by induction on n.Let $\{T^n\}$ be another such sequence. $T^0 \simeq R^0F$ is clear by the first bullet point. Next assume for $k < n$, there are natural isomorphisms $R^kF \to T^k$ that commute with all the $\delta^l, l < n$. Then given $A \in \mathcal{A}$, find an injective object I and a monomorphism $A \to I$.
Look at the long exact sequence induce by $0 \to A \to I \to I/A \to 0$: it has $R^kF(I) \to R^kF(I/A) \to R^{k+1}F(A) \to R^{k+1}F(I)$, and similarly with T^k. By the second bullet point, if $k > 0$, this gives an isomorphism $R^kF(I/A) \to R^{k+1}F(A)$. With $k = n-1$, we can then use the isomorphism $R^kF(I/A) \to T^k(I/A)$ to get an isomorphism $R^{k+1}(A) \to T^{k+1}(A)$.

There are things to check : that this isomorphism doesn't depend on the choice of I, that it's natural, and that it commutes with δ. And also, we have to look more closely at the $n = 1$ case. None of these are complicated, and we only wanted to sketch the argument here. □

To see more about this intrinsic characterization of the derived functors, one may want to look at [1] for instance.
Of course we have the dual proposition for right exact functors.

Proposition 2. *Let $\mathcal{A}$ be an abelian category with enough projectives and $F : \mathcal{A} \to \mathcal{B}$ a right exact functor; then there is a sequence $\{L_nF\}_{n\in\mathbb{N}}$ of additive functors $\mathcal{A} \to \mathcal{B}$, called the left derived functors of F, such that :*

- *$L_0F \simeq F$*
- *For $n > 1$, L_nF vanishes on projectives*
- *For any short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, there is a sequence of connecting morphisms $\delta_n : L_{n+1}F(C) \to L_nF(A)$ such that the sequence $\cdots \to L_{n+1}F(C) \to L_nF(A) \to L_nF(B) \to \cdots \to L_1F(C) \to L_0F(A) \to L_0F(B) \to L_0F(C) \to 0$ is a long exact sequence. Moreover these δ_n are natural in the exact sequence (that is, any morphism of short exact sequences induces a morphism of long exact sequences).*

Moreover these properties characterize the sequence $\{L_nF\}_{n\in\mathbb{N}}$ up to a natural isomorphism commuting with the δ_n.

We also have dimension shifting in this situation, although of course it consists in taking an epimorphism from a projective onto A.

Definition 2. The right derived functors of $(-)^G$ are called "group cohomology of G" and are denoted by $H^n(G, -)$.
The left derived functors of $(-)_G$ are called "group homology of G" and are denoted by $H_n(G, -)$.

We add a computational tool that can be taken to be a definition of derived functors :

Proposition 3. *Let $\mathcal{A}$ be an abelian category with enough injectives, and $F : \mathcal{A} \to \mathcal{B}$ a left exact functor. If $A \in \mathcal{A}$ and $A \to I_\bullet$ is an injective resolution, then there is a canonical isomorphism $H^n(F(I_\bullet)) \to R^nF(A)$.*
Dually, if $\mathcal{A}$ has enough projectives and F is right exact, then for a projective resolution $P_\bullet \to A$ we have canonical isomorphisms $H_n(F(P_\bullet)) \to L_nF(A)$.

It is well-known that categories of modules over a ring have enough injectives and projectives, so all our theorems here apply to these, in particular to the category of G-modules (even over some base ring R).

2.3 Link with topology

We will see in this subsection that (co)homology groups of a group G have a topological interpretation, which justifies their name. This topological interpretation will be very useful both when looking for some algebraic information based on topology and when looking for some topological information based on algebra. For instance it will allow us to construct some projective resolutions of $\mathbb{Z}$ over $\mathbb{Z}[G]$.
Take a nice space X with fundamental group (isomorphic to) G. Then its universal cover Y has a G action, therefore its singular chain groups $C_\bullet(Y;\mathbb{Z})$ are G-modules. Notice moreover that $C_\bullet(Y;\mathbb{Z}) \to C_\bullet(X;\mathbb{Z})$ is surjective by covering space theory, and its kernel clearly contains $g\cdot\sigma - \sigma$ for any chain σ. It is not complicated to see that $C_\bullet(Y;\mathbb{Z})_G$ is precisely $C_\bullet(X;\mathbb{Z})$ with this projection map.
It isn't complicated either to see that $C_\bullet(Y;\mathbb{Z})$ is a free (hence projective) G-module.
Therefore, if Y has no positive degree homology and is connected, so that $C_\bullet(Y;Z) \to \mathbb{Z}$ is a projective resolution, the homology of $C_\bullet(Y;Z)_G = C_\bullet(X;\mathbb{Z})$ is precisely $H_*(G;\mathbb{Z})$. If X (and thus Y) has the homotopy type of a CW-complex, requiring that the reduced homology of Y vanishes is the same as requiring that its homotopy groups are all 0 by the Hurewicz theorem, thus that Y is contractible. In particular, X has only one non trivial homotopy group.

Definition 3. A space X with the homotopy type of a CW-complex is called a $K(G,1)$ if it is connected, $\pi_1(X,x) \simeq G$ for any $x \in X$, and $\pi_n(X,x) = 0$ for any $x \in X, n > 0$.
More generally, it is a $K(G,n)$ if its only nontrivial homotopy group is $\pi_n(X,x) \simeq G$.

Thus the above discussion establishes :

Proposition 4. *Any two $K(G,1)$'s have the same singular homology, and it coincides with the group homology of G with coefficients in the trivial G-module $\mathbb{Z}$.*

Of course this works equally well if we replace $\mathbb{Z}$ by any trivial G-module M.
We can also get a topological interpretation for more general coefficient modules using local coefficient systems :

Definition 4. Let X be a space. A local coefficient system of abelian groups on X is a functor $\Pi_1(X) \to \mathbf{Ab}$.

Examples 2.

- The constant functor with value a fixed abelian group A
- Let M be an n-dimensional manifold, then let $\tilde{M}$ be the orientation covering (with fiber $H_n(M, M \setminus \{x\})$). Then for every x, the fiber over x is an abelian group, and every path in M can be lifted to a path in $\tilde{M}$ which defines a map on the fibers which depends only on the homotopy class of the original path. One easily checks that this induced map is a morphism.
- The above example can be generalized to locally constant sheaves of abelian groups (seen as étale spaces where the fibers are abelian groups) on a nice space

For a discussion of local systems and their homology, see for instance [5] (where they are introduced not as functors as we did, but the definition given there is easily seen to be equivalent). Then we have :

Proposition 5. *If M is a G-module, then by composing $\Pi_1(K(G,1)) \to \pi_1(K(G,1),*) \to G \to$ $\mathbf{Ab}$ where the first arrow from the left is any retraction, the second a fixed isomorphism, and the last corresponds to M, then we get a local coefficient system $\tilde{M}$ and $H_n(K(G,1), \tilde{M}) \simeq H_n(G, M)$.*

The proof is essentially the same.
To give similar results about group cohomology, one must use an alternate characterization.
Indeed, $H^n(G, -)$ are the derived functors of $(-)^G$. But in this category, $(-)^G$ is naturally isomorphic to $\hom_G(\mathbb{Z}, -)$ where $\mathbb{Z}$ is given the trivial G action. However, if we fix M and look at $\hom_G(-, M)$, we may derive this functor and evaluate it in $\mathbb{Z}$. It is then reasonable to ask if the two resulting groups are isomorphic.
They are, and more generally over a ring R they are denoted $\mathrm{Ext}^n_R(A, B)$. We will give a sketch of a proof later, using spectral sequences.
Accepting that fact, we see that $H^n(G, M)$ is actually $\mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, M)$, and so one can compute it by taking a projective resolution of $\mathbb{Z}$, taking $\hom_G(-, M)$, and then taking cohomology of the resulting complex. This allows us to dualize the previous discussion. In particular, similar arguments yield :

Proposition 6. *Let X be a $K(G,1)$, M a G-module. Then $H^n(G, M) \simeq H^n(K(G,1), \tilde{M})$*

These topological considerations can sometimes help us. For instance, S^1 is a $K(\mathbb{Z}, 1)$, therefore for trivial $\mathbb{Z}$-modules M, we have have $H^k(\mathbb{Z}, M) = H^k(S^1, M) = M$ if $k = 0, 1$, 0 else.
Similarly, we know that $\mathbb{R}P^\infty$ is a $K(\mathbb{Z}/2, 1)$, and so we can use it to compute $H^k(\mathbb{Z}/2, M)$, etc.
These considerations will also allow us to get more structure on (co)homology automatically from the topological structure that we already know. We could of course derive these purely algebraically, but it would essentially amount to doing the same work.
In particular we have a cup-product on cohomology : $H^p(G, M) \otimes H^q(G, N) \to H^{p+q}(G, M \otimes N)$.
When R is a ring, we get by composing with the multiplication $R \otimes R \to R$ a graded commutative graded ring $H^*(G, R)$; and for an R-module M we see that $H^*(G, M)$ is a graded $H^*(G, R)$-module.
These cup-products can be defined algebraically too, as we mentioned, using diagonal approximations for projective resolutions. We will use this fact for some computations.
When we talk about spectral sequences we will see that the Serre spectral sequence also has an interpretation in terms of group (co)homology.
We will not use this, but this topological interpretation also makes our cohomology ring $H^*(G, \mathbb{F}_p)$ a module over the Steenrod algebra (see [6] for the $p = 2$ case for instance, or [7] for an algebraic treatment of Steenrod algebras).

2.4 Cohomology of finite groups and applications

In this section we'll specialize to finite groups and give the first applications of group cohomology. It all starts with interpreting the low-dimensional cohomology groups : H^0, H^1 and H^2.
H^0 is easy : by the very definition, $H^0(G, M) = M^G$ is the invariants of M. We won't say much more about it.
To interpret H^1, H^2 we could use the standard "bar resolution" of $\mathbb{Z}$, but we feel that this wouldn't be motivated enough at this point; so we try to give a more natural way to arrive at the result.
A very important thing is that we have a forgetful functor $\mathrm{Res}_1^G : G-\mathbf{Mod} \to \mathbf{Ab}$ which is exact, and it has a right-adjoint called co-induction, denoted CoInd_H^G, which can be defined as $\mathrm{CoInd}_1^G M := \hom(\mathbb{Z}[G], M)$ with G-action given by $g \cdot f(x) = f(xg)$.

The adjunction formula $\hom_G(A, \mathrm{CoInd}_1^G B) \cong \hom(\mathrm{Res}_1^G A, B)$ together with the exactness of Res_1^G tell us that $H^k(G, \mathrm{CoInd}_1^G B) \cong H^k(1, B)$ (where 1 is the trivial group), so it's 0 for $k > 0$. This, together with the injectivity of the unit of the adjunction tells us that given a G-module A we have a short exact sequence $0 \to A \to \hom_{\mathbb{Z}}(\mathbb{Z}[G], A) \to B \to 0$ where the middle term is *acyclic* (i.e. has no higher cohomology) and the right term is just the cokernel. A quick computation with the long exact sequence in cohomology shows us that $H^1(G, A) \simeq H^0(G, B) = B^G$. We only have to interpret B^G now. But B is an explicit cokernel, and so we can explicitly see what its invariants are. The conclusion is that B^G is essentially the set of derivations $d : G \to A$, that is map d such that for all g, h, $d(gh) = d(g) + g \cdot d(h)$ (to understand the terminology, imagine that A has a trivial right G-action, then this becomes $d(gh) = d(g)h + gd(h)$ which is the Leibniz formula), modulo principal derivations, i.e. derivations of the form $g \mapsto g \cdot a - a$.
There are two consequences to that (that are related) : the first is that when A is a trivial G-module, this becomes $d(gh) = d(g) + d(h)$, so we're actually dealing with a group morphism $G \to A$ (or even, $G^{ab} \to A$ as A is abelian).
The second is that the action of G on A induces a semi-direct product $A \rtimes G$; and a derivation $G \to A$ corresponds precisely to a splitting $G \to A \rtimes G$ as is easily checked; and two derivations are equal modulo principal derivations if and only if the associated splittings are A-conjugate . In summary,

Proposition 7. *$H^1(G, A)$ is in bijection with the set of splittings of $A \rtimes G \to G$ modulo A-conjugation.*

Similar considerations (with perhaps the need to introduce the first steps of the bar resolution) show that H^2 has a similar interpretation :

Proposition 8. *$H^2(G, A)$ is in bijection with equivalence classes of extensions $0 \to A \to E \to G \to 1$; where two extensions are equivalent if there is a commutative diagram :*

$$\begin{array}{ccccc} A & \longrightarrow & E & \longrightarrow & G \\ \| & & \downarrow & & \| \\ A & \longrightarrow & E' & \longrightarrow & G \end{array}$$

We will use this and some specific properties of finite groups later to prove a special case of the Schur-Zassenhaus theorem (special case where A is abelian).
Accepting the fact that over a field k, with G a finite group, a $k[G]$-module is injective if and only if it is projective, then we can already give one application of this interpretation of H^2 (the fact that we accept is fairly standard and follows from abstract nonsensical considerations, together with the fact that a k-module is always projective and injective) :

Proposition 9. *There are no projective objects in* **FinGrp***, the category of finite groups, except for* 1.

Proof. Suppose G is projective in **FinGrp**; then any extension $0 \to A \to E \to G \to 1$ splits. Fixing an action of G on A, any two extensions of G by A that induce said action, and that are split are equivalent - therefore for any finite G-module A, $H^2(G, A) = 0$.
We will prove that this isn't possible unless G is trivial. If it isn't, let p be a prime number dividing $|G|$, and we will work over $\mathbb{F}_p$. Let $\mathbb{F}_p[G] \to \mathbb{F}_p$ be defined by $g \mapsto 1$ and consider the short exact sequence of G-modules $0 \to N \to \mathbb{F}_p[G] \to \mathbb{F}_p \to 0$.

Note that if $a \in \mathbb{F}_p[G]$ is G-invariant, then it is of the form $kP, k \in \mathbb{F}_p$ and $P = \sum_{g \in G} g$. Then kP is sent to $k|G| = 0 \in \mathbb{F}_p$, so that the induced morphism $H^0(G, \mathbb{F}_p[G]) \to H^0(G, \mathbb{F}_p)$ is zero. In particular, the long exact sequence in cohomology tells us that $H^0(G, \mathbb{F}_p)$ injects into $H^1(G, N)$: the latter must be nonzero.
Finally let $0 \to M \to P \to N \to 0$ be a short exact sequence where P is finite and projective as an $\mathbb{F}_p[G]$-module. It is therefore also injective as an $\mathbb{F}_p[G]$-module, and so its cohomology vanishes in positive dimension. From this it follows that $H^1(G, N) \to H^2(G, M)$ is an isomorphism, so that $H^2(G, M) \neq 0$, and we are done. □

Remark 1. We have used that the cohomology of an injective $\mathbb{F}_p[G]$-module vanishes in positive dimensions. Although we only introduced everything with $\mathbb{Z}$-coefficients, it is not hard to see that it also works with $\mathbb{F}_p$-coefficients when the module is an $\mathbb{F}_p$-module, and that the cohomologies are the same.

Let's now give the main ingredient for our special case of the Schur-Zassenhaus theorem :

Proposition 10. *Let R be a PID, G a finite group, A, B left RG-modules and C a right RG-module.*
If A is free as an R-module, then for $n \geq 1$, $|G|$ anihilates $\mathrm{Ext}^n_{RG}(A, B)$ and $\mathrm{Tor}^{RG}_n(C, A)$.
The same conclusion holds for $n \geq 2$ if A is not assumed free.

Sketch of proof. The proof is fairly easy and consists in noting that a projective resolution over RG is in particular a projective resolution over R if we forget the G-action, then of course these homology groups vanish over R as A is free; and then one can link the homology groups over RG to those over R by averaging over G, which explains the factor $|G|$ we get .
When A is not free, the homology groups still vanish for $n \geq 2$ because R is a PID, so the same reasoning applies, and we get annihilation for $n \geq 2$. □

In particular, $H_n(G, M)$ and $H^n(G, M)$, which are Tor and Ext groups over $\mathbb{Z}[G]$ with $\mathbb{Z}$ fall into the scope of this result, and are therefore annihilated by $|G|$ (there is another proof of this fact using the transfer map).
Even more, if $|G| : M \to M$ is an isomorphism, then it is one on (co)homology, which implies at once $H_n(G, M) = 0 = H^n(G, M)$. This is the main ingredient of the Schur-Zassenhaus theorem.

Proposition 11. *Suppose $0 \to M \to E \to G \to 1$ is an extension of finite groups, with M abelian; and $|G| \wedge |M| = 1$. Then the short exact sequence splits, $E \simeq M \rtimes G$ and any two subgroups of E of size $|G|$ are M-conjugate.*

Proof. By the hypothesis on the cardinals, any subgroup of E of size $|G|$ is a complement of M in E, therefore those subgroups correspond to splittings of the extension.
The result then follows from $H^1(G, M)$ and $H^2(G, M)$ being trivial, which they are because $|G|$ is invertible in M (because of the hypothesis on cardinals); and from our interpretations of H^1, H^2. □

So far we have a lot of theory but very little actual computations of group (co)homology. We will now give computations for $G = \mathbb{Z}/n\mathbb{Z}$ and use the Künneth formula to get our computations for all finitely generated abelian groups. The more complicated computations will need more tools, like spectral sequences, which we will develop later.

What makes the computation for $G = \mathbb{Z}/n\mathbb{Z}$ work is that we have a nice $\mathbb{Z}[G]$-projective resolution of $\mathbb{Z}$: indeed it only has $\mathbb{Z}[G]$ at each stage. Let $G = \langle g \rangle$, and let $\epsilon : \mathbb{Z}[G] \to \mathbb{Z}, g \mapsto 1$. Then the kernel of ϵ is the augmentation ideal $I(G)$ which one may check is generated by $g - 1$; and one may check that the annihilator of $g - 1$ is generated by $P = \sum_{0 \leq k < n} g^k$ which acts as ϵ so that the following sequence is exact :

$$\ldots \longrightarrow \mathbb{Z}[G] \xrightarrow{d_1} \mathbb{Z}[G] \xrightarrow{d_2} \mathbb{Z}[G] \xrightarrow{d_1} \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z}$$

where d_1 is multiplication by $g - 1$, d_2 by P. This resolution is very nicely behaved and we can also use it to get the multiplicative structure on $H^*(G, \mathbb{Z})$. This resolution also works with any coefficient module, which is another reason it's so interesting. The computation is now easy and we get

Proposition 12. *Let $\mathbb{Z}$ be the trivial $\mathbb{Z}/n\mathbb{Z}$-module. Then* $H^k(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & k = 0 \\ \mathbb{Z}/n\mathbb{Z}, & k > 0 \text{ even} \\ 0, & k \text{ odd} \end{cases}$

More generally, if M is a trivial $\mathbb{Z}/n\mathbb{Z}$-module, $H^k(\mathbb{Z}/n\mathbb{Z}, M) = \begin{cases} M, & k = 0 \\ M/nM, & k > 0 \text{ even} \\ \ker(M \xrightarrow{n} M), & k \text{ odd} \end{cases}$

In particular we see that over a finite field $\mathbb{F}_p$, the cohomology will depend on whether $p \mid n$ or not : this is to be expected given the annihilation result we proved earlier.
Now our resolution, together with a diagonal approximation, can be used to compute the cup-product. We get the following results :

Proposition 13.

1. *$H^*(\mathbb{Z}/n\mathbb{Z}, Z) = \mathbb{Z}[x]/(nx), |x| = 2$*
2. *If p is prime and $p \nmid n$, k a field of characteristic p, $H^*(\mathbb{Z}/n\mathbb{Z}, k) = k$*
3. *If $2 \mid n$ and k is a field of characteristic 2, $H^*(\mathbb{Z}/n\mathbb{Z}, k) = k[x], |x| = 1$*
4. *If $p \mid n$ is an odd prime, k a field of characteristic p, $H^*(\mathbb{Z}/n\mathbb{Z}, k) = k[x, y]/(x^2)$, $|x| = 1, |y| = 2$*

Note that these are not actual polynomial rings, these are graded commutative graded rings, $| - |$ denotes the degree of the generators.

Moreover, given our topological interpretation, we have a Künneth theorem which works as well here, and so gives us the cohomology of any finitely generated abelian group. (We focused on cohomology here, but of course this works equally well for homology).
For this example we used a nicely behaved resolution of $\mathbb{Z}$, but for more complicated groups, this is not so easy to find. This shows we need more tools if we want to compute some more cohomologies. Spectral sequences will come in handy.

3 Spectral sequences

3.1 Definition, first example: the Serre spectral sequence

One can see a spectral sequence as a "computational process" that computes successive approximations of a group (or R-module) we're trying to compute, most of the time a (co)homology group we're interested in. There are homological and cohomological versions of spectral sequences. For our purposes we'll focus on the cohomological version.

Definition 5. A spectral sequence is a sequence $\{E_r\}_{r \geq 1}$ of "pages" together with differentials $\{d_r\}_r$ where each E_r is a bigraded module $\{E_r^{p,q}\}_{p,q \in \mathbb{Z}}$, and d_r is a differential on this bigraded module, of bidegree $(r, 1-r)$ such that $E_{r+1}^{p,q}$ is the (p, q)th homology of E_r. More precisely, we have $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q+1-r}$ and $d_r^{p-r,q+r-1} : E_r^{p-r,q+r-1} \to E_r^{p,q}$, their composite is 0 and we require that $E_{r+1}^{p,q} = \ker(d_r^{p,q})/\text{im}(d_r^{p-r,q+r-1})$

Note that in this definition, the equality between the $r+1$-page and the homology of the r-page can be understood as a strict equality or as a "they are isomorphic, with a fixed isomorphism that is part of the data".
In our presentation, we will only encounter so-called *first quadrant* spectral sequences, that is, spectral sequences with $E_r^{p,q} = 0$ if $p < 0$ or $q < 0$. Thus we will define our concepts accordingly for simplicity, although there are more general definitions. In particular :

Definition 6. Let $\{E_r\}$ be a first quadrant spectral sequence. Then for fixed p, q there is r_0 such that all differentials coming in or out of $E_r^{p,q}, r \geq r_0$ are 0, so that for $r \geq r_0$, $E_r^{p,q} = E_{r+1}^{p,q}$ (= here means canonical isomorphism). We let $E_\infty^{p,q}$ denote this common eventual value.

The main definition is then :

Definition 7. Let H^* be a graded module, and $F^\bullet$ a filtration on H^*, that is for each n, $0 \subset ... \subset F^{p+1}H^n \subset F^pH^n \subset ... \subset F^0H^n \subset H^n$.
Let $\{E_r\}$ be a first quadrant spectral sequence. One says that it converges to H^*, with the filtration $F^\bullet$ if for all p, q, $E_\infty^{p,q} \simeq F^pH^{p+q}/F^{p+1}H^{p+q}$.

In the most interesting situations, the filtration will be *exhaustive* and Hausdorff, that is $F^0H^n = H^n$ and $\bigcap_k F^kH^n = 0$ (often stopping at a finite stage, that is, very often $F^{n+1}H^n = 0$). There are some interesting theorems of the form "given such data, there is a spectral sequence with a given E_2-page, and it converges to some object" - they can be used to study things about the E_2-page, or to study the object towards which it converges.
Much like for long exact sequences in cohomology where the connecting morphism isn't easy to understand, even if it is constructed explicitly, here the differentials, although constructed explicitly, are not very easy to understand, and so one often tries to gain information about them by using the information we have about the spectral sequence, such as functoriality, or multiplicativity.
The first example is the very famous Serre spectral sequence, which we will later adapt to group cohomology :

Theorem 1. *Let $F \to E \to B$ be a Serre fibration sequence, and let M be an abelian group. Let $\underline{H^q(F;M)}$ denote the local coefficient system on B that is constant on objects, and that behaves as expected on paths.*
Then there is a spectral sequence, called the Serre spectral sequence, with E_2 page given by $E_2^{p,q} = H^p(B; \underline{H^q(F;M)})$ and it converges to $H^*(E;M)$ and the filtration on this group satisfies $F^0H^n = H^n, F^{n+1}H^n = 0$ for all n.
In the future we will write this more succinctly as "there is a spectral sequence with

$$E_2^{p,q} = H^p(B; \underline{H^q(F;M)}) \implies H^{p+q}(E;M)$$

".
Moreover, this spectral sequence is functorial in the fibration (where a morphism of fibration is the obvious thing, and so is a morphism of spectral sequences), and it is multiplicative in the following sense :

1. *There is a multiplication $E_r^{p,q} \otimes E_r^{s,t} \to E_r^{p+s,q+t}$ on each page*
2. *d_r is a derivation for this multiplication, that is $d_r(xy) = d_r(x)y + (-1)^{|x|}xd_r(y)$ for homogeneous elements, where $x \in E_r^{p,q}$ and $|x| = p+q$ is the total degree*
3. *The fact that d_r is a derivation means that the multiplication on E_r induces a multiplication on its homology, which is E_{r+1} : this induced multiplication is the multiplication on E_{r+1}*
4. *If $M = R$ is a ring, and $\pi_1(B)$ acts trivially on the cohomology of F; then in the E_2-page this multiplication is, up to a sign on homogeneous elements, the composition of cup-products : $H^p(B; H^q(F;R)) \otimes H^s(B; H^t(F;R)) \to H^{p+s}(B; H^q(F;R) \otimes H^t(F;R)) \to H^{p+s}(B; H^{q+t}(F;R))$*
5. *Still under the assumptions of 4, the filtration on $H^n(E;R)$ is compatible with the cup-product, i.e. the multiplication restricted to $F^pH^n \otimes F^sH^m$ takes values in $F^{p+s}H^{n+m}$, so that it induces multiplications on the filtration quotients F^p/F^{p+1} : this multiplication coincides with the one on E_∞*

The homological version of this spectral sequence is much less rich, as there is no multiplicative structure. This multiplicative structure is very useful for computations, for instance it allows one to compute the cohomology ring of $\mathbb{C}P^n$ or $\mathbb{C}P^\infty$.
There are other results about spectral sequences, like a spectral sequence associated to a filtered complex or a double complex. These can be quite handy, for instance they allow one to prove the result that Ext can be computed as a derived functor in one variable or the other, or similarly for Tor. We include the theorem for double cochain complexes and the proof for Ext for completeness, the rest being similar : we have double chain complexes and homological spectral sequences.

Definition 8. A double cochain complex (K, d, d') is the data of : a bigraded module $K^{p,q}$; where d is a differential on $K^{\bullet,p}$ for a fixed p and d' on $K^{p,\bullet}$ for a fixed p, such that $dd' + d'd = 0$ (there is only one way to put the right indices on d, d' to make this sensible - moreover we use cohomological notations here so d, d' raise the degree).
The total complex of (K, d, d') has $\mathrm{Tot}(K)^n = \bigoplus_{p+q=n} K^{p,q}$ and $\delta = d + d'$.
It has two filtrations, ${}^IF^j\mathrm{Tot}(K)^n = \bigoplus_{p+q=n, p\geq j} K^{p,q}$, ${}^{II}F^j\mathrm{Tot}(K)^n = \bigoplus_{p+q=n, q\geq j} K^{p,q}$

Because we've only defined convergence for first quadrant spectral sequences, we have to add an unnecessary hypothesis to the following theorem

Proposition 14. *Let (K, d, d') be a double cochain complex, concentrated in nonnegative bidegrees. Then there are spectral sequences $\{^I E_r\}$ and $\{^{II} E_r\}$ with the following properties :*

1. *$^I E_1^{p,q} = H^q(K^{p,\bullet})$ and $^{II} E_1^{p,q} = H^q(K^{\bullet,p})$ with the obvious differentials (the spectral sequences are therefore first quadrant)*

2. *They both converge to $H^*(\mathrm{Tot}(K))$, $\{^I E_r\}$ with the filtration induced by $^I F$, $\{^{II} E_r\}$ with the filtration induced by $^{II} F$.*

We omit the proof of this fact. One can use exact couples for this, or dive into computations with a lot of indices. One may refer to [5] or [9] (the latter being more complete, the former more introductory) for more details. In any case, this is enough to prove that Ext can be computed with a projective resolution of the first argument, or an injective resolution of the second (or both !). We outline the proof :

Sketch of proof. Given two modules M, N, let $P_\bullet \to M$ be a projective resolution, $N \to I^\bullet$ an injective resolution. Let $K^{p,q} = \hom(P_p, I^q)$. We can clearly use the differentials of our resolutions to make this a nonnegatively graded double cochain complex (there are some signs to take care of). We therefore have two spectral sequences, $^I E$ and $^{II} E$.
Since each P_p is projective, $\hom(P_p, -)$ is exact and so taking homology with respect to the second variable is preserved by $\hom(P_p, -)$. Therefore $^I E_1^{p,q} = \hom(P_p, N)$ if $q = 0$, 0 else.
Similarly, the injectivity of the I^q ensures that $^{II} E_1^{p,q} = \hom(M, I^p)$ if $q = 0$, 0 else.
It follows that $^I E_2^{p,q}$ is Ext computed with a projective resolution, $^{II} E_2^{p,q}$ Ext computed with an injective resolution. Moreover, there are no nonzero differentials in E_2 or higher pages for degree reasons: one then easily checks that the definition of convergence and the fact that these E_2 pages are concentrated along one line imply that the two results coincide with the cohomology of the total complex, and therefore with eachother. □

This result therefore justifies using a projective resolution of $\mathbb{Z}$ (or whatever ring we're interested in) to compute $H^n(G, -)$!
We have introduced the Serre spectral sequence and given one example to show the usefulness of spectral sequences, now we will move on and see how this applies to group cohomology.

3.2 The Lyndon-Hochschild-Serre spectral sequence

Suppose $G \to Q$ is a surjective group morphism, with kernel N. Then we can in fact find a $K(G, 1)$, a $K(Q, 1)$ and a map $f : K(G, 1) \to K(Q, 1)$ that induces the given surjection on π_1. Then if we replace it by a fibration, it is easy to see from the long exact sequence of homotopy groups that the homotopy fiber is a $K(N, 1)$, so we have a fibration sequence $K(N, 1) \to K(G, 1) \to K(Q, 1)$ associated to any extension $1 \to N \to G \to Q \to 1$.
Let M be a trivial G-module, then we can consider $H^n(G, M), H^q(N, M)$. Moreover, there's something we haven't explicitly stated in our definition of group (co)homology, although it is not hard to see, it's that it is functorial in the group : covariantly for homology, contravariantly for cohomology. This simply comes from the fact that if $G \to H$ is a morphism and L an H-module, then it is also a G-module by pulling back along the morphism $G \to H$.
Now our extension induces an action of Q on N by outer automorphisms, and therefore by functoriality on $H^q(N, M)$ (there's a $^{-1}$ to put somewhere to get an action of the correct handedness, but that's not a problem) - for this we only need to check that N acts trivially on $H^q(N, M)$ by conjugation, which isn't too hard to see by expliciting the functoriality in N.
It is not a priori clear that this action should be the same as the action of $\pi_1(K(Q, 1))$ on $H^q(K(N, 1); M)$ induced by the given fibration, but it is the case, and it comes up in a few computations (to see it, one can for instance use explicit simplicial models for our $K(-, 1)$'s). In any case, the Serre spectral sequence now specializes to :

Theorem 2. *Given an extension of groups $1 \to N \to G \to Q \to 1$ and a trivial G-module M, there is a spectral sequence, called the Lyndon-Hochschild-Serre spectral sequence (LHSSS):*

$$E_2^{p,q} = H^p(Q, H^q(N, M)) \implies H^{p+q}(G, M)$$

The filtration on $H^n(G, M)$ is exhaustive and Hausdorff, and the LHSSS is multiplicative.

This simply comes from the Serre spectral sequence applied to our fibration $K(N, 1) \to K(G, 1) \to K(Q, 1)$ and our comments on the induced action.
There is of course a similar statement for group homology.
This spectral sequence tells us that if we're very good with spectral sequence, then a plan of attack to compute the cohomology of a group G is to decompose G into smaller pieces via extensions and compute their cohomologies.
Moreover, since the multiplicative structure on group cohomology coincides with the topological one, this LHSSS will also be multiplicative, in the same sense as the one described for the SSS, which can be very helpful for computations. And for the same type of reasons, this spectral sequence is also functorial in the extension.

We do not go into the details of this calculation, but using functoriality and multiplicativity of this spectral sequence, together with our knowledge of the cohomology of cyclic groups, we can compute the cohomology of the dihedral group D_8 with $\mathbb{F}_2$-coefficients (which is the only finite field that is interesting, for characteristic reasons):

Proposition 15. $H^*(D_8, \mathbb{F}_2) = \mathbb{F}_2[x, y, z]/(xy), |x| = 1, |y| = 1, |z| = 2$
It sits inside $H^*(V_4, \mathbb{F}_2) = \mathbb{F}_2[x, y], |x| = 1, |y| = 1$ *via the restriction map induced functorially by* $V_4 \hookrightarrow D_8$*, the map on rings being* $x \mapsto 0, y \mapsto x, z \mapsto y^2$

Note that here, being in characteristic 2, graded commutative implies actually commutative so these are actual polynomial rings.
It is also a bit of a lie to say that this uses only multiplicativity and functoriality, as our proof also relied on analyzing the Bockstein morphism and using the Kudo transgression theorem; but we do not wish to expand on these, at least for the time being.

3.3 The Cartan-Leray spectral sequence

This is one more spectral sequence that one can either derive algebraically or from the Serre spectral sequence. For convenience, we will use the latter option.
Given a (discrete, for our purposes) group G, there is a contractible space EG on which G acts freely. One can give a fairly easy-to-write-down simplicial model for it, but mainly we will use it for other constructions. Let X be a space on which G acts, then $X \times EG$ is homotopy equivalent to X, and G acts (diagonally) freely on it : we've made the action nicer without changing the homotopy type.

Next, consider the quotient of this new space by the action of G, denoted X_{hG} or $X \times_G EG$ (although it has nothing to do with a pullback, so the notation is a bit unfortunate). Then there are two projections one can write down : $X_{hG} \to EG/G$ and $X_{hG} \to X/G$.
Note that by covering theory, $BG := EG/G$ is a $K(G, 1)$. Moreover, $X_{hG} \to EG/G$ is a fibration with fiber X, so we have a fiber sequence $X \to X_{hG} \to K(G, 1)$, which gives us a spectral sequence $E_2^{p,q} = H^p(K(G, 1); H^q(X)) \implies H^{p+q}(X_{hG})$, which can be rewritten as $E_2^{p,q} = H^p(G, H^q(X)) \implies H^{p+q}(X_{hG})$.
It's not clear a priori that the $\pi_1 K(G, 1)$-action on $H^q(X)$ is the same as the G action on $H^q(X)$ inherited from the action of G on X, but it is the case. $H^q(X_{hG})$ is called the equivariant cohomology of X, X_{hG} being a "homotopical approximation" to X/G (it is the homotopy quotient).

When G acts freely on X, the projection $X_{hG} \to X/G$ is actually a homotopy equivalence, and therefore in this situation we get :

Theorem 3. *Let* X *be a space and* G *a (discrete) group acting freely on* X*. Then there is a spectral sequence, called the Cartan-Leray spectral sequence (CLSS) with*

$$E_2^{p,q} = H^p(G, H^q(X)) \implies H^{p+q}(X/G)$$

As usual, the filtration on $H^n(X/G)$ *is exhaustive and Hausdorff, and the CLSS is multiplicative.*

This spectral sequence is very useful when X has a particularly nice homology, as we will see in the next sections.

3.4 Computing the cohomology of S^1 inefficiently

We finally have enough tools to compute the cohomology of S^1 by correcting the mistake in the motivation section of group cohomology.
We'll simply use singular cohomology instead of de Rham cohomology, but de Rham's theorem ensures that this is the same.
For the action of $\mathbb{Z}$ on $\mathbb{R}$, the CLSS with $\mathbb{R}$-coefficients becomes $E_2^{p,q} = H^p(\mathbb{Z}, H^q(\mathbb{R};\mathbb{R})) \implies H^{p+q}(S^1;\mathbb{R})$.
But $H^q(\mathbb{R};\mathbb{R}) = 0$ for $q > 0$, so the spectral sequence is concentrated in $q = 0$, so there are no differentials and extension problems. Moreover, $H^0(\mathbb{R};\mathbb{R}) = \mathbb{R}$ and $\mathbb{Z}$ acts trivially on it. So we're looking at $H^p(\mathbb{Z}, \mathbb{R})$ with trivial action. In degree 0, this is just the invariants, so $\mathbb{R}$; in degree 1 this is $\hom(\mathbb{Z}, \mathbb{R}) \simeq \mathbb{R}$. We don't need to look at higher dimensions as S^1 is a 1-dimensional CW-complex (or manifold), and so we have the cohomology of S^1.
Of course this is a very inefficient method of computing this cohomology, but it might not be as inefficient if we're interested in a manifold M with an appropriate action of group G and are trying to compute $H^*(M/G)$: group cohomology tells us how far this is from the naive $H^*(M)^G$.

3.5 More computations, some applications : free actions on a sphere

We can use the CLSS to gain information on spaces and/or on groups. Here we give some examples of this with free actions on a sphere.
Indeed if G acts freely on the sphere S^n, the E_2-page of the CLSS looks very nice : $H^p(G, H^q(S^n))$ is concentrated on two lines, $q = 0, n$. Moreover, if G acts sufficiently nicely (e.g. if G is finite), we have some information on S^n/G, and therefore we can gain information about $H^p(G)$.
Some basic algebraic topology shows that when n is even, G can only be $\mathbb{Z}/2\mathbb{Z}$, but when n is odd, much more than that can happen, but we can still give some restrictions. In fact we have :

Proposition 16. *Let G be a finite group acting freely on S^n (n odd) and k a commutative ring. Then there is $x \in H^{n+1}(G, k)$ such that cup-product with x induces an isomorphism $H^p(G, k) \to H^{p+n+1}(G, k)$ for $p > 0$ and a surjection for $p = 0$.*
Moreover, for $p < n$, $H^p(G, k) \simeq H^p(S^n/G, k)$, and there is a short exact sequence

$$0 \to H^n(G, k) \to H^n(S^n/G, k) \to \ker(x) \to 0$$

where $\ker(x)$ *denotes the kernel of multiplication by x from $H^0 \to H^{n+1}$*

This follows from studying the associated CLSS and using the fact that S^n/G will be an n-dimensional manifold (or CW-complex).
In particular this restricts the class of groups that can act freely on a sphere (or some n-dimensional CW-complex/manifold homotopy equivalent to the sphere, as the proof would be the same), and [10] provides a partial converse to this restriction.
The first application of this is :

Proposition 17. *Let G be a finite abelian group acting freely on the sphere S^n. Then G is cyclic. In particular, if G is a finite group acting freely on S^n, any abelian subgroup is cyclic.*

Outline . If G is finite abelian it can be written as a product of cyclic groups of prime powers. If it is not itself cyclic then for some prime p, it has a direct summand $\mathbb{Z}/p^k\mathbb{Z} \times \mathbb{Z}/p^l\mathbb{Z}$ with $l, k > 0$. We then compute the cohomology of G with coefficients in $\mathbb{F}_p$ with the Künneth isomorphism and check that all summands of the form $\mathbb{Z}/q^r\mathbb{Z}, q \neq p$ don't come into play, only the ones that are powers of p do.
It is then easy to see (still with the Künneth formula, and our computation for cohomology of cyclic groups) that if there are two or more summands, the cohomology of G is not periodic, and thus G cannot act freely on a sphere. Therefore there's only one summand for each prime p, so G is cyclic. □

The second application will be a new computation of the cohomology of a group : the quaternions Q_8.
Indeed Q_8 sits inside the quaternion algebra that is, as a topological space, just $\mathbb{R}^4$; more precisely it sits inside $S^3 \subset \mathbb{R}^4$ and acts multiplicatively on it; this action is clearly continuous and free.
This action is orientation preserving, so that S^3/Q_8 is orientable, and so Poincaré duality together

with the interpretation of H^1 allow us to compute low-degree cohomology groups for S^3/Q_8, and therefore by the proposition for Q_8. This works for both $\mathbb{Z}, \mathbb{F}_2$-coefficients.
Working a bit more with this spectral sequence and with some dimension-shifting we get all the lower cohomology groups : H^0 through H^4, both with $\mathbb{Z}, \mathbb{F}_2$-coefficients. Precisely,

Lemma 1. $H^p(Q_8, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & p = 0 \\ \mathbb{Z}/8\mathbb{Z}, & p > 0, 4 \mid p \\ 0, & p \text{ congruent to } 1,3 \text{ mod } 4 \\ (\mathbb{Z}/2\mathbb{Z})^2, & \text{else} \end{cases}$

$H^p(Q_8, \mathbb{F}_2) = \begin{cases} \mathbb{F}_2, & p = 0,3 \mod 4 \\ \mathbb{F}_2^2, & p = 1,2 \mod 4 \end{cases}$

Finally the multiplicativity of the CLSS and the analysis of the two Bockstein morphisms associated to $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{F}_2$ allow us to derive the multiplication on $H^*(Q_8, \mathbb{F}_2)$. Just like for D_8 we do not go into the details of this computation, we just give the end result :

Proposition 18. *$H^*(Q_8, \mathbb{F}_2) \simeq \mathbb{F}_2[x, y, \eta]/(xy + x^2 + y^2, x^2y + xy^2)$ with $|x| = |y| = 1, |\eta| = 4$*

One can take a look at [11] which does some computations for small 2-groups and introduces other computational tools and structures on the cohomology rings such as Steenrod operations and Massey products.

3.6 Other tools

There are of course many other tools in the stduy of group cohomology. One of them, very interesting, is the Eilenberg stable elements formula. It allows one to reduce the computation of $H^*(G, \mathbb{F}_p)$ to the computation of $H^*(P, \mathbb{F}_p)$ where P are the p-subgroups of G, together with the computation of some maps. It allows one for instance to compute the cohomology of the symmetric groups $\mathfrak{S}_3, \mathfrak{S}_4$ with $\mathbb{F}_2$ or $\mathbb{F}_3$ coefficients, based on our earlier computations for cyclic groups, and for D_8, Q_8.
We will not say more about this, except that this formula naturally brings about an interesting category, the category of nontrivial p-subgroups of G, and it encourages one to study limits over this category.
When studying these limits, some classifying spaces of categories or posets related to G also naturally appear, and this leads to very interesting mathematics. For more information on this, one may consult [12]; and for a seminal paper on homotopy properties of posets of p-subgroups of G, one may see [13] for instance.

3.7 Noetherianity

In this section, we give one complete proof. We only give our proof for an odd prime p but it can easily be adapted to $p = 2$. We rely on two results that we do not prove :

Lemma 2. *Let C_p be a cyclic group of order an odd prime p and k a field of characteristic p . Then*

$$H^*(C_p, k) \simeq k[y] \otimes_k \bigwedge_k [x]$$

where $|y| = 2, |x| = 1$ and $\beta(x) = y$ (β is the Bockstein morphism)

Theorem 4 (corollary of Kudo's transgression theorem)**.** *Let $0 \to H \to G \to Q \to 0$ be an extension of groups such that Q acts trivially on $H^*(H, \mathbb{F}_p)$.*
Let $x \in H^k(H, \mathbb{F}_p)$ that is transgressive, that is, $d_r(x) = 0$ for all $r \leq k$, where d_r is the differential of the LHSSS associated to the extension.
Then $\beta(x)$ is also transgressive, and so are all x^{p^k} (more generally, one may apply all Steenrod operations and stay transgressive)

The first result is a simple computation, the second is actually easier to prove than the whole transgression theorem of Kudo. Now the result we wish to prove is that for all finite p-groups G, $H^*(G, \mathbb{F}_p)$ is noetherian (we could also prove that it is finitely generated over $\mathbb{F}_p$). Our proof follows the appendix of [14].

Theorem 5. *Let G be a finite p-group. Then $H^*(G, \mathbb{F}_p)$ is noetherian.*

Remark 2. The proof works equally well over any field k of characteristic p, and in fact we can adapt it to see that $H^*(G, k)$ is finitely generated over k as a graded commutative algebra.
In what follows we will denote $H^*(K)$ cohomology with coefficients in $\mathbb{F}_p$ unless explicitly stated otherwise.

Proof. We proceed by induction on $|G|$. It is clear when $|G| = 1$ or p, by our computation of $H^*(C_p)$.
Let G be a finite p-group and assume the result holds for all p-groups of cardinality $< |G|$. Let $g \in Z(G)$ have order p (we can find such a G as $Z(G)$ is non-trivial, and by Cauchy's lemma), and consider $Q = G/\langle g \rangle$, and $\langle g \rangle \simeq C_p$. We then have a LHSSS of the form

$$E_2^{p,q} = H^p(Q, H^q(C_p)) \implies H^{p+q}(G)$$

Note that $H^q(C_p)$ is finitely generated and, by the choice of g, the action of Q is trivial on it, so that $E_2^{p,q} = H^p(Q) \otimes H^q(C_p)$.
Take the generator $x \in H^1(C_p) \simeq E_2^{0,1}$. For positional reasons, it is transgressive, therefore, by Kudo's transgression theorem, so is its Bockstein $y \in H^2(C_p) \simeq E_2^{0,2}$ and so are the powers of the latter : $y^{p^k} \in E_2^{0,2p^k}$, $k \geq 0$, i.e. $d_r(y^{p^k}) = 0$ for $r \leq 2p^k$.

Let A_r denote the graded ring $E_r^{*,0}$ (in particular $A_2 \simeq H^*(Q)$). Then, again for positional reasons, A_{r+1} is a quotient of A_r by a homogeneous ideal, so for all $r \geq 2$, A_r is a quotient of A_2 by a homogeneous ideal I_r.
$|Q| < |G|$ by choice of g, so the induction hypothesis applies and $H^*(Q)$ is noetherian, therefore so is A_2. But the sequence of ideals (I_r) is nondecreasing, therefore it must be constant after some rank, say r_0. This means that the map $A_r \to A_{r+1}$ is injective for $r \geq r_0$.
In particular, for $r \geq r_0$, any differential hitting A_r must be 0 : let k be big enough so that $2p^k + 1 \geq r_0$, then $d_{2p^k+1}(y^{p^k}) = 0$, and so for all r, $d_r(y^{p^k}) = 0$.
Let $z = y^{p^k}$. It follows that z survives until the E_∞-page. Moreover, by inspection, E_2 is a finitely-generated $A_2[z]$-module : it is generated by $\{1, x, y, xy, y^2, ..., y^{p^k-1}, xy^{p^k-1}\}$.
In particular, $A_2[z]$ is noetherian so E_2 is a noetherian module.

Let $B_r \subset E_2$ denote the sub-$A_2[z]$-module of those elements that become coboundaries in the E_r-page (it is a sub-$A_2[z]$-module because every $x \in A_2[z]$ has $d_k(x) = 0$ for all k). Then clearly (B_r) is a nondecreasing sequence of submodules, therefore it stabilizes.
This implies that there is some r_1 such that for $r \geq r_1$, $d_r = 0$, and so $E_r = E_\infty$.

E_2 being finitely generated over the noetherian ring $A_2[z]$ immediately implies by induction that each E_r is also finitely generated (as subquotients of E_2), and so $E_r = E_\infty$ implies that E_∞ is finitely generated as an $A_2[z]$-module.
z survived until the E_∞-page, and so it is the image under $H^*(G) \to H^*(C_p)$ of some $\overline{z}$.
Then $A_2[z]$ acts on E_∞ via the action of $A_2[\overline{z}]$ on $H^*(G)$. One may then check that the finite generation of E_∞ over $A_2[z]$ implies that of $H^*(G)$ over $A_2[\overline{z}]$, which is also noetherian, so in conclusion $H^*(G)$ is noetherian.
To check that, one uses that the filtration on H^* is finite. Indeed, let $x_1 ..., x_n$ be lifts of generators of E_∞ over $A_2[z]$, and let $x \in H^p(G)$. Then x belongs to some filtration slice, and so up to a lower filtration quotient, $x = \sum_i \lambda_i x_i$. By induction, the difference of the two is also generated by the x_i, because it is in a lower filtration slice, and so x is generated by the x_i's. □

We have a corollary of this :

Corollary 1. Let M be a finite-dimensional $\mathbb{F}_p$-vector space with a G-action (G finite p-group). Then $H^*(G, M)$ is finitely generated over $H^*(G)$

Proof. We prove the result by induction on the dimension of M. When M has dimension 1, the action must be trivial (G is a p-group, so the cardinality of the fixed point set of M is congruent to the cardinality of M modulo p, that is, it's 0, so the submodule of fixed points can't be reduced

to $\{0\}$), and so $H^*(G, M)$ is just $H^*(G)$ and the property is trivial.
Next, assume the result holds for all dimensions $< \dim_{\mathbb{F}_p} M$.
We find $M' \leq M$ such that the action of G on M/M' is trivial, and such that $\dim_{\mathbb{F}_p} M/M' = 1$. This can be done by induction, by noting that M^G is never trivial (same argument as above for the $\dim M = 1$ case), and so we may look at M/M^G to find such a submodule, and then $(M/M^G)/(M'/M^G) \simeq M/M'$). Having such a submodule M' we have a short exact sequence $0 \to M' \to M \to M/M' \to 0$ which yields a long exact sequence in cohomology.

The subgroup of $H^*(G, M/M')$ consisting of the image of $H^*(G, M)$ is clearly a sub-$H^*(G)$-module. $H^*(G, M/M')$ is finitely generated over $H^*(G)$ by the dim $= 1$ case, and the latter is noetherian by what we proved earlier, so that this submodule is finitely generated, say by $y_1, ..., y_n$. Find lifts $\overline{y_i}$ of these in $H^*(G, M)$.
By the induction hypothesis, $H^*(G, M')$ is finitely generated : let $x_1, ..., x_m$ be some generators and let $i_*(x_i)$ denote their images in $H^*(G, M)$. Then we claim that $H^*(G, M)$ is generated by $\{\overline{y_1}, ..., \overline{y_n}, i_*(x_1), ..., i_*(x_m)\}$ over $H^*(G)$. This is clear from the long exact sequence in cohomology.
□

4 Model categories - reinterpreting derived functors

In this section, we introduce some abstract homotopy theory, and explain how derived functors can be interpreted in a more general setting, and how the derived functors we defined earlier are just shadows of others that contain more "homotopical information".
We won't go into to many details; our basic framework is that of homotopical categories as presented in [15], and when specialized to model categories, we base our presentation on [16].
The point of view we will take here is that we have a category $\mathscr{C}$ and some class $\mathscr{W}$ of arrows that we want to consider to be isomorphisms, although they are not actually isomorphisms. This leads to the theory of localization. But many functors do not behave nicely with respect to this localization, because they don't see this class $\mathscr{W}$ as close enough to isomorphisms : this leads to the theory of derived functors.
Model structures are then there to "assist" in forming these derived functors : they provide means to ensure their existence, and actually compute them.
The first definition is :

Definition 9. Let $\mathscr{C}$ be a category, $\mathscr{W}$ a subcategory that contains all objects of $\mathscr{C}$ (we say it is a *wide* subcategory) and satsifies the 2-out-of-6 property : for any composable triple of arrows (f, g, h), if $g \circ f$ and $h \circ g$ are in $\mathscr{W}$, then so are $f, g, h, h \circ g \circ f$.
Then $(\mathscr{C}, \mathscr{W})$ is called a *homotopical category* and $\mathscr{W}$ is called its class of weak equivalences.

Some authors choose to only require the 2-out-of-3 property :

Definition 10. Let $\mathscr{C}$ be a category, $\mathscr{W}$ a subcategory; we say that it satisfies the 2-out-of-3 property if for all composable (f, g), we have that if any 2 of $f, g, g \circ f$ are in $\mathscr{W}$ then so is the third.

The 2-out-of-6 property implies the 2-out-of-3 property, but it is stronger. When model categories are defined, we will see that their definition only requires the 2-out-of-3 property, but actually the 2-out-of-6 property follows from the other axioms in that case.

Examples 3.

- For any category $\mathscr{C}$, the class $\mathscr{W}$ of all isomorphisms satisfies the 2-out-of-6 property
- By the above example, for any categories $\mathscr{C}, \mathscr{D}$ and any functor $F : \mathscr{C} \to \mathscr{D}$, the class $\mathscr{W}$ of arrows in $\mathscr{C}$ that gets sent to isomorphisms by F satisfies the 2-out-of-6 property. More generally, if $(\mathscr{D}, \mathscr{V})$ is a homotopical category, the class of arrows in $\mathscr{C}$ that get sent to $\mathscr{V}$ has the 2-out-of-6 property.
- In particular, we have that homotopy equivalences or weak equivalences of topological spaces, chain homotopy equivalences or quasi-isomorphisms of chain complexes form classes of weak equivalences on the appropriate categories

These last examples are the motivating examples.
When we have a homotopical category $(\mathscr{C}, \mathscr{W})$, we wish to invert the arrows in $\mathscr{W}$, in the best way possible. This leads to the following definition of localization (which is a "2-categorical universal property") :

Definition 11. Let $\gamma : \mathscr{C} \to \mathscr{C}[\mathscr{W}^{-1}]$ be a functor such that $\gamma(\mathscr{W})$ is included in the class of all isomorphisms of $\mathscr{C}[\mathscr{W}^{-1}]$. Then the couple $(\mathscr{C}[\mathscr{W}^{-1}], \gamma)$ is called a localization of $\mathscr{C}$ at $\mathscr{W}$ if it has the following property :
for any category $\mathscr{D}$, the pullback functor $\gamma^* : \mathrm{Fun}(\mathscr{C}[\mathscr{W}^{-1}], \mathscr{D}) \to \mathrm{Fun}(\mathscr{C}, \mathscr{D})$ is fully faithful, and has as essential image the subcategory of functors $F : \mathscr{C} \to \mathscr{D}$ such that for all $f \in \mathscr{W}$, $F(f)$ is an isomorphism.

Of course a localization, if it exists, is unique up to equivalence, and the equivalence is itself unique up to unique isomorphism. For small categories it is easy to construct the localization using zig-zags of morphisms; for large categories this can be a problem as we can go from a locally small category to a non-locally small one, and so it somehow rests on our set-theoretic foundations. Model categories will be homotopical categories with more structure, and this additional structure will ensure the existence of a locally small localization. For now, we won't worry too much about it, and state things conditional on the existence of the localization.

4.1 Derived functors, homotopically

Let $(\mathscr{C}, \mathscr{W}), (\mathscr{D}, \mathscr{V})$ be two homotopical categories, with localizations $\mathscr{C}[\mathscr{W}^{-1}]$ and $\mathscr{D}[\mathscr{V}^{-1}]$ (by a slight abuse of notation, we omit the name of the localizing functor) and $F : \mathscr{C} \to \mathscr{D}$ a functor between them. In very nice situations, F sends weak equivalences to weak equivalences (F is then called *homotopical*) and so the following diagram has a filler :

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{F} & \mathscr{D} \\ \downarrow & & \downarrow \\ \mathscr{C}[\mathscr{W}^{-1}] & \dashrightarrow & \mathscr{D}[\mathscr{V}^{-1}] \end{array}$$

But in practice, this rarely happens, and derived functors are a way to correct this. A total derived functor of F will be a functor between the localizations that approximates F in the "best possible way". This is formalized via Kan extensions (see [16] or [17]) of the appropriate handedness. In particular left derived functors sit on the left of F, so correspond to right Kan extensions, and right derived functors sit on the right of F so correspond to left Kan extensions (it can be a bit confusing)

Definition 12. Let $(\mathscr{C}, \mathscr{W}), (\mathscr{D}, \mathscr{V})$ be two homotopical categories with localizations $(\mathscr{C}[\mathscr{W}^{-1}], \gamma), (\mathscr{D}[\mathscr{V}^{-1}], \delta)$ and $F : \mathscr{C} \to \mathscr{D}$ a functor. A *total left derived functor* $\mathbf{L}F$ of F is a right Kan extension of $\delta \circ F$ along γ.
Explicitly, it means a functor $\mathbf{L}F : \mathscr{C}[\mathscr{W}^{-1}] \to \mathscr{D}[\mathscr{V}^{-1}]$ together with a natural transformation $\mathbf{L}F \circ \gamma \to \delta \circ F$ that is universal :

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{F} & \mathscr{D} \\ {\scriptstyle\gamma}\downarrow & \nearrow\!\!\!\!\nearrow & \downarrow{\scriptstyle\delta} \\ \mathscr{C}[\mathscr{W}^{-1}] & \xrightarrow[\mathbf{L}F]{} & \mathscr{D}[\mathscr{V}^{-1}] \end{array}$$

Dually, a *total right derived functor* $\mathbf{R}F$ of F is a left Kan extension of $\delta \circ F$ along γ. Explicitly, this is a functor $\mathbf{R}F : \mathscr{C}[\mathscr{W}^{-1}] \to \mathscr{D}[\mathscr{V}^{-1}]$ together with a natural transformation $\delta \circ F \to \mathbf{R}F \circ \gamma$ that is universal.

Definition 13. A *left derived functor* $\mathbb{L}F$ of F is a homotopical functor $\mathbb{L}F : \mathscr{C} \to \mathscr{D}$ together with a natural transformation $\lambda : \mathbb{L}F \to F$ such that the factorization of $\delta \circ \mathbb{L}F$ through $\mathscr{C}[\mathscr{W}^{-1}]$ (which exists, as $\mathbb{L}F$ is homotopical), together with $\delta\lambda$ is a total left derived functor of F. In symbols, one could write : $\mathbf{L}F \circ \gamma = \delta \circ \mathbb{L}F$.In other words it is a lift along δ of $\mathbf{L}F \circ \gamma$.
There is of course the dual notion of *right derived functor*.

In nice situations, these are guaranteed to exist, and (still under some nice assumptions) compose correctly, that is $\mathbb{L}G \circ \mathbb{L}F$ is a left derived functor of $G \circ F$ (although this does not always hold). We won't say more about this, however, the interested reader should look up paragraph 2.2 in [15], to learn about left deformations of functors between homotopical categories.

4.2 Model categories

We essentially follow [16]. The attentive reader will notice that there is a slight gap in that [16] doesn't assume the functoriality of the factorizations we will introduce, whereas for instance Riehl's theory of deformations ([15]) requires the deformation to be functorial. This is not an issue, for a lot of it can be adapted to fit in this non-functorial framework, and that which cannot will most of the time imply that we only have total derived functors, and not derived functors. We won't say more about this.
A model category will be a category $\mathscr{C}$ with a class of weak equivalences $\mathscr{W}$ and classes C, F of cofibrations and fibrations satisfying certain axioms. They will allow us to define classes of fibrant and cofibrant objetcs that will allow for effective computations of derived functors.
The names are here in analogy with the topological situation, but we can think of them algebraically as well : there is an analogy between cofibrations and monomorphisms, cofibrant objects and complexes of projectives; or (depending on the situation of interest) an analogy between fibrations and epimorphisms, fibrant objects and complexes of injectives. With these analogies in mind :

Definition 14. Let $\mathscr{C}$ be a category, and let $(\mathscr{W}, C, F)$ be three wide subcategories. Arrows in C are called cofibrations, in F fibrations, in $\mathscr{W}$ weak equivalences, in $C \cap \mathscr{W}$ acyclic cofibrations, and those in $F \cap \mathscr{W}$ acyclic fibrations. This data constitutes a *model structure* on $\mathscr{C}$ (the whole thing is called a model category) if it satisfies :

1. $\mathscr{C}$ is finitely complete and cocomplete
2. $\mathscr{W}$ satisfies the 2-out-of-3 property
3. If f is a retract of g (in the arrow category) and g is a fibration (resp. cofibration, resp. weak equivalence) then so is f
4. Given a commutative diagram $\begin{array}{ccc} A & \longrightarrow & B \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ C & \longrightarrow & D \end{array}$ where i is a cofibration, p a fibration, then if either of them is also a weak equivalence, there is a diagonal making the whole thing commute : $\begin{array}{ccc} A & \longrightarrow & B \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ C & \longrightarrow & D \end{array}$
5. Any map f can be factored as an acyclic cofibration i followed by a fibration p, and as a cofibration i' followed by an acyclic fibration p' $(f = pi = p'i')$

Notation 1. Let $\mathscr{C}$ be a model category (by abuse of notation, we omit the names of the classes of weak equivalences, fibrations, cofibrations). Then by finite (co)completeness, $\mathscr{C}$ has an initial object, which will be denoted $\emptyset$ and a final object denoted $*$.

Definition 15. Let $\mathscr{C}$ be a model category. An object X is called cofibrant if the unique map $\emptyset \to X$ is a cofibration; it is called fibrant if the unique map $X \to *$ is a fibration.

Remark 3. Note that the notion of model category is "self-dual", in that if $(\mathscr{C}, \mathscr{W}, C, F)$ is a model category, then so is $(\mathscr{C}^{op}, \mathscr{W}^{op}, F^{op}, C^{op})$. This is called the opposite model category.
In this correspondance, fibrant objects are cofibrant in the opposite model category, and vice versa.
What this also implies is that any result that holds on all model categories has a dual, which we don't need to prove as it follows from applying said result to the opposite model category.

One can then mimick a lot of topological constructions in this more general setting; for instance one can define homotopies between maps through cylinder-objects that act somewhat similarly to $X \times I$ or through path-objects that act similarly to X^I; and prove that when all objects in question are replaced by fibrant-cofibrant objects (which correspond more or less to CW-complexes in the topological setting) everything behaves well.
In particular we get a Whitehead theorem :

Proposition 19. *Let $\mathscr{C}$ be a model category. Suppose A, X are fibrant and cofibrant, and let $f : A \to X$ be a map. Then f is a weak equivalence if and only if it is a homotopy equivalence.*

This, together with other considerations, allows to prove a very important result :

Theorem 6. *Let $\mathscr{C}$ be a model category, let $\mathscr{C}_{cf}$ be the full subcategory of fibrant-cofibrant objects. Finally, let $\pi\mathscr{C}_{cf}$ denote the category whose objects are those of $\mathscr{C}_{cf}$ and whose arrows are homotopy classes of maps between these. Then it is a well-defined category and there is a fibrant-cofibrant replacement functor $\gamma : \mathscr{C} \to \pi\mathscr{C}_{cf}$ that exhibits $\pi\mathscr{C}_{cf}$ as the localization of $\mathscr{C}$ with respect to $\mathscr{W}$.*

Remark 4. We mentioned it earlier, but to state this theorem without changing our definitions, $\mathscr{W}$ would have to have the 2-out-of-6 property. It turns out that this actually follows from the axioms, although we only required the 2-out-of-3 property.

Remark 5. If we had required functorial factorizations in the axioms for a model category (as some authors do - see [3] for instance), then there would be a fibrant-cofibrant replacement functor $\mathscr{C} \to \mathscr{C}_{cf}$ that, when composed with $\mathscr{C}_{cf} \to \pi\mathscr{C}_{cf}$ would be a localization functor. In our setting, however, fibrant-cofibrant replacement and the associated functor is only defined up to homotopy. This can play a role in the theory of derived functors : whether or not we get a derived functor instead of a total derived functor may depend on it.

Notation 2. In the context of model categories at least, the localized category is usually denoted $\mathrm{Ho}(\mathscr{C})$, short for "homotopy category".

This theorem in particular implies that for a model category, a localization always exists, and is locally small, so there are no size issues here.
It also generalizes the fact that if we take the naive homotopy category of CW-complexes, we get something equivalent to the homotopy category of topological spaces (or Kan complexes and simplicial sets).

This theorem also allows us to get a theorem for the existence of derived functors :

Theorem 7. *Let $\mathscr{C}, \mathscr{D}$ be model categories, $F : \mathscr{C} \to \mathscr{D}$ a functor. Assume F takes weak equivalences between cofibrant objects to weak equivalences. Then F has a total left derived functor $\mathbf{L}F$. Moreover if we let γ denote the localization for $\mathscr{C}$, δ the one for $\mathscr{D}$, $t : \mathbf{L}F \circ \gamma \to \delta \circ F$ denote the transformation that exhibits $\mathbf{L}F$ as a total left derived functor, we have that t_X is an isomorphism for any cofibrant object X.*

Remark 6. By duality, we have a similar theorem for total right derived functors, where the word "cofibrant" is replaced by "fibrant".
In practice, this theorem implies that to compute $\mathbf{L}F(X)$, one takes a cofibrant replacement $Y \to X$ and computes $F(Y)$. Then $\mathbf{L}F(X)$ is isomorphic to $F(Y)$ in the homotopy category (we're abusing notation and writing $F(Y)$ instead of $\delta F(Y)$). This should be thought of as analogous to computing left derived functors in homological algebra, where one takes a projective resolution before applying F.

There is now a result that allows one to use the machinery of model categories on homological algebra; see [16] for a proof :

Theorem 8. *Let R be a ring, let $\mathbf{Ch}_R$ be the category of nonnegatively (homologically) graded chain complexes of R-modules. Then there is a model structure on $\mathbf{Ch}_R$, called the projective model structure, with $\mathscr{W}$ = the class of quasi-isomorphisms, C the class of degreewise monomorphisms that have a degreewise projective cokernel, F the class of maps that are epimorphisms in positive degree.*

In this model structure, any object is fibrant, and cofibrant objects are complexes of projectives. If one uses the construction of left derived functors with a cofibrant replacement with this model structure, one finds the usual notion of derived functors for right exact functors on $R - \mathbf{Mod}$, except that we don't take homology, so we have more information.
More precisely, when we have a right exact functor F defined on $R - \mathbf{Mod}$, if we extend it to $\mathbf{Ch}_R$ and derive it we get a functor $\mathbf{L}F$ such that $H_i\mathbf{L}F \simeq L_iF$; famous examples include the derived tensor product $- \otimes^{\mathbf{L}} B$ and the derived hom called $\mathrm{Ext}(-, B)$.
We can tell a similar story by taking the nonnegatively (cohomologically) graded chain complexes and taking the so-called *injective model structure*, which allows us to deal similarly with right-derived functors.

Note that the hypotheses of the theorem for the existence of derived functors are satisfied here, as a quasi-isomorphism between (nonnegatively graded) complexes of projectives is a chain homotopy equivalence, and is therefore automatically preserved.

One other example to show that the homotopy category contains the higher dimensional information about modules is the following :

Example 1. Let A, B be R-modules, n, m nonnegative integers. let $K(A, n)$ be the chain complex that has A concentrated in degreee n, similarly for $K(B, m)$.
Then $\hom_{\mathrm{Ho}(\mathbf{Ch}_R)}(K(A,n), K(B,m)) \simeq \mathrm{Ext}^{n-m}(A, B)$

The stories are a bit more subtle for more general types of chain complexes (unbounded for instance). One can see [21] for discussions of these topics.
One of the points of this more general framework is that it allows us to transfer topological intuition in homological algebra, for instance with homotopy (co)limits. It also allows for a better understanding of the derived category : we have only hinted at the basics of model category theory but much more can be said about them (we haven't even mentioned the theory of Quillen equivalences).

5 Other topics

In this last section, I will simply give a very quick overview of other topics I studied during this semester.

5.1 Some modular representation theory

With group cohomology in mind and under my belt, I studied some of the structure of kG and its (stable) category of modules when G is a p-group and $\mathrm{char} k = p$.
More specifically I studied specific properties of kG such as its locality, and some constructions relating to the stable module category, such as the functor $\Omega^1(M)$ defined as the kernel of a projective cover of M (or, stably, as the kernel of any epimorphism from a projective module to M) and its iterates.
The stable module category is defined to have as objects (finitely generated) kG-modules and $\hom_{\mathbf{stMod}}(A, B) = \hom(A, B)/\mathrm{Proj}(A, B)$ where $\mathrm{Proj}(A, B)$ is the subgroup of maps that factor through some projective module. Since kG is local, projective modules over it are free and so the stable module category is a way to look at the module category by erasing all the "easy" information (free modules). If $M \oplus P \cong N \oplus Q$ as modules, where P, Q are projective, then $M \simeq N$ in the stable module category (and the converse is also true), which somehow explains the name (think about stably isomorphic vector bundles for instance).
This category contains some amount of homotopical information about G-modules : for instance if we look at iterates of the functor Ω^1 (defined loosely above), denoted Ω^n, we have results such as $\mathrm{Ext}^n_{kG}(M, M') \cong \hom_{\mathbf{stMod}}(\Omega^n(M), M')$, which yields one of many interpretations of the Yoneda composition on ext-modules.

I also had a glimpse at the theory of support varieties of G-modules, and how they relate to the representation theory of G over k. One may have a look at [19], [20] for instance : the support variety of M is defined via its ext-modules : we look at $\mathrm{Ext}^*_{kG}(M, M) = \bigoplus_n \mathrm{Ext}^n_{kG}(M, M)$ which is a finitely generated $H^*(G, k)$ algebra when it's endowed with Yoneda composition. We then consider it as a $H^\bullet(G, k)$-module (where $H^\bullet(G, k)$ is the commutative algebra spanned by even degree elements if p is odd, and $H^*(G, k)$ if $p = 2$) and look at this module's annihilator : the support variety $V_G(M)$ is defined as the subvariety of $\mathrm{Spec}(H^\bullet(G, M))$ defined by this annihilator.

This allows one to use some tools of algebraic geometry to study kG-modules. There are results relating properties of the support variety and properties of the module : for instance M is projective if and only if its support variety is reduced to a point.

5.2 Some equivariant homotopy theory

I studied the paper [13] on some aspects of the homotopy theory of certain G-complexes, and Benson's book [18] on equivariant homotopy theory.
More explicitly I had a look at the equivariant Whitehead theorem; and at Quillen's conjecture relating the homotopy of the space of nontrivial elementary p-subgroups of a group G and the existence of a nontrivial normal p-subgroup of G.
Equivariant homotopy theory consists in looking at (nice) spaces X together with continuous G-actions for some group G and equivariant maps between those, up to homotopy. Of course, everything has to be equivariant, including homotopies. The equivariant Whitehead theorem, in this setting, gives an interesting criterion (similar to Whitehead's theorem) for when a map between two (nice) G-spaces X, Y is a G-homotopy equivalence : it suffices for it to be a homotopy equivalence when restricted to each $X^H \to Y^H$, H subgroup of G.
The spaces I looked at were so-called subgroup complexes : $|S_p(G)|, |A_p(G)|$ for some group G where $| - |$ denotes the geometric realization of these posets (seen as simplicial sets via the nerve functor) and $S_p(G)$ is the poset of nontrivial p-subgroups of G, and $A_p(G)$ is the poset of nontrivial elementary p-subgroups of G.
For instance Quillen shows in [13] that the inclusion $|A_p(G)| \to |S_p(G)|$ is a homotopy equivalence, and it's easy to apply the equivariant Whitehead theorem to check that this is a G-equivalence.
While studying this, I encountered Quillen's theorems A and B, which are also very important theorems in the field.
Quillen's conjecture relates the topology of $|A_p(G)|$ to properties of G : it states that if $|A_p(G)|$ is contractible then G has a normal p-subgroup (the converse is known to be true and the conjecture is known in certain cases).

5.3 p-completion and p-compact groups

I studied parts of the paper [14], which establishes a sort of dictionary between the theory of compact Lie groups and the theory of p-compact groups.
For instance, it establishes the existence of a maximal p-compact torus for any p-compact group, which is to be seen as analogous to the existence of maximal tori in compact Lie groups.
To study this paper I had to study p-completions as well, and also studied their algebraic counterparts .
p-completion is a functor from spaces to themselves which allows one to "focus" on information which is "p-local" : precisely a map $X \to Y$ is a $H_*(-, \mathbb{F}_p)$ equivalence if and only if its p-completion is a homotopy equivalence; and p-complete spaces are (roughly) spaces that are homotopy equivalent to their p-completion.
On the other hand, loop spaces are spaces that are homotopy equivalent to some ΩY (space of based loops in Y), they are a homotopical analogue of groups (because we can concatenate loops, there is a "neutral" loop and there are "inverse" loops - all this up to homotopy).
A p-compact group is then the following data : (X, BX, e) where X, BX are spaces, $e : \Omega BX \to X$ is a homotopy equivalence, BX is p-complete and X "looks like" a finite CW-complex from the point of view of mod p homology.
It turns out (that's what [14] explains) that these have analogous properties to compact Lie groups "if we were to look at them from the point of view of p", and that in fact some properties of compact Lie groups can be explained via the study of these p-compact groups (indeed, any compact Lie group is a finite loop space, so it "suffices" to be able to piece things back from p-local data to global data to get information on finite loop spaces from information on p-compact groups). An example of this sort of technique is the main theorem of [14] : if X is a finite loop space then for each prime p, $H^*(BX, \mathbb{F}_p)$ is a finitely generated algebra. This recovers in particular the theorem that states this in the special case where X is a compact Lie group, which was proved in [23].

6 Index: notions and notations

In this section we define, in alphabetical order, the notions and notations I have used but not defined in the report.

6.1 Notions

- Category (Abelian): see e.g. [17] and [4]
- Covering (Universal) : see for instance [22]
- Functor (Additive, Adjoint) : see [17] and [4]
- Fundamental group(oid) of a space : see [22]
- Kan extension : see [15] and [17]
- Natural transformation : see [17]
- PID : Principal ideal domain

6.2 Notations

- **Ab**: the (abelian) category of abelian groups and group morphisms between them.
- $C_\bullet(X; A)$: complex of singular chains of a space X with coefficients in the abelian group A
- **FinGrp** : the category of finite groups and group morphisms between them.
- $\Pi_1(X)$: the fundamental groupoid of the space X
- $\mathbb{Z}[G]$ ($RG, R[G]$): the group ring of G (group algebra over R)
- $|x|$: degree of x, where x is a homogeneous element of a graded object

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