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Group completion is a completion

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Abstract

Classically, group completion relates the singular homology of a topological monoid M to the singular homology of ΩBM [BP72, Qui94, MS76]. Following Nikolaus [Nik17], we present a higher categorical proof of the group completion theorem for $\mathbb{E}_1$-monoids and generalized homology theories.

The technical ingredient in this proof is the theory of Bousfield localizations in the setting of stable and locally presentable ∞-categories, in particular the special case of Ore localization of $\mathbb{E}_1$-ring spectra. We give an introduction to this theory after reviewing the corresponding theory for ordinary categories.

The higher categorical approach allows a simple proof of a well-known (but only recently proved in full [RW13]) theorem which connects group completion to Quillen's plus construction. In particular, this theorem yields the striking result $\Omega^\infty \mathbb{S} \simeq \mathbb{Z} \times B\Sigma_\infty^+$.

Resumé

Klassisk gruppefuldstændiggørelse relaterer den singulære homologi af en topologisk monoid M til den singulære homologi af ΩBM [BP72, Qui94, MS76]. Vi præsenterer et bevis fra Nikolaus [Nik17] for gruppefuldstændiggørelsessætningen for $\mathbb{E}_1$-monoider og generaliserede homologiteorier.

Den tekniske ingrediens i dette bevis er teorien om Bousfieldlokalisering af stabile og lokalt præsentable ∞-kategorier, især specialtilfældet af Orelokalisering af $\mathbb{E}_1$-ringspektra. Vi introducerer denne teori efter at have gennemgået den tilsvarende teori for almindelige kategorier.

Den højere kategoriske tilgang tillader et simpelt bevis for en velkendt (men først for nylig bevist til fulde [RW13]) sætning, der forbinder gruppefuldstændiggørelse til Quillens pluskonstruktion. Specielt medfører denne sætning det slående resultat $\Omega^\infty \mathbb{S} \simeq \mathbb{Z} \times B\Sigma_\infty^+$.

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I would also like to thank participants in the Topics in Algebraic Topology seminar course for helpful comments on my presentation of Nikolaus's article. Finally, I wish to thank my friends at the institute who, throughout the writing process, accompanied me on an unhealthy number of trips to the coffee machine in the break room.

0 Introduction

In algebraic topology, every loop space has a canonical multiplication map which is associative up to (coherent) homotopy: loop concatenation. Given an $\mathbb{E}_1$-monoid, meaning a space with a multiplication map which is associative up to coherent homotopy, it is therefore natural to ask whether the space is actually a loop space.

Loop concatenation always comes with inverses. It follows that in order for an $\mathbb{E}_1$-monoid to be a loop space, it must have inverses. A classical theorem of Stasheff says that failing to have inverses is the only obstruction to delooping. In modern terms, any $\mathbb{E}_1$-group is a loop space.

The delooping of $\mathbb{E}_1$-groups is functorial, and the functor is defined more generally for $\mathbb{E}_1$-monoids. This functor is denoted by B since it extends the well-known classifying space functor $G \mapsto BG$, originally defined for topological groups. Hence every $\mathbb{E}_1$-monoid M has an associated $\mathbb{E}_1$-group ΩBM, and $M \simeq \Omega BM$ if and only if M is an $\mathbb{E}_1$-group. The group completion theorem describes the relationship between M and ΩBM in general.

0.1 A brief history of group completion

The first group completion theorem appeared in [BP72]. In the introduction to that paper, Barratt and Priddy write that they discovered the theorem in their efforts to understand (what is now called) the Barratt-Priddy-Quillen theorem (see § 2.5 below). Quillen gave another early proof of the theorem, finally published as an appendix in [Qui94]. Both proofs are based on spectral sequence calculations.

McDuff and Segal gave a more conceptual proof in [MS76], which remains a standard reference. The idea of their proof is to exhibit a homology fiber sequence $X \to X_M \to BM$ for each space X on which M (a topological monoid) acts via homology equivalences. When X is taken to be a certain well-understood space M_∞, one finds that the middle term of this sequence is contractible, hence M_∞ is homology equivalent to ΩBM. This approach is inspired by Segal's proof of the delooping theorem for $\mathbb{E}_1$-groups in [Seg74] (see § 2.1 below). More than a decade later, Moerdijk published a proof [Moe89] based on the McDuff-Segal approach which avoids the use of spectral sequences completely, making for a pleasantly abstract proof.

Here we follow the approach in Nikolaus's exposition article [Nik17]. The proof given there has several advantages compared to older proofs. For one, it justifies the use of the term "group completion" by showing that the functor $M \mapsto \Omega BM$ is a completion (or localization) in Bousfield's sense, whence the title of this project. This insight allows elegant proofs of both the main group completion theorem in its proper setting of $\mathbb{E}_1$-monoids (rather than strict topological monoids!) and the theorem conneting group completion to the plus construction, which was not proved in full before [RW13].

0.2 Prerequisites

This exposition is aimed at advanced students with a background in homotopy theory. Specifically, we assume familiarity with simplicial sets, homotopy (co)limits and (ring) spectra.

We use language and results from higher category theory and higher algebra. One goal of this exposition is to demonstrate the *power* of this framework by proving a classical (and historically quite difficult) result using the machinery of Bousfield localization of ∞-categories (following [Nik17]). Ample textbook references are given, especially to Lurie's books: *Higher Topos Theory* (HTT) [Lur09] and *Higher Algebra* (HA) [Lur17]. We believe that readers who are unfamiliar with higher nonsense can still appreciate the ideas that underpin this project, all of which are classically motivated. In particular, we urge such readers to think of (co)limits in ∞-categories (e.g. the ∞-category of spaces) as homotopy (co)limits (HTT 4.2.4.1).

0.3 Notation and terminology

- We fix a base Grothendieck universe whose elements are referred to as *small* sets. The category of small sets is denoted by Sets.
- Following Lurie, we always think of ∞-categories simply as *quasicategories*, meaning simplicial sets which have the right extension property with respect to inner horn inclusions $\Lambda^n_i \subseteq \Delta^n$ $(0 < i < n)$.
- The *∞-category of spaces*, denoted $\mathcal{S}$, is the homotopy coherent nerve (as in HTT 1.1.5) of the category of Kan complexes enriched over Kan complexes via the standard mapping complex construction.
- In keeping with older conventions (but breaking with Lurie), we use the term *locally presentable category* (resp. *∞-category*) instead of simply "presentable category" (resp. "∞-category").

1 Localization

In this section, we set up the localization machinery that will be needed in § 2. In § 1.1, we present the theory of Bousfield localization for locally presentable categories. This motivates the corresponding theory for ∞-categories which we study in § 1.2. In both settings, the proofs of the main theorems are based on small object arguments, and are therefore elegant and conceptual. Finally, we specialize to the case of Ore localization of ring spectra in § 1.3.

1.1 Localization of ordinary categories

Fix a category $\mathcal{C}$ and a set of morphisms W in $\mathcal{C}$. Recall that a localization of $\mathcal{C}$ at W is a functor $q\colon \mathcal{C} \to \mathcal{C}[W^{-1}]$ which (1) sends every morphism in W to an isomorphism and (2) is initial among functors satisfying (1) in the enriched sense, meaning that

$$\operatorname{Fun}(\mathcal{C}[W^{-1}], \mathcal{D}) \xrightarrow{q^*} \operatorname{Fun}^{\sim W}(\mathcal{C}, \mathcal{D})$$

is an equivalence of categories for all categories $\mathcal{D}$, where $\operatorname{Fun}^{\sim W}(\mathcal{C}, \mathcal{D})$ denotes the full subcategory of $\operatorname{Fun}(\mathcal{C}, \mathcal{D})$ spanned by functors that invert morphisms in W.

In general the localization $\mathcal{C}[W^{-1}]$ does not have a nice description. Even when $\mathcal{C}$ is locally small and W is a small set, we cannot expect the localization to be locally small.

On the other hand, many localizations which occur in nature are well-behaved and admit a description in terms of so-called *local objects* which live inside the original category. This idea is due to Adams, but the set-theoretic issues were overcome by Bousfield [Bou75].

1.1 Definition. An object $c \in \mathcal{C}$ is *W-local* if the induced map

$$\hom_{\mathcal{C}}(y, c) \xrightarrow{f^*} \hom_{\mathcal{C}}(x, c)$$

is a bijection for all $f\colon x \to y$ in W. The full subcategory of $\mathcal{C}$ spanned by W-local objects is denoted by $\mathcal{C}_W$.

1.2 Definition. A morphism $f\colon x \to y$ is a *W-equivalence* if

$$\hom_{\mathcal{C}}(y, c) \xrightarrow{f^*} \hom_{\mathcal{C}}(x, c)$$

is a bijection for each $c \in \mathcal{C}_W$. The set of W-equivalences is denoted by $\widetilde{W}$.

Intuitively, a W-equivalence is a map which looks like an isomorphism to the W-local objects. Indeed:

1.3 Proposition (Whitehead's theorem). *Suppose $f\colon d \to d'$ is a W-equivalence where d and d' are W-local. Then f is an isomorphism.*

Proof. By assumption $\hom_{\mathcal{C}}(d', c) \xrightarrow{f^*} \hom_{\mathcal{C}}(d, c)$ is an bijection for all $c \in \mathcal{C}_W$. But here d and d' are also in $\mathcal{C}_W$, and since the latter is a full subcategory of $\mathcal{C}$, the hom-sets identify with hom-sets in $\mathcal{C}_W$. The conclusion follows from Yoneda's lemma. □

Clearly every element of W is a W-equivalence. The opposite inclusion is false in general. Due to the following proposition, it is important to understand the relationship between W and $\widetilde{W}$.

1.4 Proposition. *Suppose $L\colon \mathcal{C} \to \mathcal{C}_W$ is left adjoint to the inclusion $\mathcal{C}_W \subseteq \mathcal{C}$. Then*

(1) *A map f in $\mathcal{C}$ is a W-equivalence if and only if Lf is an isomorphism.*

(2) *The map $L\colon \mathcal{C} \to \mathcal{C}_W$ exhibits $\mathcal{C}_W$ as the localization of $\mathcal{C}$ at $\widetilde{W}$.*

Proof. For $f\colon x \to y$ and any $c \in \mathcal{C}_W$, the adjunction supplies a commutative diagram:

$$\begin{array}{ccc} \hom_{\mathcal{C}_W}(Ly, c) & \xrightarrow{(Lf)^*} & \hom_{\mathcal{C}_W}(Lx, c) \\ \downarrow\wr & & \downarrow\wr \\ \hom_{\mathcal{C}}(y, c) & \xrightarrow{f^*} & \hom_{\mathcal{C}}(x, c) \end{array}$$

By definition, the bottom arrow is a bijection for all $c \in \mathcal{C}_W$ if and only if f is a W-equivalence. By Yoneda's lemma, the top arrow is a bijection for all $c \in \mathcal{C}_W$ if and only if Lf is an equivalence.

We prove (2). Let u be the unit of the adjunction. We claim that $u_x\colon x \to Lx$ is a W-equivalence for all $x \in \mathcal{C}$. Since u is the unit of the adjunction, we have that

$$\hom_{\mathcal{C}_W}(Lx, c) \to \hom_{\mathcal{C}}(Lx, c) \xrightarrow{u_x^*} \hom_{\mathcal{C}}(x, c)$$

is a bijection for all $x \in \mathcal{C}$ and $c \in \mathcal{C}_W$. Since $\mathcal{C}_W$ is a full subcategory, the first map is already a bijection, showing the claim.

Let $\mathcal{D}$ be any category. We will show that the functor $\mathrm{Fun}(\mathcal{C}_W, \mathcal{D}) \xrightarrow{L^*} \mathrm{Fun}^{\sim\widetilde{W}}(\mathcal{C}, \mathcal{D})$ is fully faithful and essentially surjective.

To see that L^* is essentially surjective, suppose $F\colon \mathcal{C} \to \mathcal{D}$ is a functor which inverts $\widetilde{W}$. Denoting the inclusion $\mathcal{C}_W \subseteq \mathcal{C}$ by i, we claim that F is naturally isomorphic to $L^*(Fi) = FiL$. Indeed, $Fu\colon FiL \to F$ is a natural isomorphism since F inverts every component of u, all of which lie in $\widetilde{W}$ by what we have just seen.

To see that L^* is fully faithful, we may assume that the composition Li is the identity on $\mathcal{C}_W$.[1] If $\eta\colon FL \to GL$ is a natural transformation where F and G are functors from $\mathcal{C}_W$ to $\mathcal{D}$, it follows that $\eta = L^*(\eta i)$. This shows that every natural transformation from F to G lies in the image of L^*. The proof of injectivity is similar. □

We also have a helpful criterion for the existence of such a left adjoint.

1.5 Definition. Let $x \in \mathcal{C}$. A map $f\colon x \to c$ is a *W-localization* of x if f is a W-equivalence and c is W-local.

1.6 Proposition. *The inclusion $\mathcal{C}_W \subseteq \mathcal{C}$ has a left adjoint if and only if each $x \in \mathcal{C}$ has a W-localization $u\colon x \to c$.*

Proof. Assume first that the inclusion has a left adjoint L and let u be the unit of the adjunction. We saw in the proof of the Proposition 1.4 that each component $u_x\colon x \to Lx$ is a W-equivalence, and by assumption Lx is W-local, so u_x is a W-localization of x.

Now assume that there is a W-equivalence $u_x\colon x \to c_x$ for each $x \in \mathcal{C}$. We wish to define a functor $L\colon \mathcal{C} \to \mathcal{C}_W$ by $Lx = c_x$ on objects such that the collection $\{u_x\colon x \to Lx\}$ forms the unit of an adjunction between L (on the left) and the inclusion $\mathcal{C}_W \subseteq \mathcal{C}$ (on the right). We must define L on morphisms. But since Ly is W-local and u_x is a W-equivalence, we have

$$\hom_{\mathcal{C}}(Lx, Ly) \xrightarrow{u_x^*} \hom_{\mathcal{C}}(x, Ly),$$

so for each $f\colon x \to y$ there is a unique dashed arrow such that

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ \downarrow{\scriptstyle u_x} & & \downarrow{\scriptstyle u_y} \\ Lx & \overset{Lf}{\dashrightarrow} & Ly \end{array}$$

[1] This follows from the proof of Proposition 1.6.

commutes. Uniqueness implies that L respects composition and takes identity morphisms to identity morphisms. Hence L is a functor and by construction $\{u_x\}$ is a natural transformation from the identity functor on $\mathcal{C}$ to the composition of L and the inclusion of $\mathcal{C}_W$ back into $\mathcal{C}$. Furthermore, it follows from the argument at the start of this proof (but in the opposite direction) that $\{u_x\}$ forms the unit of an adjunction between L (on the left) and the inclusion $\mathcal{C}_W \subseteq \mathcal{C}$ (on the right). □

From now on, assume for simplicity that $\mathcal{C}$ contains small colimits. Recall that no assumptions were placed on the set W. On the other hand, we will see that $\widetilde{W}$ has certain restrictive properties. The main theorem of this subsection shows that $\widetilde{W}$ is generated by W under these properties as long as we make some mild assumptions on $\mathcal{C}$ and W.

1.7 Definition. A set of morphisms S in $\mathcal{C}$ is *weakly saturated* if

(1) S contains all isomorphisms.

(2) (Closure under cobase-change) Given a pushout

$$\begin{array}{ccc} x & \longrightarrow & x' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ y & \longrightarrow & y' \end{array}$$

in $\mathcal{C}$, if $f \in S$ then $f' \in S$.

(3) (Closure under transfinite composition) If λ is an ordinal and $x\colon \lambda \to \mathcal{C}$ is a functor such that for every $\alpha \in \lambda$ with $\alpha \neq 0$, the map

$$\varinjlim_{\beta<\alpha} x(\beta) \to x(\alpha)$$

is in S, then the canonical map $x(0) \to \varinjlim_{\beta<\lambda} x(\beta)$ is in S.

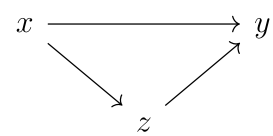
(4) (Closure under retracts) Given a commutative diagram

$$\begin{array}{ccccc} a & \longrightarrow & x & \longrightarrow & a \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ b & \longrightarrow & y & \longrightarrow & y \end{array}$$

where both horizontal compositions are identity morphisms, if $g \in S$ then $f \in S$.

1.8 Definition. A weakly saturated set S is *saturated* if it has the following additional property:

(5) (Two-out-of-three) Given a commutative diagram



if two of the arrows lie in S, then the third arrow also lies in S.

Remark. Note that an intersection of (weakly) saturated sets is again (weakly) saturated.

The set of isomorphisms in $\mathcal{C}$ is saturated. Clearly it is then the *smallest* saturated set. More generally, if $\mathcal{D}$ is another category which contains small colimits and $F\colon \mathcal{C} \to \mathcal{D}$ is a functor which preserves small colimits, then the set of morphisms in $\mathcal{C}$ which are sent to isomorphisms in $\mathcal{D}$ is a saturated set.

1.9 Proposition. *The set of W-equivalences is saturated.*

Proof. For each $z \in \mathcal{C}$, we view $F_z = \hom_{\mathcal{C}}(-, z)$ as a functor from $\mathcal{C}$ to $\mathrm{Sets}^{\mathrm{op}}$. Note that F_z preserves colimits. Hence

$$S_z = \{f\colon x \to y \mid \hom_{\mathcal{C}}(y, z) \xrightarrow{f^*} \hom_{\mathcal{C}}(y, z) \text{ is an iso}\}$$

is a saturated set for each $z \in \mathcal{C}$. It follows that the intersection

$$\bigcap_{c \in \mathcal{C}_W} S_c = \{f\colon x \to y \mid \hom_{\mathcal{C}}(y, c) \xrightarrow{f^*} \hom_{\mathcal{C}}(y, c) \text{ is an iso for all } c \in \mathcal{C}_W\}$$

is a saturated set. But this intersection is precisely $\widetilde{W}$. □

Since intersections of saturated sets are again saturated, the following definition makes sense:

1.10 Definition. Let S be any set of morphisms in $\mathcal{C}$. The *saturation* of S is the smallest saturated set containing S, denoted by $\mathrm{sat}(S)$. Similarly, the *weak saturation* of S is the smallest weakly saturated set containing S, denoted by $\mathrm{sat}_w(S)$.

Remark. Clearly $\mathrm{sat}_w(S) \subseteq \mathrm{sat}(S)$.

We can now state the main theorem of this subsection:

1.11 Theorem (Bousfield localization)**.** *Suppose that the category $\mathcal{C}$ is locally presentable and contains a terminal object. Assume also that the set of morphisms W is small. Then*

(1) *For each $x \in \mathcal{C}$, there is a map $f\colon x \to c$ with $f \in \operatorname{sat}(W)$ and $c \in \mathcal{C}_W$.*

(2) *The inclusion $\mathcal{C}_W \subseteq \mathcal{C}$ has a left adjoint.*

(3) *The set of W-equivalences $\widetilde{W}$ equals the saturation $\operatorname{sat}(W)$.*

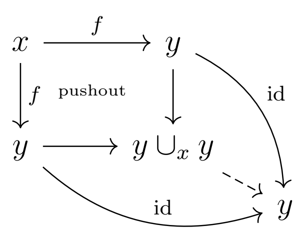
The proof uses the small object argument, which in its original form is due to Quillen. We refer the reader to HTT A.1.2.5 for a proof of the version used here.

1.12 Theorem (Small object argument)**.** *Suppose $\mathcal{C}$ is locally presentable and that S is a small set of morphisms in $\mathcal{C}$. Then any morphism $f\colon x \to y$ in $\mathcal{C}$ has a factorization*

$$x \xrightarrow{\lambda} z \xrightarrow{\rho} y$$

where λ lies in the weak saturation $\operatorname{sat}_w(S)$ and ρ has the right lifting property with respect to S.

Proof of Theorem 1.11. We first prove (1). For each $f\colon x \to y$ in $\mathcal{C}$, the *fold map* of f is the dashed arrow coming from the universal property applied to



Denote the fold map of f by ∇f.

Put

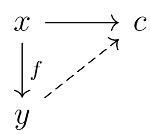
$$W' = W \cup \{\nabla f \mid f \in W\}.$$

This is again a small set, so Theorem 1.12 applies. Letting $*$ denote a terminal object of $\mathcal{C}$, this implies in particular that for each $x \in \mathcal{C}$, the map $x \to *$ factors as

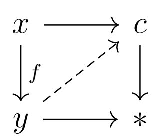
$$x \xrightarrow{\lambda} c \to *$$

where λ is in the saturation of W' and $c \to *$ has the right lifting property with respect to W'. Note that $\nabla f \in \mathrm{sat}(W)$ for each $f \in W$ by conditions (1), (2) and (5) for saturated sets. Hence the saturation of W' is just the saturation of W, so $\lambda \in \mathrm{sat}(W)$.

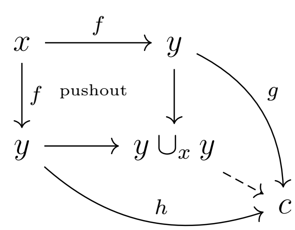
We claim that c is W-local. Note that by definition c is W-local if and only if for every solid arrow diagram



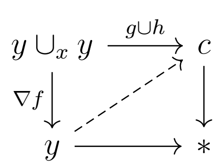
with $f \in W$, there exists a unique dashed arrow making the diagram commute. This extension problem is equivalent to the lifting problem:



We know that $c \to *$ has the right lifting property with respect to $W \subseteq W'$, so certainly we can always find a dashed arrow solution to this lifting problem. We claim that such solutions are unique. Suppose g and h both solve the lifting problem for the same map $x \to c$. Then they fit into the diagram:



Let $g \cup h$ denote the dashed arrow coming from the universal property as shown in the diagram. Since $c \to *$ has the right lifting property with respect to fold maps, we can solve the following lifting problem:



But this implies that $g = h$, showing uniqueness.

We claim that (2) follows from what we have just proved. Indeed, we know that $\widetilde{W}$ is saturated and that $W \subseteq \widetilde{W}$, so it follows that $\mathrm{sat}(W) \subseteq \widetilde{W}$. Hence the existence of morphisms $f : x \to c$ in $\mathrm{sat}(W)$ where c is W-local implies the existence of a left adjoint by Proposition 1.6.

Denote the left adjoint by L. Recall from the proof of Proposition 1.6 that the W-equivalences $f : x \to c$, which in our case come from the saturation $\mathrm{sat}(W)$, form the unit of the adjunction between L and the inclusion. Denote the collection of these maps by $\{u_x\}$. For each map $f : x \to y$ in $\mathcal{C}$, naturality gives a commutative diagram

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ \downarrow{\scriptstyle u_x} & & \downarrow{\scriptstyle u_y} \\ Lx & \xrightarrow{Lf} & Ly \end{array}$$

If $f \in \widetilde{W}$, we know from Proposition 1.4 (1) that Lf is an isomorphism, so in particular it lies in the saturated set $\mathrm{sat}(W)$. But then the two-out-of-three property implies that $f \in \mathrm{sat}(W)$. This shows that $\widetilde{W} \subseteq \mathrm{sat}(W)$, which proves (3) since we know from above that the opposite inclusion holds. □

Remark. Typically, the set of morphisms W that one wishes to localize is already saturated, but fails to be *small*. In that case, the goal is to find a small set $W_0 \subseteq W$ whose saturation is all of W. If this is possible, W is said to be *of small generation*.

1.2 Localization of ∞-categories

The machinery from the previous subsection carries over to ∞-categories. In terms of definitions and theorems, this passage from ordinary categories to ∞-categories is predictable: simply replace hom-sets with mapping spaces and bijections with equivalences. The *proofs*, however, are often different for ∞-categories.

Fix an ∞-category $\mathcal{C}$ and a set of morphisms W in $\mathcal{C}$.

1.13 Definition. An object $c \in \mathcal{C}$ is *W-local* if the induced map

$$\mathrm{Map}_{\mathcal{C}}(y, c) \xrightarrow{f^*} \mathrm{Map}_{\mathcal{C}}(x, c)$$

is an equivalence for all $f\colon x \to y$ in W. The full subcategory of $\mathcal{C}$ spanned by W-local objects is denoted by $\mathcal{C}_W$.

As in the previous subsection, the goal is to find a functor $L\colon \mathcal{C} \to \mathcal{C}_W$ which is left adjoint to the inclusion $\mathcal{C}_W \subseteq \mathcal{C}$ in the ∞-categorical sense.

Adjunctions are covered in HTT 5.2, especially 5.2.2 and 5.2.3. We will use a different definition than Lurie, which has the advantage of being recognizable to students of classical category theory – without having to be "straightened" first. Lurie proves that this definition agrees with the unstraightened definition in HTT 5.2.2.8.

1.14 Definition. Given a pair of parallel functors

$$\mathcal{C} \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{C}$$

between ∞-categories, we say that f is a *left adjoint* to g if there exists a natural transformation $u\colon \mathrm{id}_{\mathcal{C}} \to g \circ f$ such that for every pair of objects $c \in \mathcal{C}$ and $d \in \mathcal{D}$, the induced map on mapping spaces

$$\mathrm{Map}_{\mathcal{D}}(f(c), d) \xrightarrow{g_*} \mathrm{Map}_{\mathcal{C}}(gf(c), g(d)) \xrightarrow{u(c)^*} \mathrm{Map}_{\mathcal{C}}(c, g(d))$$

is an equivalence. In this case, we also say that g is a *right adjoint* to f, and that u is a *unit transformation*.

Roughly speaking, ∞-categorical left adjoints (resp. right adjoints) behave as one would expect:

(a) Left adjoints preserve colimits and right adjoints preserve limits (HTT 5.2.3.5);

(b) If it exists, a left resp. right adjoint to a functor is unique up to contractible choice (HTT 5.2.6.2);

(c) A version of Freyd's adjoint functor theorem holds, i.e. for *locally presentable ∞-categories*, every functor which preserves small colimits is a left adjoint, and every accessible functor which preserves limits is a right adjoint (HTT 5.5.2.9).

With prerequisites out of the way, descent gives us a recognition principle for endofunctors which are left adjoint to the inclusion of their image:

1.15 Proposition (HTT 5.2.7.4). *Let $L\colon \mathcal{C} \to \mathcal{C}$ be a functor with essential image $L\mathcal{C}$. Then the following are equivalent:*

(1) *When viewed as a functor from $\mathcal{C}$ to $L\mathcal{C}$, the functor L is a left adjoint to the inclusion $L\mathcal{C} \subseteq \mathcal{C}$.*

(2) *There exists a natural transformation $u\colon \Delta^1 \times \mathcal{C} \to \mathcal{C}$ from $\mathrm{id}_{\mathcal{C}}$ to L such that for all $x \in \mathcal{C}$ the morphisms $L(u(x)), u(L(x))\colon Lx \to LLx$ of are equivalences in $\mathcal{C}$.*

Proof. As in the proof of Proposition 1.6, the key point is that in the sequence

$$\mathrm{Map}_{L\mathcal{C}}(Lx, Ly) \to \mathrm{Map}_{\mathcal{C}}(Lx, Ly) \xrightarrow{u(x)^*} \mathrm{Map}_{\mathcal{C}}(x, Ly),$$

the first map is already an equivalence since $L\mathcal{C}$ is by definition a full subcategory of $\mathcal{C}$. It follows that the composition is an equivalence if and only if $u(c)^*$ is an equivalence, and so u is a unit transformation if and only if this holds for each $x \in \mathcal{C}$ and each $Ly \in L\mathcal{C}$.

Assume that u is a unit transformation, so by our discussion the map $u(c)^*$ is always an equivalence. If also $x \in L\mathcal{C}$ (so that we are always really in $L\mathcal{C}$), then the Yoneda lemma implies that $u(x)$ is an equivalence. In particular, this shows that $u(L(x))$ is an equivalence. Naturality implies that we have commutative diagram

$$\begin{array}{ccc} x & \xrightarrow{u(x)} & Lx \\ {\scriptstyle u(x)}\downarrow & & \downarrow{\scriptstyle u(Lx)} \\ Lx & \xrightarrow[L(u(x))]{} & LLx \end{array}$$

Since $u(x)^*$ is an equivalence (hence injective on π_0), we see that $L(u(x)) \simeq u(Lx)$, so in particular $L(u(x))$ is also an equivalence.

Assume now that u is some natural transformation satisfying the condition in (2). We must show that

$$\mathrm{Map}_{\mathcal{C}}(Lx, Ly) \xrightarrow{u(x)^*} \mathrm{Map}_{\mathcal{C}}(x, Ly)$$

is an equivalence for all $x \in \mathcal{C}$ and all $Ly \in L\mathcal{C}$, i.e. an isomorphism in the homotopy category of spaces. By the Yoneda lemma, it suffices to show that

$$\hom_{\mathcal{H}}(K, \mathrm{Map}_{\mathcal{C}}(Lx, Ly)) \xrightarrow{[u(x)^*]_*} \hom_{\mathcal{H}}(K, \mathrm{Map}_{\mathcal{C}}(x, Ly))$$

is a bijection for every space K. But using the exponential rule, this map identifies with

$$\hom_{h\,\mathrm{Fun}(K,\mathcal{C})}(\delta(Lx), \delta(Ly)) \to \hom_{h\,\mathrm{Fun}(K,\mathcal{C})}(\delta x, \delta(Ly)),$$

where $\delta\colon \mathcal{C} \to \mathrm{Fun}(K, \mathcal{C})$ is the functor which sends c to the constant functor at c. Since we can replace $\mathcal{C}$ with $\mathrm{Fun}(K, \mathcal{C})$, it therefore suffices to show that

$$\hom_{h\mathcal{C}}(Lx, Ly) \xrightarrow{[u(x)]^*} \hom_{h\mathcal{C}}(x, Ly)$$

is a bijection for all $x \in \mathcal{C}$ and all $Ly \in L\mathcal{C}$.

We first check surjectivity. Let $f: x \to Ly$. By naturality of u, we have

$$\begin{array}{ccc} x & \xrightarrow{f} & Ly \\ {\scriptstyle u(x)}\downarrow & & \wr\downarrow{\scriptstyle u(Ly)} \\ Lx & \xrightarrow{Lf} & LLy \end{array}$$

Here our assumption says that $u(Lx)$ is invertible, so choosing an inverse $(u(Lx))^{-1}$, we can write

$$f \simeq (u(Lx))^{-1} \circ Lf \circ u(x) = u(x)^* \left((u(Lx))^{-1} \circ Lf\right).$$

As for injectivity, note that for each $g\colon Lx \to Ly$ we have

$$\begin{array}{ccc} Lx & \xrightarrow{g} & Ly \\ {\scriptstyle u(Lx)}\downarrow\wr & & \wr\downarrow{\scriptstyle u(Ly)} \\ LLx & \xrightarrow{Lg} & LLy \end{array}$$

It follows that

$$\begin{aligned} g &\simeq (u(Ly))^{-1} \circ Lg \circ u(Lx) \\ &\simeq (u(Ly))^{-1} \circ Lg \circ L(u(x)) \circ (L(u(x)))^{-1} \circ u(Lx) \\ &\simeq (u(Ly))^{-1} \circ L\left(g \circ u(x)\right) \circ (L(u(x)))^{-1} \circ u(Lx) \end{aligned}$$

Here everything except the middle factor of $L\left(g \circ u(x)\right)$ is invertible by assumption. It follows that g is uniquely determined by $g \circ u(x)$, hence that $u(x)^*$ is injective. □

As before, we are interested in morphisms which look like equivalences to the W-local objects.

1.16 Definition. A morphism $f\colon x \to y$ in $\mathcal{C}$ is a *W-equivalence* if

$$\mathrm{Map}_{\mathcal{C}}(y, c) \xrightarrow{f^*} \mathrm{Map}_{\mathcal{C}}(x, c)$$

is an equivalence for each $c \in \mathcal{C}_W$. The set of W-equivalences is denoted by $\widetilde{W}$.

Here is the analog of the characterization of W-equivalences from Proposition 1.4:

1.17 Proposition. *Suppose $L\colon \mathcal{C} \to \mathcal{C}_W$ is left adjoint to the inclusion $\mathcal{C}_W \subseteq \mathcal{C}$. Then a map $f \in \mathcal{C}$ is a W-equivalence if and only if Lf is an equivalence.*

Proof. This follows from the adjunction as in the proof of Proposition 1.4. □

There is also a direct analog of Proposition 1.6.

1.18 Definition. Let $x \in \mathcal{C}$. A map $f\colon x \to c$ is a *W-localization* of x if f is a W-equivalence and c is W-local.

1.19 Proposition (HTT 5.2.7.8)**.** *The inclusion $\mathcal{C}_W \subseteq \mathcal{C}$ has a left adjoint if and only if each $x \in \mathcal{C}$ has a W-localization $u\colon x \to c$.*

Replacing mapping spaces with hom-sets and equivalences with bijections, the argument from the proof of Proposition 1.6 can be used to show that the components of the unit transformation which accompanies a left adjoint to the inclusion are W-equivalences.

As for the other direction, we cannot (at least naively) use the technique from the proof of the classical counterpart to this proposition, since a functor between ∞-categories consists of more than its values on objects and morphisms, and a natural transformation between such functors is more than a collection of morphisms. Instead, we will use a trick from the unstraightened world to show that the unstraightened condition for the existence of a left adjoint is satisfied. Readers who are unfamiliar with the theory of (un)straightening can skip the proof.

Proof of Proposition 1.19. Define $\mathcal{D}$ to be the full subcategory of $\Delta^1 \times \mathcal{C}$ spanned by objects of the form (x, i) with $x \in \mathcal{C}_W$ if $i = 1$. The projection $p\colon \mathcal{D} \to \Delta^1$ is the cartesian fibration associated to the inclusion $\mathcal{C}_W \subseteq \mathcal{C}$ via unstraightening.

We claim that a morphism $f\colon x \to c$ with $c \in \mathcal{C}_W$ is a W-equivalence if and only if the associated morphism $f\colon (x, 0) \to (c, 1)$ in $\mathcal{D}$ is p-cocartesian. According to the dual of HTT 2.4.4.3, $f\colon (x, 0) \to (c, 1)$ is p-cocartesian if and only if the diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{D}}((c,1),(y,i)) & \xrightarrow{f^*} & \mathrm{Map}_{\mathcal{D}}((x,0),(y,i)) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\Delta^1}(1,i) & \longrightarrow & \mathrm{Map}_{\Delta^1}(0,i) \end{array}$$

is homotopy cocartesian for all $(y, i) \in \mathcal{D}$.

If $i = 0$, then both mapping spaces on the left are empty and the bottom righthand corner is contractible, so there is nothing to check. Therefore, assume $i = 1$. Then the spaces appearing in the bottom row are both contractible. Also, since $i = 1$, the y appearing in the top row always belongs to $\mathcal{C}_W$. The top horizontal map then identifies with

$$\mathrm{Map}_{\mathcal{C}}(c, y) \xrightarrow{f^*} \mathrm{Map}_{\mathcal{C}}(x, y),$$

and the diagram is homotopy cocartesian if and only if this map is an equivalence. By definition, this holds for all $y \in \mathcal{C}_W$ if and only if f is a W-equivalence. Hence a W-equivalence $x \to c$ with $c \in \mathcal{C}_W$ corresponds to a cocartesian lift with source $(x, 0)$ of the unique morphism in Δ^1.

We have shown that there exists W-equivalences from every object of $\mathcal{C}$ to a W-local object if and only if the cartesian fibration $\mathcal{D} \to \Delta^1$, which is the unstraightening of the inclusion $\mathcal{C}_W \subseteq \mathcal{C}$, is also a cocartesian fibration. But this is precisely the unstraightened condition for the inclusion to have a left adjoint (HTT 5.2.2.1). □

Remark. In the situation of the previous proposition, the proof of HTT 5.2.2.8 actually shows that the morphisms $f\colon x \to c$ form the unit of the adjunction, just as in the classical case. We will need this in the proof of Theorem 1.22.

Again the relationship between W and $\widetilde{W}$ is, modulo set-theoretic assumptions, that $\widetilde{W}$ is a kind of saturation of W. Lurie proves a general theorem of this sort (HTT 5.5.4.15), but using a stronger notion of saturation than we have been using. We stick to the weaker notions of saturation and weak saturation introduced above, which are defined correspondingly for ∞-categories. The argument from the proof of Proposition 1.9 carries over to show that $\widetilde{W}$ is saturated.

Again we will base our proof on the small object argument. We need a version of this result for locally presentable ∞-categories.

1.20 Definition. A morphism $g\colon x \to y$ in $\mathcal{C}$ has the *right lifting property* with respect to a morphism $f\colon c \to d$ if every square

$$\begin{array}{ccc} c & \longrightarrow & x \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ d & \longrightarrow & y \end{array}$$

can be extended to a 3-simplex

$$\begin{array}{ccc} c & \longrightarrow & x \\ \downarrow{\scriptstyle f} & \nearrow & \downarrow{\scriptstyle g} \\ d & \longrightarrow & y \end{array}$$

More generally, the morphism g has the *right lifting property* with respect to a set of morphisms S in $\mathcal{C}$ if it has the right lifting property with respect to every element of S.

1.21 Theorem (Small object argument [Lur18, 12.4.2.1])**.** *Suppose that the ∞-category $\mathcal{C}$ is locally presentable and that S is a small set of morphisms in $\mathcal{C}$. Every morphism $f: x \to y$ in $\mathcal{C}$ has a factorization*

$$x \xrightarrow{\lambda} z \xrightarrow{\rho} y$$

where λ lies in the weak saturation $\mathrm{sat}_w(S)$ and ρ has the right lifting property with respect to S.

The right lifting property has an equivalent definition in terms of mapping spaces. Namely, $g\colon x \to y$ has the right lifting property with respect to $f\colon c \to d$ if and only if the induced map

$$\mathrm{Map}_{\mathcal{C}_{/y}}(d, x) \xrightarrow{f^*} \mathrm{Map}_{\mathcal{C}_{/y}}(c, x) \tag{1.1}$$

is surjective on path-components for all maps $d \to y$. This property is too weak for our purposes because we have to produce a W-equivalence, meaning a map that induces *equivalences* on the relevant mapping spaces.

In the previous subsection, we adjoined fold maps to W to turn a surjective map into a bijection. The same trick works here. But we need the induced map on mapping spaces to be a bijection on homotopy groups in *all* degrees, not just on π_0. In order to achieve this, we make some simplifying assumptions that are satisfied in our applications.

Say that a set of morphisms S in a stable ∞-category is *closed under upward shifts* if $f \in S$ implies $\Sigma f \in S$. Then:

1.22 Theorem (Bousfield localization)**.** *Suppose that the ∞-category $\mathcal{C}$ is locally presentable and stable. Assume also that the set of morphisms W is small and closed under upward shifts. Then*

(1) *For each $x \in \mathcal{C}$, there is a map $f\colon x \to c$ with $f \in \mathrm{sat}(W)$ and $c \in \mathcal{C}_W$.*

(2) *The inclusion $\mathcal{C}_W \subseteq \mathcal{C}$ has a left adjoint.*

(3) *The set of W-equivalences $\widetilde{W}$ equals the saturation $\mathrm{sat}(W)$.*

Proof. Form W' by adjoining fold maps to W. For each $x \in \mathcal{C}$, the small object argument produces a factorization

$$x \xrightarrow{\lambda} c \to 0,$$

where 0 is the zero object of $\mathcal{C}$, λ lies in the saturation $\mathrm{sat}(W') = \mathrm{sat}(W)$, and $c \to 0$ has the right lifting property with respect to W'. We claim that c is W-local. Since 0 is, in particular, a terminal object, the lifting problem (1.1) is equivalent to the extension problem

$$\mathrm{Map}_{\mathcal{C}}(d, x) \xrightarrow{f^*} \mathrm{Map}_{\mathcal{C}}(c, x)$$

for $f\colon c \to d$ in W'. The fold map trick shows that this map is a bijection on π_0. Since W is closed under upward shifts, the maps

$$\mathrm{Map}_{\mathcal{C}}(\Sigma^n d, x) \xrightarrow{(\Sigma^n f)^*} \mathrm{Map}_{\mathcal{C}}(\Sigma^n c, x)$$

are also surjections on path components for all $n \geq 0$. But these maps identify with

$$\Omega^n \operatorname{Map}_{\mathcal{C}}(d, x) \xrightarrow{\Omega^n(f^*)} \Omega^n \operatorname{Map}_{\mathcal{C}}(c, x),$$

so we conclude that f induces bijections on higher homotopy groups as desired.

Now (2) and (3) follow from (1) as in the proof of Theorem 1.11. $\square$

1.3 Ore localization of ring spectra

A special case of the theory developed above is the localization of a ring spectrum at a subset of its ring homotopy groups. This theory, which seemingly is due to Lurie, is the subject of HA 7.2.3. Here we follow the introduction from the appendix in [Nik17].

Fix an $\mathbb{E}_1$-ring spectrum A and a subset $S \subseteq \pi_* A$. As in algebra, we want to find a universal map $i\colon A \to S^{-1}A$ having $i_*(S) \subseteq \pi_*(A')^\times$. As we will soon see, such a universal map always exists, and under certain conditions we can describe how the process of localization affects $\pi_* A$.

1.23 Definition. A left A-module spectrum M is called *S-local* if S acts invertibly on $\pi_* M$. Let $\mathrm{LMod}_A^{\sim S} \subseteq \mathrm{LMod}_A$ denote the full subcategory spanned by S-local modules.

1.24 Theorem. *The inclusion* $\mathrm{LMod}_A^{\sim S} \subseteq \mathrm{LMod}_A$ *has a left adjoint* L*. Furthermore, the left adjoint is given by* $L = S_{\mathrm{mod}}^{-1} A \otimes_A -$*, where* $S_{\mathrm{mod}}^{-1} A$ *is a uniquely determined* (A, A)*-bimodule with a map* $A \to S_{\mathrm{mod}}^{-1} A$ *of* (A, A)*-bimodules.*

Proof. We claim that the S-local modules are precisely the W_S-local modules (in the sense of the previous subsection), where

$$W_S = \{\Sigma^{-n} A \xrightarrow{\Sigma^{-n} R_s} \Sigma^{-n} A \mid n \in \mathbb{Z}, s \in S\}.$$

Here R_s denotes multiplication by s from the right. By definition, M is W_S-local if and only if

$$\operatorname{Map}_A(\Sigma^{-n} A, M) \xrightarrow{{R_s}^*} \operatorname{Map}_A(\Sigma^{-n} A, M)$$

is an equivalence for all n and s. Of course, here $R_s^* = (L_s)_*$. In LMod_A, we can identify the mapping space $\operatorname{Map}_A(-, -)$ with the infinite delooping of the *mapping spectrum* $\operatorname{map}_A(-, -)$ (see [Gep19, 3.2]). As in algebra, $\operatorname{map}_A(A, -) \simeq \mathrm{id}$. But then

$$\begin{array}{ccc}
\Omega^\infty \operatorname{map}_A(\Sigma^{-n}A, M) & \xrightarrow{L_{s*}} & \Omega^\infty \operatorname{map}_A(\Sigma^{-n}A, M) \\
\wr\downarrow & & \wr\downarrow \\
\Omega^\infty\Sigma^n \operatorname{map}_A(A, M) & \xrightarrow{\Sigma^n(L_{s*})} & \Omega^\infty\Sigma^n \operatorname{map}_A(A, M) \\
\wr\downarrow & & \wr\downarrow \\
\Omega^\infty\Sigma^n M & \xrightarrow{\Sigma^n L_s} & \Omega^\infty\Sigma^n M
\end{array}$$

Here the bottom arrow is an equivalence for all n if and only if $L_s\colon M \to M$ is an equivalence, proving our claim that the S-local modules are exactly the W_S-local modules. The ∞-category LMod_A is locally presentable (HA 4.2.3.7), so Theorem 1.22 implies that the inclusion $\mathrm{LMod}_A^{\sim S} \subseteq \mathrm{LMod}_A$ has a left adjoint L.

Clearly the subcategory $\mathrm{LMod}_A^{\sim S} \subseteq \mathrm{LMod}_A$ is closed under colimits, so the composition

$$\mathrm{LMod}_A \xrightarrow{L} \mathrm{LMod}_A^{\sim S} \hookrightarrow \mathrm{LMod}_A,$$

preserves colimits. By abuse of notation, we denote this composition by L. But then HA 4.8.4.1 (or more specifically HA 7.1.2.4) implies that

$$L \simeq S^{-1}_{\mathrm{mod}}A \otimes_A -$$

for some uniquely determined (A, A)-bimodule $S^{-1}_{\mathrm{mod}}A$. It follows immediately that $L(A) \simeq L \simeq S^{-1}_{\mathrm{mod}}A \otimes_A A \simeq S^{-1}_{\mathrm{mod}}A$. The map $A \to S^{-1}_{\mathrm{mod}}A$ is simply the unit of the adjunction. □

1.25 Corollary. *The bimodule $S^{-1}_{\mathrm{mod}}A$ has a unique A-algebra structure such that the canonical map $i\colon A \to S^{-1}_{\mathrm{mod}}A$ is a map of algebras.*

With this structure,

(1) *The map i is the localization of A at S as an A-algebra, i.e. for each A-algebra B,*

$$\mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_1}(A)}(S^{-1}_{\mathrm{mod}}A, B) \to \mathrm{Map}^{\sim S}_{\mathrm{Alg}_{\mathbb{E}_1}(A)}(A, B),$$

where $\mathrm{Map}^{\sim S}_{\mathrm{Alg}_{\mathbb{E}_1}(A)}(A, B)$ is the subspace of $\mathrm{Map}^{\mathrm{inv}(S)}_{\mathrm{Alg}_{\mathbb{E}_1}(A)}(A, B)$ spanned by maps which invert S.

(2) *Suppose that A is an R-algebra for some $\mathbb{E}_1$-ring spectrum R. Then i is the localization of A at S as an R-algebra.*

Proof. Since $L = S_{\mathrm{mod}}^{-1} \otimes_A -$ is a localization, it follows from Proposition 1.15 that the canonical maps

$$L(A) = S_{\mathrm{mod}}^{-1}A \rightrightarrows S_{\mathrm{mod}}^{-1}A \otimes_A S_{\mathrm{mod}}^{-1}A = LL(A)$$

are homotopic equivalences, so $S_{\mathrm{mod}}^{-1}A$ is idempotent in the monoidal ∞-category of (A, A)-bimodules and it follows abstractly that $S_{\mathrm{mod}}^{-1}A$ is an algebra in this ∞-category, i.e. pick an inverse $S_{\mathrm{mod}}^{-1}A \otimes_A S_{\mathrm{mod}}^{-1}A \to S_{\mathrm{mod}}^{-1}A$ as the multiplication map (see e.g. HA 4.8.2.9).

The abstract arguments in the proof of Theorem 1.24 can be used to show the existence of a localization of A at S as an A-algebra, which we denote by $A \to S_{\mathrm{alg}}^{-1}A$. We have a diagram:

$$\begin{array}{ccc} & & S_{\mathrm{mod}}^{-1}A \\ & \nearrow & \lambda \uparrow \downarrow \rho \\ A & \searrow & \\ & & S_{\mathrm{alg}}^{-1}A \end{array}$$

Universal properties give unique dashed arrows in both directions. Uniqueness implies that $\lambda \circ \rho \simeq \mathrm{id}$ and $\rho \circ \lambda \simeq \mathrm{id}$. Hence $S_{\mathrm{mod}}^{-1}A \simeq S_{\mathrm{alg}}^{-1}A$. The same argument shows that $S_{\mathrm{alg}}^{-1}A$ is the localization of A at S as an R-algebra. □

From now on, we let $S^{-1}A$ denote the localization of A at S in whatever sense is relevant to the context. According to the corollary, this does not lead to ambiguity.

Theorem 1.24 says that the localization of A at S always exists. However, we saw that this was a purely formal fact. We would like to understand the process of localization more concretely. In particular, we would like to understand how localization influences homotopy groups.

The problem of understanding localization concretely also shows up in algebra. To prepare for the main theorem of this subsection, we need a brief digression to the classical situation. Here the analog of Theorem 1.24 is:

1.26 Proposition. *Let S be a subset of a graded ring A_*. Let LMod_{A_*} denote the category of graded left A_*-modules, and let $\mathrm{LMod}_{A_*}^{\sim S}$ denote the full subcategory of S-local modules, i.e. modules on which S acts invertibly. Then the inclusion $\mathrm{LMod}_{A_*}^{\sim S} \subseteq \mathrm{LMod}_{A_*}$ has a left adjoint.*

Proof. As in the proof of Theorem 1.24, one shows that the S-local modules are exactly the W_S-local modules where

$$W_S = \{A_* \xrightarrow{-\cdot s} A_* \mid s \in S\}.$$

Then Theorem 1.11 implies the existence of the desired left adjoint. □

Of course, in algebra we are used to imposing certain restrictions on S (maybe even on A_*, but this is unnecessary). Under such restrictions, it is possible to describe the localization $S^{-1}A_*$ as a ring of left fractions with denominators in S. We would not like our restrictions to be *too* restrictive, however. The Ore condition strikes a good balance:

1.27 Definition. Let A_* be a graded ring. A subset $S \subseteq A_*$ satisfies the *left Ore condition* if

(1) S is multiplicatively closed and contains the identity.

(2) For all $a \in A_*$ and $s \in S$, there is $a' \in A_*$ and $s' \in S$ so that $as' = sa'$.

(3) If $as = 0$ where $a \in A_*$ and $s \in S$, then there is $t \in S$ with $ta = 0$.

1.28 Proposition. *Suppose $S \subseteq A_*$ satisfies the left Ore condition. Let M_* be a left A_*-module. Define a relation $\sim$ on $M_* \times S$*

$$(m, s) \sim (m', s') \quad \textit{iff} \quad \exists\, a, a' \in A_* : am = a'm' \quad \textit{and} \quad as = a's'.$$

The relation $\sim$ is an equivalence relation. Furthermore, there is a unique A_*-module structure on the quotient $S^{-1}M_* = (M_* \times S) / \sim$ so that the map $M_* \to S^{-1}M_*$ given by $a \mapsto [(a, 1)]$ is a module homomorphism exhibiting $S^{-1}M_*$ as the universal localization of M_* at S.

1.29 Definition. A map $f\colon M_* \to M'_*$ of graded left A_*-modules is an *S-nil equivalence* if

(1) For each $m' \in M'_*$ there is $s \in S$ so that $sm' \in \operatorname{im}(f)$.

(2) For each $m \in \ker(f)$ there is $s \in S$ so that $sm = 0$.

The following algebraic lemma will come in handy later:

1.30 Lemma. *Let $S \subseteq A_*$ be a multiplicatively closed subset containing the identity.*

(1) *S satisfies the left Ore condition if and only if right multiplication $A_* \xrightarrow{-\cdot s} A_*$ is an S-nil equivalence for each $s \in S$.*

(2) *Assume that S satisfies the left Ore condition. Let $f\colon M_* \to M'_*$ be a map of left A_*-modules. Then there is a commutative diagram*

$$\begin{array}{ccc} & & M'_* \\ & \overset{f}{\nearrow} & \uparrow\wr \\ M_* & \longrightarrow & S^{-1}M_* \end{array}$$

where $M_* \to S^{-1}M_*$ is the universal localization if and only if f is an S-nil equivalence and M'_* is S-local.

Proof. (1) follows directly from definitions. As for (2), it follows from the proof of Proposition 1.26 that the universal localization $M_* \to S^{-1}M_*$ lies in the weak saturation of $W_S = \{A_* \xrightarrow{-\cdot s} A_*\}$. But W_S is contained in the class of S-nil equivalences, which is easily seen to be saturated. Hence $M_* \to S^{-1}M_*$ must in particular be an S-nil equivalence. By the two-out-of-three property, the induced map $S^{-1}M_* \to M'_*$ is an S-nil equivalence if and only if f is an S-nil equivalence. Since S-nil equivalences between S-local modules are just isomorphisms, we conclude that f is an S-nil equivalence if and only if the induced map $S^{-1}M_* \to M'_*$ is an isomorphism. □

We now return to the world of higher algebra. Let A be an $\mathbb{E}_1$-ring spectrum again.

1.31 Definition. A map $f\colon M \to M'$ of left A-module spectra is an *S-nil equivalence* if the induced map of $\pi_*(M) \to \pi_*(M')$ of left π_*A-modules is an S-nil equivalence.

1.32 Lemma. *The class of S-nil equivalences in LMod_A is saturated.*

Proof. Obviously the class of S-nil equivalences has the two-out-of-three property and contains all equivalences. Closure under retracts and pushouts follows from the long exact sequence of homotopy groups. Closure under transfinite composition follows from the fact that the sphere spectrum is compact. □

1.33 Theorem (Lurie). *Suppose $S \subseteq \pi_*(A)$ satisfies the left Ore condition. For every left A-module M, there is a unique isomorphism*

$$\varphi\colon S^{-1}\pi_*(M) \xrightarrow{\sim} \pi_*(S^{-1}M)$$

so that

$$\begin{array}{ccc} & & \pi_*(S^{-1}M) \\ & \overset{f}{\nearrow} & \uparrow{\scriptstyle \varphi\,\wr} \\ \pi_*(M) & \longrightarrow & S^{-1}\pi_*(M) \end{array}$$

commutes. Here the bottom arrow is the localization of $\pi_*(M)$ at S as a $\pi_*(A)$-module.

Proof. In the proof of Theorem 1.24 we saw that the localizations $M \to S^{-1}M$ come from the Bousfield localization of LMod_A at the set

$$W_S = \{\Sigma^{-n}A \xrightarrow{\Sigma^{-n}R_s} \Sigma^{-n}A \mid n \in \mathbb{Z}, s \in S\}.$$

Since S is assumed to satisfy the left Ore condition, Lemma 1.30(1) says that $W_S \subseteq \{S\text{-nil equivalences}\}$. Saturating both sides and using Lemma 1.32, we find $\mathrm{sat}(W_S) \subseteq \{S\text{-nil equivalences}\}$. But then Theorem 1.22 implies that the localization $M \to S^{-1}M$ is an S-nil equivalence, and since $\pi_*(S^{-1}M)$ is S-local, the result follows from Lemma 1.30(2). □

2 Group completion

This section forms the heart of our exposition. We first prove a classical delooping theorem for $\mathbb{E}_1$-groups[2] in § 2.1. Using this result, we prove the group completion theorem in § 2.2. The construction in § 2.3 allows us to understand the group completion theorem via a stable equivalence at the level of spaces, and in § 2.4 we see that the plus construction makes this equivalence an actual equivalence. Finally, we describe a classical example in § 2.5

[2]In older literature, these are called *grouplike*.

2.1 Delooping $\mathbb{E}_1$-groups

Categorical products endow the category of Kan complexes enriched over Kan complexes with a symmetric monoidal structure. This structure lifts to a symmetric monoidal structure on the ∞-category of spaces. Throughout what follows, an $\mathbb{E}_1$-monoid (resp. $\mathbb{E}_\infty$-monoid) will mean an $\mathbb{E}_1$-algebra (resp. $\mathbb{E}_\infty$-algebra) in the symmetric monoidal ∞-category of spaces. For the reader's convenience, we have spelled out these definitions in more elementary terms in the appendix.

Since the ∞-category of spaces contains small colimits, the following definition makes sense:

2.1 Definition (HTT 6.1.2.12)**.** Let $A_\bullet$ be a simplicial space. The *geometric realization* of $A_\bullet$ is the colimit

$$\|A_\bullet\| = \varinjlim_{N\Delta^{\mathrm{op}}} A_\bullet.$$

When $A_\bullet = M_\bullet$ is an $\mathbb{E}_1$-monoid, we instead refer to the geometric realization of $M_\bullet$ as the *classifying space* of $M_\bullet$ and, abusing notation, we write $BM = \|M_\bullet\|$.

Let $M_\bullet$ be an $\mathbb{E}_1$-monoid. We have a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{d_0} & M_0 \\ \downarrow{\scriptstyle d_1} & & \downarrow \\ M_0 & \longrightarrow & BM \end{array}$$

where $M_0 \to BM$ is simply the canonical map into the colimit. Here $M_0 \simeq *$, and since by definition ΩBM is the pullback of $* \to BM \leftarrow *$, we get a canonical map of $\mathbb{E}_1$-monoids $M \to \Omega BM$.

We are interested in the question of when $M \to \Omega BM$ is an equivalence. In particular, this would display M as a loop space in a canonical way. Of course, we cannot always expect $M \to \Omega BM$ to be an equivalence since ΩBM is, in classical terms, a H-group whereas M is only a H-space – i.e. we have not assumed that the $\mathbb{E}_1$-structure on M comes with inverses. The $\mathbb{E}_1$-structure on M endows $\pi_0 M$ with a canonical monoid structure, and at this level our observation becomes that $\pi_0 M$ is not always a group, whereas $\pi_0(\Omega BM)$ is always a group. If $\pi_0 M$ is a group, we say that M is an $\mathbb{E}_1$-*group*. A theorem going back at least to [Sta63] states that the failure of $\pi_0 M$ to be a group is the *only* obstruction to $M \to \Omega BM$ being an equivalence:

2.2 Theorem (Stasheff)**.** *Let M be an $\mathbb{E}_1$-monoid. The canonical map $M \to \Omega BM$ is an equivalence if and only if M is an $\mathbb{E}_1$-group.*

Proof. As discussed above, the "only if" statement is trivial. To prove the nontrivial "if" statement, we follow Segal's proof from [Seg74]. First note that $\pi_0 M$ is a group if and only if the *shear maps*

$$M \times M \xleftarrow[\sim]{(d_0,d_2)} M_2 \xrightarrow{(d_0,d_1)} M \times M \quad \text{and} \quad M \times M \xleftarrow[\sim]{(d_0,d_2)} M_2 \xrightarrow{(d_2,d_1)} M \times M$$

are equivalences. These are the maps $M \times M \to M \times M$ mapping (m, m') to (mm', m) and (m, mm') respectively. Note that the first shear map is an equivalence if and only if the diagram

$$\begin{array}{ccc} M_2 & \xrightarrow{d_1} & M \\ \downarrow{\scriptstyle d_0} & & \downarrow{\scriptstyle d_0} \\ M & \xrightarrow{d_1} & M_0 \end{array}$$

is a pullback. The same observation holds for the second shear map, and it follows more generally (using simplicial identities and the Segal condition) that

$$\begin{array}{ccc} M_{n+1} & \xrightarrow{\psi^*} & M_{m+1} \\ \downarrow{\scriptstyle d_0} & & \downarrow{\scriptstyle d_0} \\ M_n & \xrightarrow{\theta^*} & M_m \end{array} \tag{2.1}$$

is a pullback where $\theta\colon [m] \to [n]$ is any map and $\psi\colon [m+1] \to [n+1]$ is the map defined by $\psi(0) = 0$ and $\psi(k) = \psi(k-1) + 1$.

Recall that the simplicial space A has a *path-space*

$$PA_\bullet = A_\bullet \circ P\colon N\Delta^{\mathrm{op}} \to \mathcal{S}$$

where $P\colon N\Delta \to N\Delta$ is the functor taking $[n]$ to $[n+1]$ and $\theta\colon [m] \to [n]$ to the map $\psi\colon [m+1] \to [n+1]$ defined as in diagram (2.1). Moreover, the coface operator d^0 defines a natural transformation $\mathrm{id} \to P$, and thus a canonical map $PA_\bullet \to A_\bullet$. Note that by definition $PA_n = A_{n+1}$. Crucially, $\|PA_\bullet\| \simeq *$. To prove the theorem, it therefore suffices to show that

$$\begin{array}{ccc} M_1 = PM_0 & \longrightarrow & \|PM\| \\ \downarrow & & \downarrow \\ M_0 & \longrightarrow & BM \end{array}$$

is a pullback. But according to the following lemma (specifically the $n = 0$ case of the conclusion), this follows from the fact that the diagram (2.1) is a pullback (for all θ). □

2.3 Lemma ([Seg74, 1.6]). *Let $f\colon A'_\bullet \to A_\bullet$ be a map of simplicial spaces so that*

$$\begin{array}{ccc} A'_n & \xrightarrow{\theta^*} & A'_m \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_m} \\ A_n & \xrightarrow{\theta^*} & A_m \end{array}$$

is a pullback for each $\theta\colon [m] \to [n]$. Then for each n, the diagram

$$\begin{array}{ccc} A'_n & \longrightarrow & \|A'_\bullet\| \\ \downarrow{\scriptstyle f_n} & & \downarrow \\ A_n & \longrightarrow & \|A_\bullet\| \end{array}$$

is a pullback.

2.2 The group completion theorem

Let M be an $\mathbb{E}_1$-monoid, let R be an $\mathbb{E}_\infty$-ring spectrum (e.g. $H\mathbb{Z}$ or $\mathbb{S}$), and let π_R be the image of the Hurewicz map $\pi_0 M \to R_0 M$.

2.4 Theorem (Group completion). *Suppose that π_R satisfies the left Ore condition in R_*M. Then the canonical map $M \to \Omega BM$ induces an isomorphism of rings*

$$\pi_R^{-1} R_*(M) \xrightarrow{\sim} R_*(\Omega BM).$$

Proof. We claim that the functor $\Omega B\colon \mathrm{Mon}_{\mathbb{E}_1}(\mathcal{S}) \to \mathrm{Mon}_{\mathbb{E}_1}(\mathcal{S})$ is a localization with essential image $\mathrm{Grp}_{\mathbb{E}_1}(\mathcal{S})$. The canonical maps $M \to \Omega BM$ form the components of a natural transformation, so according to Proposition 1.15 it suffices to check that both canonical maps $\Omega BM \rightrightarrows \Omega B\Omega BM$ are equivalences. But since ΩBM is grouplike, this follows from Theorem 2.2.

We now claim that $R[M] \to R[\Omega BM]$ exhibits $R[\Omega BM]$ as the localization of $R[M]$ at π_R as an $\mathbb{E}_1$-algebra. Note that ordinarily $R[-]$ is a functor from $\mathcal{S}$ to LMod_R, explicitly defined as $R \otimes \Sigma^\infty_+(-)$. At this level we have adjunctions

$$\mathrm{LMod}_R \underset{}{\overset{R\otimes -}{\leftrightarrows}} \mathrm{Sp} \underset{\Omega^\infty}{\overset{\Sigma^\infty_+}{\leftrightarrows}} \mathcal{S}$$

All of these ∞-categories are symmetric monoidal. The functors $R \otimes -$ and Σ^∞_+ are monoidal, so their adjoints are lax monoidal by easy nonsense. The upshot is that we can promote everything to the level of $\mathbb{E}_1$-algebras:

$$\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{LMod}_R) \underset{}{\overset{R\otimes -}{\leftrightarrows}} \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp}) \underset{\Omega^\infty}{\overset{\Sigma^\infty_+}{\leftrightarrows}} \mathrm{Mon}_{\mathbb{E}_1}(\mathcal{S})$$

The claim that $R[M] \to R[\Omega BM]$ is a localization follows from the adjunction. The theorem now follows from Theorem 1.33. □

2.3 The $(-)_\infty$ construction

Assume from now on that M is an $\mathbb{E}_\infty$-monoid. This implies that the Ore condition comes for free. More importantly, it permits the following construction:

2.5 Construction. Let X be a left M-space. Since M is an $\mathbb{E}_\infty$-monoid, each multiplication $X \xrightarrow{m} X$ $(m \in M)$ is a map of left M-spaces. Explicitly, for each $m, n \in M$ the $\mathbb{E}_\infty$-structure on M produces a canonical homotopy $h\colon n\cdot(m\cdot -) \simeq m\cdot(n\cdot -)$ which is determined by the path $\tau_{(m,n)}\colon t \mapsto h(t, e)$, where e denotes the unit of M. But then

$$\begin{array}{ccc} X & \xrightarrow{m} & X \\ \downarrow{\scriptstyle n} & \swarrow\!\!\swarrow_{\tau_{(m,n)}} & \downarrow{\scriptstyle n} \\ X & \xrightarrow{m} & X \end{array}$$

is a commutative diagram in the ∞-categorical sense, where the 2-cell is multiplication by $\tau_{(m,n)}$. Hence the colimit

$$X_m = \varinjlim(X \xrightarrow{m} X \xrightarrow{m} X \to \dots)$$

may be taken in the category of left M-spaces, and is therefore again a left M-space.

Once and for all, fix a well-ordered generating set T of M. For each finite subset $S \subseteq T$, we may write $S = \{m_1, \dots, m_n\}$ such that $m_1 < \cdots < m_n$. Inductively, define

$$X_S = (X_{\{m_1,\dots,m_{n-1}\}})_{m_n}.$$

Finally, put

$$X_\infty = \varinjlim_S X_S,$$

where the colimit is taken over finite subsets $S \subseteq T$.

Obviously the construction is well-defined and functorial. In particular, $i\colon M \to \Omega BM$ induces $M_\infty \to (\Omega BM)_\infty$. Since the action of M on ΩBM is by equivalences, there is a canonical equivalence $(\Omega BM)_\infty \simeq \Omega BM$. Precomposing with the canonical map $M \to M_\infty$, we get a sequence of canonical maps:

$$M \to M_\infty \to \Omega BM.$$

This sequence is our next object of study. Note that $M_\infty \to \Omega BM$ is a stable equivalence (i.e. induces isomorphisms on stable homotopy) by the group completion theorem. In general, however, this map will not be a true equivalence. We will study an intersting counterexample in § 2.5. First, however, we want to know when the map is an equivalence.

2.6 Construction. For each $n \geq 2$, the fact that M is an $\mathbb{E}_\infty$-monoid implies that the multiplication map $M^n \to M$ factors through the homotopy coinvariants $(M^n)_{h\Sigma_n}$ of the action which permutes coordinates.[3]

For each $m \in M$, let $(m, \dots, m)\colon B\Sigma_n \to (M^n)_{h\Sigma_n}$ be the map induced by $(m, \dots, m)\colon * \to M^n$. Then m defines a map

$$\phi(m)\colon B\Sigma_n \to (M^n)_{h\Sigma_n} \to M \to M_\infty.$$

2.7 Proposition. *The following are equivalent:*

(1) *The canonical action of $\pi_0 M$ on M_∞ is via equivalences.*

(2) *The map $M \to M_\infty$ is a localization with respect to left M-spaces on which $\pi_0 M$ acts via equivalences.*

[3]Outside the world of higher nonsense, a roundabout way of saying that a monoid M (in Sets) is commutative is that the n-fold multiplication map $M^n \to M$ factors through the coinvariants $(M^n)_{\Sigma_n}$ of the action which permutes coordinates.

(3) *The canonical map $M_\infty \to \Omega BM$ is an equivalence.*

(4) *The fundamental groups of M_∞ at all basepoints are abelian.*

(5) *The fundamental groups of M_∞ at all basepoints are hypoabelian, meaning that they have no nontrivial perfect subgroups.*

(6) *For each $m \in T$, the map*

$$\phi(m)_*\colon \Sigma_3 \to \pi_1(M, m^3) \to \pi_1(M_\infty, m^3)$$

sends (123) *to zero.*

(7) *For each $m \in T$ there is some $n \geq 2$ such that*

$$\phi(m)_*\colon \Sigma_n \to \pi_1(M, m^n) \to \pi_1(M_\infty, m^n)$$

sends $(12\dots n)$ *to zero.*

Proof of Proposition 2.7. Note that (3) ⇒ (4), (4) ⇒ (5) and (6) ⇒ (7) are trivial. Furthermore, (4) ⇒ (6) follows from the algebraic fact that (123) lies in the commutator subgroup of Σ_3, and similarly (5) ⇒ (7) follows from the fact that $(12\dots n)$ lies in the largest perfect subgroup of Σ_n for odd $n \geq 5$. It now suffices to show (1) ⇒ (2), (2) ⇒ (3) and (7) ⇒ (1).

Of the remaining implications, we first prove (1) ⇒ (2). Say that a left M-space X is *$\pi_0 M$-local* if $\pi_0 M$ acts invertibly on X. By assumption M_∞ is $\pi_0 M$-local, so it suffices to show that the induced map on mapping spaces

$$\mathrm{Map}(M_\infty, X) \to \mathrm{Map}(M, X)$$

is an equivalence for every $\pi_0 M$-local space X. (Here the mapping spaces are taken in the ∞-category of left M-spaces $\mathrm{LMod}_M(\mathcal{S})$, but we suppress this in our notation.) Due to the recursive definition M_∞, we may assume that the set of generators T from earlier consists of a single point $m \in M$. But then

$$\begin{aligned}\mathrm{Map}(M_\infty, X) &= \mathrm{Map}(\varinjlim(M \xrightarrow{m} M \xrightarrow{m} \dots), X)\\ &\simeq \varprojlim\left(\mathrm{Map}(M, X) \xrightarrow{m^*} \mathrm{Map}(M, X) \xrightarrow{m^*} \dots\right)\\ &\simeq \varprojlim\left(\mathrm{Map}(M, X) \xrightarrow{m_*} \mathrm{Map}(M, X) \xrightarrow{m_*} \dots\right).\end{aligned}$$

In the last expression, note that m_* is an equivalence since X is $\pi_0 M$-local. Hence the limit identifies with $\mathrm{Map}(M, X)$ and this identification comes from the canonical map $M \to M_\infty$.

To show that (2) $\Rightarrow$ (3), it suffices to show that the localization of M (at $\pi_0 M$) as a left M-space is automatically its localization (at $\pi_0 M$) as an $\mathbb{E}_1$-monoid, and hence coincides with the group completion of M. But this follows from the same nonsense as in the proof of Corollary 1.25.

It remains to be seen that (7) $\Rightarrow$ (1). Again we assume the generating set T of M consists of a single generator m. To show (1), it clearly suffices to show that m acts invertible on M_∞. In other words, we must show that the map induced on colimits by

$$\begin{tikzcd} M \arrow[r, "m"] \arrow[d, "m"] & M \arrow[r, "m"] \arrow[d, "m"] \arrow[dl, Leftarrow, "\tau"] & M \arrow[r, "m"] \arrow[d, "m"] \arrow[dl, Leftarrow, "\tau"] & \dots \\ M \arrow[r, "m"] & M \arrow[r, "m"] & M \arrow[r, "m"] & \dots \end{tikzcd} \tag{2.2}$$

is an equivalence. Here τ denotes the symmetry path $\tau_{(m,m)}$ from earlier, which is a loop here. For once we have included the diagonals and 2-cells, since these will be relevant to our argument. There are no issues with higher coherence since the indexing category is $(\mathbb{N}, \leq)$.

We claim that the map induced by

$$\begin{tikzcd} M \arrow[r, "m"] \arrow[dr, "\mathrm{id}"] & M \arrow[r, "m"] \arrow[d, Rightarrow, "\mathrm{id}"] \arrow[dr, "\mathrm{id}"] & M \arrow[r, "m"] \arrow[d, Rightarrow, "\mathrm{id}"] \arrow[dr, "\mathrm{id}"] & \dots \\ M \arrow[r, "m"] & M \arrow[r, "m"] & M \arrow[r, "m"] & \dots \end{tikzcd}$$

is an inverse to the map induced by (2.2). In both directions, the composition is induced by

$$\begin{tikzcd} M \arrow[r, "m"] \arrow[dr, "m"] & M \arrow[r, "m"] \arrow[d, Rightarrow, "\tau"] \arrow[dr, "m"] & M \arrow[r, "m"] \arrow[d, Rightarrow, "\tau"] \arrow[dr, "m"] & \dots \\ M \arrow[r, "m"] & M \arrow[r, "m"] & M \arrow[r, "m"] & \dots \end{tikzcd} \tag{2.3}$$

Note that the loop τ corresponds to the image of the transposition (12) under $\Sigma_2 \to \pi_1(M, m^2)$. More generally, the composition of $n-1$ consecutive 2-cells in diagram (2.3) corresponds to the image of $(12\dots n)$ under $\Sigma_n \to \pi_1(M, m^n)$. Our assumption says that this 2-cell is equivalent to the identity 2-cell for some $n \geq 2$. It follows that the map coming from diagram (2.3) is the identity. $\square$

2.8 Corollary. *Suppose that M has hypoabelian fundamental groups at every basepoint. Then $M_\infty \to \Omega BM$ is an equivalence, so in particular M_∞ has abelian fundamental groups.*

Proof. For each $m \in T$ and $n \geq 5$, the first map in the composition $\Sigma_n \to \pi_1(M, m^n) \to \pi_1(M_\infty, m^n)$ must annihilate $(12\ldots n)$, since the latter lies in the largest perfect subgroup of Σ_n. The conclusion now follows from Theorem 2.7(7). □

2.4 Connection with the plus construction

Classically (see [HH79]), the plus construction on a space X is a map $i\colon X \to X^+$ such that

- The map i is *acyclic*, meaning that it induces isomorphisms on homology in all local coefficient systems.
- For each basepoint $x_0 \in X$, the induced map $\pi_1(X, x_0) \to \pi_1(X^+, i(x_0))$ is surjective and its kernel is the largest perfect subgroup of $\pi_1(X, x_0)$.

The second condition implies in particular that X^+ has hypoabelian fundamental groups at all basepoints.

Since local coefficient systems completely detect obstructions to lifting problems (see [Whi12]), this definition characterizes $X \to X^+$ as the unit of an adjunction. Here the right adjoint is the inclusion $\mathcal{S}^{\mathrm{hypo}} \subseteq \mathcal{S}$ of the full subcategory spanned by spaces whose fundamental groups are hypoabelian at all basepoints and the left adjoint is the plus construction functor $(-)^+$. When viewed as a functor from spaces to spaces, the plus construction preserves products, so in particular it preserves $\mathbb{E}_\infty$-monoids.

The following theorem is hard-won in the classical approach to group completion, but follows easily from our setup:

2.9 Theorem (Randal-Williams [RW13], Nikolaus [Nik17])**.** *Both canonical maps*

$$(M^+)_\infty \leftarrow M_\infty \to \Omega BM$$

are plus constructions. In particular,

$$(M^+)_\infty \simeq (M_\infty)^+ \simeq \Omega BM.$$

Proof. We start by checking that $M_\infty \to (M^+)_\infty$ is a plus construction. First note that $(M^+)_\infty$ has hypoabelian (even abelian!) fundamental groups by Corollary 2.8. Hence all that remains is to prove that for each space X with hypoabelian fundamental groups, the induced map

$$\mathrm{Map}_{\mathcal{S}}((M^+)_\infty, X) \to \mathrm{Map}_{\mathcal{S}}(M_\infty, X)$$

is an equivalence. For simplicity, we again assume that the generating set T consists of a single generator. But then

$$\begin{aligned}\mathrm{Map}_{\mathcal{S}}((M^+)_\infty, X) &= \mathrm{Map}_{\mathcal{S}}(\varinjlim(M^+ \xrightarrow{m} M^+ \xrightarrow{m} \dots), X) \\ &\simeq \varprojlim \left(\mathrm{Map}_{\mathcal{S}}(M^+, X) \xrightarrow{m^*} \mathrm{Map}_{\mathcal{S}}(M^+, X) \xrightarrow{m^*} \dots\right) \\ &\simeq \varprojlim \left(\mathrm{Map}_{\mathcal{S}}(M, X) \xrightarrow{m^*} \mathrm{Map}_{\mathcal{S}}(M, X) \xrightarrow{m^*} \dots\right) \\ &\simeq \mathrm{Map}_{\mathcal{S}}(M_\infty, X),\end{aligned}$$

proving the claim.

We now show that $M_\infty \to \Omega BM$ is a plus construction. Functoriality gives a commutative diagram

$$\begin{array}{ccc} (M_\infty)^+ & \longrightarrow & \Omega BM \\ \downarrow \wr & & \downarrow \wr \\ (M^+)_\infty & \xrightarrow{\sim} & \Omega B(M^+) \end{array}$$

Here the lefthand vertical arrow is an equivalence by what we have proved; the bottom arrow is an equivalence by Corollary 2.8; and the righthand vertical arrow is a stable equivalence by the group completion theorem, and thus an equivalence since both spaces are loop spaces. □

2.5 The Barratt-Priddy-Quillen theorem

In this subsection, we apply the theory from above to show that each path-component of the stable sphere $\Omega^\infty\mathbb{S}$ is equivalent to $B\Sigma_\infty^+$, the plus construction on the classifying space of the group of permutations of $\mathbb{Z}_{\geq 0}$ which fix all but finitely many elements. In order to arrive at this beautiful and surprising theorem, we must first recall the Barratt-Priddy-Quillen theorem.

Let SymMonCat (resp. SymMonGrpd) denote the category of small symmetric monoidal categories (resp. groupoids) and monoidal functors. The forgetful functor

$$\mathrm{SymMonGrpd} \to \mathrm{SymMonCat}$$

has a right adjoint $(-)^{\sim}$. Explicitly, if $\mathcal{C}$ is a small symmetric monoidal category, we define $\mathcal{C}^{\sim}$ to be the monoidal subcategory of $\mathcal{C}$ containing all objects of $\mathcal{C}$, but having as morphisms only the isomorphisms of the latter. The counit of this adjunction is the inclusion $\mathcal{C}^{\sim} \subseteq \mathcal{C}$.

Consider the sequence of adjunctions

$$\mathrm{Sets} \underset{U}{\overset{}{\rightleftarrows}} \mathrm{SymMonCat} \underset{(-)^{\sim}}{\overset{U}{\leftrightarrows}} \mathrm{SymMonGrpd} \underset{N}{\overset{h}{\leftrightarrows}} \mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{S}) \underset{U}{\overset{\Omega B}{\leftrightarrows}} \mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{S}),$$

where the functors labelled U are forgetful.

2.10 Definition. *Algebraic K-theory* is the composite functor

$$K = \Omega B N(-)^{\sim} \colon \mathrm{SymMonCat} \to \mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{S}).$$

2.11 Theorem (Barratt-Priddy-Quillen)**.** *There is an equivalence of $\mathbb{E}_\infty$-groups*

$$\Omega^\infty \mathbb{S} \simeq K(\mathrm{Fin}),$$

where Fin *denotes the skeletal category of finite sets with the symmetric monoidal structure of disjoint union.*

Sketch of proof. The forgetful functor $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{S}) \to \mathcal{S}$ has a left adjoint F which at the level of objects is given by

$$F \colon X \mapsto \coprod_{n \geq 0} (X^n)_{h\Sigma_n},$$

where Σ_n acts on X^n by coordinate permutation. Note that

$$F(*) \simeq \coprod_{n \geq 0} B\Sigma_n.$$

Since ΩB is left adjoint to the forgetful functor $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{S}) \to \mathrm{Mon}_{\mathbb{E}_\infty}$, the composite functor $\Omega B \circ F\colon \mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{S}) \to \mathcal{S}$ is left adjoint to the forgetful functor from $\mathbb{E}_\infty$-groups to spaces. Hence

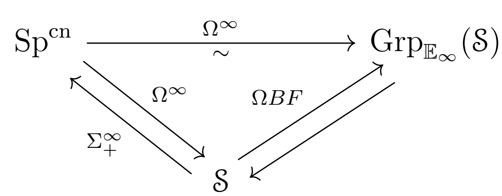
$$\Omega BF(*) \simeq \Omega B \coprod_{n\geq 0} B\Sigma_n \tag{2.4}$$

is the free $\mathbb{E}_\infty$-group on a single generator.

There is another way to compute the free $\mathbb{E}_\infty$-group on a single generator. A classical fact states that connective spectra are the same as $\mathbb{E}_\infty$-groups (see [Seg74, 3.4]). More precisely,

$$\Omega^\infty\colon\ \mathrm{Sp}^{\mathrm{cn}} \xrightarrow{\sim} \mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{S}).$$

is an equivalence (HA 5.2.6.26). From the commutative diagram



we see that

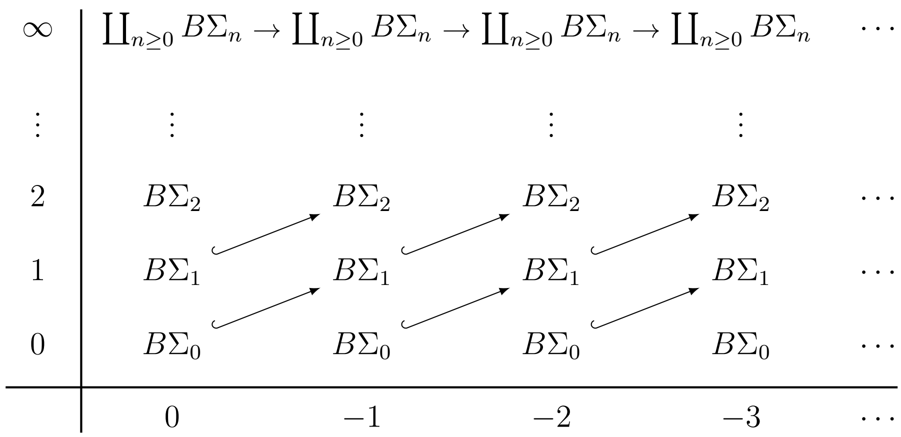
$$\Omega BF(*) = \Omega^\infty \Sigma^\infty_+(*) = \Omega^\infty \mathbb{S}. \tag{2.5}$$

Combining (2.4) and (2.5), we find that $\Omega^\infty \mathbb{S} \simeq \Omega B \coprod_{n\geq 0} B\Sigma_n$. But by definition $K(\mathrm{Fin}) = \Omega BN(\mathrm{Fin}^\sim) = \Omega B \coprod_{n\geq 0} B\Sigma_n$. □

2.12 Corollary. *There is an equivalence of spaces*

$$\mathbb{Z} \times B\Sigma_\infty^+ \simeq \Omega^\infty \mathbb{S}.$$

Proof. The $\mathbb{E}_1$-monoid $M = \coprod_{n\geq 0} B\Sigma_n$ is generated by the basepoint m_0 in $B\Sigma_1$. As may be seen at the symmetric monoidal groupoid level, this basepoint acts via shifting M, meaning that multiplication by m_0 embeds $B\Sigma_n$ in $B\Sigma_{n+1}$. The particular embedding $B\Sigma_n \hookrightarrow B\Sigma_{n+1}$ is induced by the injection $\Sigma_n \hookrightarrow \Sigma_{n+1}$ which sends an n-permutation σ to the $(n+1)$-permutation which acts by σ on the first n elements while fixing the last element. Hence $M_\infty \simeq \mathbb{Z} \times B\Sigma_\infty$, where the n-th summand is contributed by the diagonal starting at the cell labelled n in the following picture:



The conclusion now follows from Theorem 2.9. □

A Appendix

Let Fin_* denote the category of finite pointed sets and pointed maps. For each $n \geq 0$, let $\langle n \rangle$ denote the set $\{0, \ldots, n\}$ with 0 as its basepoint; also, for each $1 \leq i \leq n$ let $\rho^i \colon \langle n \rangle \to \langle 1 \rangle$ denote the morphism in Fin_* that sends i to 1 and everything else to 0.

A.1 Definition. An $\mathbb{E}_\infty$-*monoid* is a functor $M \colon N(\mathrm{Fin}_*) \to \mathcal{S}$ such that

(1) The space $M\langle 0 \rangle$ is contractible.

(2) The induced maps $(\rho^i)_* \colon M\langle n \rangle \to M\langle 1 \rangle$ exhibit $M\langle n \rangle$ as the n-fold product of $M\langle 1 \rangle$.

A.2 Definition. The *cut functor* is the faithful contravariant functor $\mathrm{cut} \colon \Delta \to \mathrm{Fin}_*$ given

- on objects by sending $[n]$ to $\langle n \rangle$;
- on morphisms by sending $\theta \colon [m] \to [n]$ to the map $f \colon \langle n \rangle \to \langle m \rangle$ determined by

$$\begin{aligned} f^{-1}(0) &= \{i \leq \theta(0)\} \cup \{\theta(m) < i \leq n\} \\ f^{-1}(k) &= \{\theta(j-1) < i \leq \theta(j)\} \end{aligned}$$

  for $0 < k \leq n$.

The cut functor allows us to think of Δ^{op} as a subcategory of Fin_* with the same objects but far fewer morphisms. However, the morphisms ρ^i which appear in Definition A.1 are contained in (the image of) Δ^{op}. Explicitly, $\rho^i = \mathrm{cut}(\rho_i)$ where $\rho_i\colon [1] \to [n]$ is the morphism that sends 0 to $i-1$ and 1 to i.

A.3 Definition. An $\mathbb{E}_1$-*monoid* is a functor $M\colon N\Delta^{\mathrm{op}} \to \mathcal{S}$ such that

(1) The space $M[0]$ is contractible.

(2) The induced maps $(\rho_1)^*\colon M[n] \to M[1]$ exhibit $M[n]$ as the n-fold product of $M[1]$.

More generally, a functor $A\colon N\Delta^{\mathrm{op}} \to \mathcal{S}$ is just a (homotopy coherent) *simplicial space*. To emphasize this, we write $A_\bullet = A$ and $A_n = A[n]$.

Let $M_\bullet$ be an $\mathbb{E}_1$-monoid. Intuitively, the fact that $\rho_1, \dots, \rho_n$ exhibit M_n as the n-fold product of M_1 suggests that we should think of $M_\bullet$ as a kind of bar construction on M_1. That is, forgetting higher coherence we can picture $M_\bullet$ as a diagram

$$\cdots \;\; \substack{\longrightarrow \\ \vdots \\ \longrightarrow} \;\; M_n \;\; \substack{\longrightarrow \\ \vdots \\ \longrightarrow} \;\; \cdots \;\; \substack{\longrightarrow \\ \vdots \\ \longrightarrow} \;\; M_2 \;\; \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} \;\; M_1 \;\; \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} \;\; M_0 \simeq *$$

where the indicated arrows are face and degeneracy operators. Thus the middle arrow $d_1\colon M_2 \to M_1$ should be though of as *multiplication*. Functoriality of $M_\bullet$ encodes the homotopy coherent associativity of this multiplication.

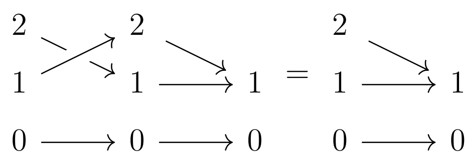
Similarly, choosing an equivalence $M_0 \simeq *$ equips M_1 with a basepoint via the degeneracy operator $M_0 \to M_1$, and simplicial identities show that this basepoint behaves like a homotopy coherent unit with respect to the multiplication map.

We will think of M_1 as the *underlying monoid* of $M_\bullet$ and write $M = M_1$.

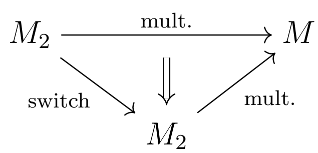
An $\mathbb{E}_\infty$-monoid can be forgetfully viewed as an $\mathbb{E}_1$-monoid by precomposing with the cut functor. The additional structure on $\mathbb{E}_\infty$-monoids comes from the fact that there are more maps in Fin_*. Concretely, the extra maps are those which are not order-preserving. For instance, there is a switching morphism $\langle 2 \rangle \to \langle 2 \rangle$:

$$\begin{array}{ccc} 2 & \searrow\!\!\!\!\nearrow & 2 \\ 1 & & 1 \\ 0 & \longrightarrow & 0 \end{array}$$

Composing with the multiplication morphism, we find that



By functoriality, when we pass to spaces we therefore have a commutative diagram in the ∞-categorical sense:



Along with the remaining structure, this encodes homotopy coherent commutativity of the multiplication in an $\mathbb{E}_\infty$-monoid.

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